

Fundamentals
of the Theory
of Electricity

I. E. Tamm



Mir Publishers
Moscow



И. Е. Тамм

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of Electricity**

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Contents

Preface to the Ninth Russian Edition	11
From the Preface to the First Russian Edition	12
From the Preface to the Third Russian Edition	13
From the Preface to the Eighth Russian Edition	14
List of Most Important Symbols	15
Introduction	19
1 Electric Field of Fixed Charges in the Absence of Dielectrics	23
1.1. Coulomb's Law	23
1.2. Electric Field	27
1.3. Gauss's Law	29
1.4. Electric Field of Charged Surfaces	33
1.5. Conductors in an Electric Field	38
1.6. Sources of an Electric Field. Surface Divergence	41
1.7. Work of Electric Forces. Its Independence of the Shape of the Path. Continuity of the Tangential Components of the Vector \mathbf{E}	45
1.8. Potential of an Electrostatic Field	50
1.9. Capacitance. Capacitors	56
1.10. Gradient of Electrostatic Potential. Lines of Force	60
1.11. Poisson and Laplace Equations	65
1.12. Potential of Space and Surface Charges	70
1.13. Typical Problems of Electrostatics	77
1.14. Electrical Double Layer	81
1.15. Energy of Interaction of Electric Charges	86
1.16. Energy of an Electric Field	90
1.17. Ponderomotive Forces	96
1.18. Determining the Ponderomotive Forces from the Expression for Energy	99
1.19. Instability of Electrical Systems. Constraints	104
2 Dielectrics	108
2.1. Dielectrics. Electric Moment and Potential of a Neutral Molecule. Polarization of a Dielectric	108
2.2. Free and Bound Charges. Potential of an Electric Field When Dielectrics Are Present. Dependence of Polarization on the Field	113

2.3. Electric Displacement Vector. Differential Equations of a Field in an Arbitrary Medium. Induction Lines	117
2.4. Electric Field in a Homogeneous Dielectric	122
2.5. Direct Calculation of a Field When a Dielectric Is Present (in Very Simple Cases)	125
2.6. Micro- and Macroscopic Values of Physical Quantities	130
2.7. Derivation of Equations for the Field in Dielectrics by Averaging the Microscopic Field	134
2.8. Two Classes of Dielectrics. Quasi-Elastic Dipoles	137
2.9. Difference of the Field Acting on a Dipole from the Mean One	139
2.10. Polarization of Dielectrics Whose Molecules Have a Constant Electric Moment. Temperature Dependence of Permittivity	144
2.11. Energy of the Electric Field in Dielectrics.	150
2.12. Energy Transformations Connected with the Polarization of Dielectrics. Free Energy of an Electric Field	154
2.13. Ponderomotive Forces in Dielectrics	162
2.14. Reduction of Body Forces to Tensions	170
2.15. Stress Tensor of an Electric Field	175
3 Steady Electric Current	184
3.1. Current in Metals. Ohm's and Joule's Laws. Voltage	184
3.2. Current Density. Differential Form of Ohm's and Joule's Laws	188
3.3. Conditions of Steadiness of Currents. Continuity Equation. Current Filaments	191
3.4. Extraneous Electromotive Forces. Quasilinear Currents. Kirchoff's Second Law	195
3.5. Conversion of Energy in a Current Circuit. Contact E.M.F.'s.	200
3.6. Fundamental Concepts of the Electron Theory of Metals. Tolman's Experiments	206
3.7. Electron Theory of Electrical Conductivity. Difficulties of the Classical Theory. Sommerfeld's Theory	210
4 Ponderomotive Interaction of Steady Currents and Their Magnetic Field (in the Absence of Magnetizing Media)	218
4.1. The Magnetic Field of Currents	218
4.2. Interaction of Current Elements. The Electromagnetic Constant.	222
4.3. Transition from Line Currents to Currents Having a Finite Cross Section	226
4.4. Lorentz Force	229
4.5. Vector Potential of a Magnetic Field	234
4.6. Differential Equations of a Magnetic Field. Circulation of Magnetic Field Intensity	239
4.7. Potential Fields and Solenoidal Fields. Comparison of Differential Equations for an Electric and a Magnetic Fields	241
4.8. Boundary Conditions in the Magnetic Field of Currents. Surface Currents. Surface Curl. Field of an Infinite Solenoid	242
4.9. Ponderomotive Forces Acting on a Current Loop in a Magnetic Field. Potential Function of a Current in an External Magnetic Field	248

4.10. Ponderomotive Interaction of Currents. Mutual Induction	252
4.11. Self-Inductance. Total Potential Function of a System of Currents . . .	258
4.12. Magnetic Lines of Force	262
4.13. Topology of a Vortex (Magnetic) Field. Conditional Barriers	268
4.14. Magnetic Sheets. Their Equivalence to Currents	272
4.15. Magnetic Moment of a Current. Elementary Currents and Magnetic Dipoles	278
4.16. Direct Determination of the Field of Elementary Currents and the Forces Acting on Them	282
4.17. Evolution of Notions of the Nature of Magnetism. Spin of Electrons . .	290
4.18. Absolute (Gaussian) and Other Systems of Units. The Electromagnetic Constant	294
5 Magnetics (Magnetizable Media)	302
5.1. Magnetization of Magnetics. Molecular Currents and Conduction Currents	302
5.2. Vector Potential of a Magnetic Field in the Presence of Magnetics. Mean Density of Space and Surface Molecular Currents	306
5.3. Differential Equations of the Macroscopic Magnetic Field in Magnetics. Magnetic Field Intensity in Magnetics and Magnetic Induction Vector .	311
5.4. Dependence of Magnetization on Magnetic Field Intensity. Para-, Dia-, and Ferromagnetics	314
5.5. Complete System of Equations for the Field of Steady Currents. Homo- geneous Magnetic Medium	317
5.6. Mechanical Forces Acting on Currents in a Magnetic Field. Interaction of Currents	319
5.7. Ponderomotive Forces Acting on Magnetics in a Magnetic Field	323
5.8. Supplement to the Derivation of the Macroscopic Equations for a Magnetic Field in Magnetics	325
5.9. Mechanism of Magnetization of Magnetics. Larmor's Theorem	329
5.10. Diamagnetism	335
5.11. Paramagnetism	337
5.12. More Precise Definitions and Additions to the Theory of Magnetization. The Part of Spin. Gyromagnetic Phenomena	343
5.13. Ferromagnetism. Weiss Molecular Field	348
5.14. Equations of the Field in Idealized Ferromagnetics (Conventional Variant). Permanent Magnets	356
5.15. Another Variant of the Equations of the Magnetic Field in Idealized Ferromagnetics. The Equivalence of Electric Currents and Permanent Magnets	362
5.16. Ponderomotive Forces Acting on Permanent Magnets in an External Magnetic Field	371
6 Quasistationary Electromagnetic Field	377
6.1. Induction of Currents in Moving Conductors	377
6.2. Law of Electromagnetic Induction. Ohm's Law for Varying Currents . .	382
6.3. Quasistationary Currents. Differential Equations for Varying Currents . .	386

6.4. Transformations of Energy in the Field of Varying Currents. Energy of Magnetic Interaction of Currents. Lenz's Law	389
6.5. Simple Applications of the Varying Current Theory. Transformer	395
6.6. Energy of a Magnetic Field. Energy Meaning of Inductances	403
6.7. Transformation of Energy in the Magnetization of Para- and Diamagnetics. Free Energy of a Magnetic Field	411
6.8. Determination of the Ponderomotive Forces of a Magnetic Field from the Expression for Energy	415
6.9. Stress Tensor of a Magnetic Field	421
6.10. Vortices of an Electric Field	424
6.11. Dependence of Electric Voltage on Integration Path. Voltage of Alternating Current	427
6.12. Equation of Continuity	432
6.13. Displacement Currents	434
6.14. A Capacitor in the Circuit of a Quasistationary Current. Electric Oscillations	441
6.15. The Skin Effect	446
7 Varying Electromagnetic Field in a Stationary Medium and Its Propagation. Electromagnetic Waves	455
7.1. System of Maxwell's Equations for Macroscopic Electromagnetic Field	455
7.2. Poynting's Theorem. Energy Flow	461
7.3. Unambiguity of the Solutions of Maxwell's Equations	467
7.4. Differential Equations for the Potentials of an Electromagnetic Field	470
7.5. Solution of the Wave Equation and the d'Alembert Equation	474
7.6. Delayed and Advanced Potentials. Gauge Invariance	481
7.7. Velocity of Propagation of Electromagnetic Disturbances. Conditions for a Quasistationary State	488
7.8. Oscillator. Delayed Potentials of an Oscillator Field	492
7.9. Field of an Oscillator. Its Radiation	501
7.10. Electromagnetic Nature of Light. Plane Waves in a Dielectric	513
7.11. Reflection and Refraction of Plane Waves in Dielectrics	518
7.12. Propagation of Waves in a Conducting Medium. Reflection of Light from a Metal Surface	527
7.13. Light Pressure. Momentum of an Electromagnetic Field	532
7.14. Electromagnetic Angular Momentum. A Particular Case of a Static Field	539
7.15. Stress Tensor and Ponderomotive Forces of an Electromagnetic Field	543
7.16. An Example of Non-Quasistationary Currents: Waves along a Cable	549
7.17. Approximate Theory of Fast-Varying Currents. "Telegraph Equation"	559
7.18. Free Energy of Ferromagnetics. Hysteresis	564
7.19. General Characteristic of the Theories of Short-Range and Long-Range Interaction	571
8 Electromagnetic Phenomena in Slowly Moving Media	576
8.1. Differential Equations of a Field in Moving Media	576
8.2. Convection Current. Polarization and Magnetization of Moving Media	581

8.3. Ohm's Law and Electromagnetic Induction in Moving Conductors. Unipolar Induction	588
8.4. A Dielectric Moving in an Electromagnetic Field	595
8.5. Propagation of Light in Moving Dielectrics. Fresnel Drag Coefficient. Reflection from a Moving Mirror	597
8.6. Transformations of Frame of Reference. Relative Nature of Difference Between Electric and Magnetic Fields	602
9 Appendix. Vector Analysis	608
A.1. Vector Algebra	608
A.2. Vector and Scalar Fields. Gradient	610
A.3. Vector Flux Through a Surface	616
A.4. Gauss's Theorem. Divergence	619
A.5. Circulation of a Vector. Curl of a Vector. Stokes's Theorem	626
A.6. Derivative of a Vector with Respect to Direction	634
A.7. The Nabla. Second Derivatives. Derivatives of a Product	635
A.8. Integral Relationships. Green's Theorem	642
A.9. The Most Important Formulas of Vector Analysis	645
Fundamental Formulas in the SI and Gaussian Systems of Units	648
Supplements	651
S.1. Superconductivity (to Sec. 3.7)	651
S.2. Antiferromagnetism and Ferrites (to Sec. 5.12)	651
S.3. Dispersive Media. Spatial Dispersion (to Sec. 7.2)	652
S.4. Anisotropic Media (to Sec. 7.2)	653
S.5. Vavilov-Cerenkov Effect (to Sec. 7.9)	654
S.6. Plasma (to Sec. 7.12)	654
Solutions of Problems	657
Name Index	673
Subject Index	674

Preface to the Ninth Russian Edition

Fundamentals of the Theory of Electricity is a classical course in electrodynamics written by Igor E. Tamm (1895-1971), one of the most outstanding and well-known Soviet physicists.

During the author's life, the book ran into eight editions and for over forty years has been effectively used by a very broad circle of readers. Therefore the Publishers' (Nauka, Moscow) proposal to produce a new edition seemed quite natural from the very beginning. Nevertheless, I considered myself obliged, in addition, to get the opinion of a considerable number of specialists on the value of the book at present. The result of these inquiries left no doubts that the appearance of a new edition of I.E. Tamm's book would indeed be completely justified. This does not mean, naturally, that the course requires no alterations and amendments. On the contrary, as Igor Tamm himself stressed in the preface to the last (eighth) Russian edition given below that appeared during his lifetime, the book "requires significant revision". Actually, however, Igor Tamm found it possible to introduce only a small number of amendments into the book. The less are the grounds to make any appreciable changes at present when we cannot take into account the wishes and opinion of the author himself. In general, as we know, attempts to modernize classical courses that have already recommended themselves do not bring real success as a rule—the original course loses its integrity and harmoniousness, while a new quite modern course also does not appear.

For these reasons, the present edition is in essence a reprint of the preceding eighth one. Minor corrections have been made where needed, a few words have been changed or newly introduced into the text (particularly in the supplements), and some remarks have been added.

In conclusion, I would like to note with gratitude the appreciable aid rendered to me by B.M. Bolotovskiy and V. A. Ugarov in preparing the present edition. We all hope that the present edition of Igor Tamm's book, like the preceding ones, will be of great benefit to all its readers.

1976

Vitali L. Ginzburg

From the Preface to the First Russian Edition

The present book is intended for persons acquainted with differential and integral calculus and with vector algebra. The fundamentals of vector analysis are set out in the text as needed.

The main object of this book is to find out the *physical meaning and content* of the fundamental laws and postulates of the theory of electricity. In comparison with this object, only a subordinate part is relegated to formal and logical harmony, and to a strict and ordered treatment.

Not trying to achieve completeness of discussion, I omitted even comparatively important questions if they dropped out of the general line of the exposition (for example, thermoelectric phenomena and electrolysis). On the other hand, I permitted myself to treat in somewhat greater detail than is usually the custom some questions (for example in the theory of dielectrics and magnetics). I did not set out technical applications of the theory, but tried as far as possible to prepare the reader to pass over directly to studying the applied theory of electricity.

Owing to the undeveloped state of the Russian scientific *terminology*, I was forced to introduce two new terms: the "extraneous" electromotive force and "magnetic". In addition, I permitted myself to deviate from the generally adopted, but obsolete and irrational use of the terms "free" and "bound" charge. Finally, I call the Gaussian (symmetrical) system of units, unlike the electrostatic and electromagnetic ones, simply the absolute system (without the adjective).

Most of the problems given in the book are an organic part of the text; the solutions of many problems are needed in order to understand the following material.

1929

Igor E. Tamm

From the Preface to the Third Russian Edition

In preparing the third edition, I considerably revised the book, but retained its general plan and nature of treatment. In other words, I did not set myself the task of writing a new book.

In essence, Chapters 1 and 3 alone were subjected only to minor alterations.

Chapters 2 and 4 were virtually completely rewritten to ensure greater harmony and strictness.

In Chapter 7, I have additionally considered advanced potentials, gauge invariance, the radiation of a magnetic dipole, the propagation of light in metals, the pressure of light, the momentum and angular momentum of an electromagnetic field, and the electromagnetic stress tensor.

Finally, a new chapter has been written for the third edition — Chapter 8 — about electromagnetic phenomena in slowly moving media. The theory, which is true only up to terms of the order of magnitude of u^2/c^2 , covers many important applications and at the same time does not require the direct use of the theory of relativity, which we assume our reader not to be acquainted with. We had to use directly only the dependence of the polarization and magnetization of a moving medium on the field intensity from the theory of relativity. But with respect to transformations of the frame of reference, we had to limit ourselves in Sec. 8.6 only to the statement of the question and a brief indication of how it is solved in the theory of relativity.

1944

Igor E. Tamm

From the Preface to the Eighth Russian Edition

I wrote this book at the end of the twenties. At present it requires significant revision and ought to contain the results of the development of science having a significance both for theory and practice. I was unable to do this, however.

In connection with the republishing of the book, I was forced to limit myself, apart from minor alterations in the text, to adding a number of remarks or supplements at the end of the book in which the most significant new results of research are noted and the corresponding references are given.

1965

Igor E. Tamm

List of Most Important Symbols

Vectors are denoted by boldface upright type (for example \mathbf{R}); the same symbol set in lightface italics (for example R) signifies the *numerical value* of the relevant vector.

Integrals of any multiplicity are denoted by a single integral sign \int and are distinguished only by the designation of the element of integration: the element of a volume (triple integral) dV , that of a surface (double integral) dS , and that of a line (single integral) ds .

The symbol \oint stands for an integral over a *closed* surface or a *closed* contour.

For the meaning of the subscripts a and q at the symbols grad, div, etc. see the appendix.

The formulas and equations of vector analysis given in the appendix are referred to throughout the book by their relevant number preceded by the letter A, for example (A.17). If several equations have been given the same number, then each of them is referred to by the corresponding subscript at the right of the equation number, for example (A.43₃).

A	constant
A	vector potential
a	acceleration; constant; radius of ring current
B	magnetic induction
b	constant
C	capacitance; circulation of vector; constant
C'	capacitance of unit length
c	constant; electromagnetic constant; velocity of light in a vacuum
D	electric displacement
d	distance
E	electric field intensity
\mathcal{E}	electromotive force; voltage
e	base of natural logarithms; charge of electron
F	force (ponderomotive)
f	density of forces
G	momentum
g	density of electromagnetic momentum

H	magnetic field intensity
<i>h</i>	Planck's constant
<i>I</i>	current; moment of inertia
<i>i</i>	imaginary number ($i = \sqrt{-1}$)
i	density of surface currents; unit vector along <i>x</i> -axis
<i>j</i>	generalized coordinate
j	electric current density; unit vector along <i>y</i> -axis
<i>k</i>	Boltzmann constant; wavelength constant
k	unit vector along <i>z</i> -axis
<i>L</i>	curve (particularly closed contour); mutual inductance; self-inductance
L	angular momentum
<i>l</i>	distance; length
M	magnetization vector
<i>m</i>	magnetic (fictitious) charge; mass (of an electron)
<i>N</i>	number
<i>N_A</i>	Avogadro constant
N	moment of couple of forces
<i>n</i>	number; refractive index
n	unit vector of a normal
o	angular velocity of precession
P	Hertz vector; polarization of dielectric
p	electric dipole moment
<i>Q</i>	heat liberated by current in unit time
<i>q</i>	charge; unit power of a current
<i>q'</i>	charge per unit length
q	movement
<i>R</i>	distance; radius; resistance
R	radius-vector
<i>r</i>	distance; radius; reflection coefficient
<i>S</i>	area
<i>s</i>	damping factor; distance; path
S	Poynting's vector
<i>s</i>	displacement; element of length
<i>T</i>	absolute temperature; kinetic energy
T	force (tension); stress tensor
<i>t</i>	length; time
t	unit tangent vector
<i>U</i>	energy; internal energy; potential function of magnetic field of currents
<i>u</i>	constant; velocity; volume density of electric energy
u	velocity
<i>V</i>	potential energy; volume
v	velocity
<i>W</i>	work
α	angle; longitude (azimuth); polarizability per unit volume
β	angle; polarizability of molecule
γ	angle
Δ	increment
δ	depth of penetration of wave; increment
ϵ	permittivity
Θ	generalized force
θ	angle; polar angle
ω	coefficient of elasticity of a dipole; conductivity

λ	free path of an electron; wavelength
μ	permeability
$\boldsymbol{\mu}$	magnetic moment
ρ	density
Σ	total energy flux
σ	density of surface charge; surface area
τ	density (mass of unit volume); strength of layer
Φ	flux of vector; magnetic flux
ϕ	angle; electric potential; function; phase shift
χ	magnetic susceptibility
Ψ	flux of magnetic induction; free energy of system
ψ	free energy of unit volume; function; scalar potential of magnetic field
Ω	solid angle
ω	angle; angular velocity
∇	Hamiltonian operator "nabla"
∇^2, Δ	Laplacian

Introduction

According to modern views, the atoms of all bodies are built of electrically charged particles—relatively light electrons charged negatively and relatively heavy atomic nuclei charged positively. Bodies that are neutral from an electrical viewpoint appear so to us only because the negative charge of their electrons equals the positive charge of the atomic nuclei in them. Hence, the influence of the opposite charges is mutually neutralized (at least at distances sufficiently great compared with the distance between the separate electric particles that are a part of a neutral body). The redistribution of electric charges and, particularly, an electric current are due to the motion of electric particles, mainly electrons and not atomic nuclei because an atom of a chemical element always includes a certain number of “outer” electrons that are comparatively weakly bound to the massive central atomic nucleus and comparatively easily detached from it.

The negative charge of an electron equals 4.80×10^{-10} absolute electrostatic units of electricity (or 1.60×10^{-19} coulombs). The positive charges of atomic nuclei in their absolute value equal integral factors of this so-called elementary charge, which is an indivisible atom of electricity. The charges of various atomic nuclei vary from one (hydrogen) to ninety two (uranium) elementary charges*. The lightest atomic nucleus is that of hydrogen, called a proton; the mass of a proton (1.67×10^{-24} g) is about 2000 times greater than that of an electron (9.11×10^{-28} g). The geometrical dimensions of both atomic nuclei and electrons are so small in comparison with

* At present, a number of new heavy elements after uranium in the Periodic Table have been prepared artificially.

the mean distances between these particles in atoms and molecules that when treating an overwhelming majority of physical and chemical phenomena we can consider both atomic nuclei and electrons to be material points characterized by a definite electric charge and a definite mass. How the atomic nuclei are built up of more elementary particles (protons and neutrons) is significant only for a limited group of physical phenomena relating to the field of nuclear physics. We shall not consider them. The so-called nuclear forces determining the interaction of the particles (protons and neutrons) in atomic nuclei are also of significance only for this group of phenomena.

On the basis of the above ideas, the task confronting modern physics is the determination of the electrical structure of all the substances encountered in nature (the number, arrangement, and nature of motion of the electric particles in them) and the derivation of laws of physical and chemical phenomena from the fundamental laws of interaction of electric charges and from the laws of their motion (which are of a quantum nature in the microworld). The only exception must be made for the phenomena for which the forces of gravitation and nuclear forces are significant because it is only these forces that do not result in the interaction of electric charges.

The first step in solving this task should be the determination of the laws of interaction of electric charges and the laws of an electromagnetic field. Almost all the ways and means of observation and measurement used in practice are too rough to allow us to detect the existence of separate particles of electricity. The smallest electric charges that can be observed by such ways and means contain many thousands of millions of particles of electricity separated from one another by negligible distances. Using such a summary or *macroscopic** investigation of electrical phenomena on a scale lending itself to direct observation, we can, without introducing an appreciable error into the results of our reasoning, take no account whatsoever of the atomistic structure of electricity and use the notion of continuously extended electric charges. In other words, we can consider that electric charges continuously fill the charged portions of material bodies (the so-called "space charges"). Simplifying our task in this

* From the Greek "makros"—large and "skopeō"—I see.

way, we are only following the example of mechanics. Since the studying of the mechanics of material bodies with account taken of the atomistic structure of matter involves considerable mathematical difficulties, the theory of elasticity, hydrodynamics, and aerodynamics operate with an idealized notion of continuously extended material bodies. Such a substitution is quite lawful within certain rather broad limits, and the results obtained when considering continuous media may be applied to real bodies with an intermittent (discrete) structure.

Following the historical course of development of electrodynamics, we shall begin with a treatment of the macroscopic theory of electromagnetic phenomena based on the notion of the continuous distribution of electric charges. As soon as we accumulate a certain amount of information, we shall pass over to a parallel treatment of the fundamental ideas of the *microscopic* theory, which is based on account being taken of the atomistic structure of electricity (the so-called “electron theory”). We shall show that the approximate macroscopic laws of summary phenomena follow from the more accurate microscopic laws of elementary phenomena. It must be borne in mind, however, that a more or less complete and strict exposition of the microscopic theory inevitably has to be based on the *quantum theory* in its modern form. Since we cannot presume that our reader has a fundamental knowledge of quantum mechanics, we shall have to limit ourselves mainly to a treatment of only the questions of the microscopic theory that can be considered with sufficient accuracy within the scope of classical (prequantum) physics.

In Chapter 7, we shall again limit ourselves to a consideration of the macroscopic theory of a field. The results obtained will permit us to formulate the complete system of the basic laws of this theory (the formulation of an appreciable part of them is contained in the so-called Maxwell equations). This system of basic laws or “postulates” plays the same part in electrodynamics as Newton’s “axioms” do in classical mechanics. Particularly, the correctness of these main postulates of macroscopic electrodynamics (like the correctness of Newton’s axioms) can be substantiated in the most convincing way not by the inductive method (which is the only one that can be used in finding fundamental laws, but which, however, cannot give an absolutely strict proof of their correctness), but by agreement

with experimental results of the entire complex of corollaries following from the theory and covering all the laws of a macroscopic electromagnetic field. In Chapter 7, we shall consider some of these corollaries relating in particular to electromagnetic waves.

Finally, in Chapter 8, we shall treat electromagnetic phenomena in moving media, limiting ourselves to the case when the velocity of these media is small compared with that of light.

1

Electric Field of Fixed Charges in the Absence of Dielectrics

1.1 Coulomb's Law

We shall assume in this chapter that an electric field of fixed charges contains no other material bodies except for conductors of electricity.

1. The *theory* of the electrostatic field is based on Coulomb's law that summarizes *experimental data*. This law states that two charged bodies with *infinitely small dimensions* (two *point charges*) repel each other if they have like charges and attract each other if they have unlike charges. The force of their interaction F is proportional to q_1q_2/R_{12}^2 :

$$F \sim \frac{q_1q_2}{R_{12}^2} \quad (1.1)$$

where q_1 and q_2 = charges of the first and second bodies, respectively

R_{12} = distance between them.

If there are more than two charges, then a force of the type shown by expression (1.1) will be exerted on a charge by all the other charges. Particularly, if the charges q_1 and q_2 are in air, kerosene, or some other non-conducting medium, then apart from the direct interaction of the charges q_1 and q_2 according to Coulomb's law (1.1), account must be taken of the interaction of these charges q_1 and q_2 with the charges of the electrons and atomic nuclei in the neutral molecules of the medium.

In the present chapter, we shall limit ourselves to a consideration of an electrostatic field in a vacuum. A perfect vacuum cannot naturally be achieved in experiments, and a certain, even though negligible, amount of air always remains in the vessels being evacuated. This does not at all mean, however, that the laws of an electric field in a vacuum cannot be studied experimentally. By investigating the change in the nature of the field as the pressure of the air diminishes, we can establish the limit which the properties of the field tend to (for example the forces of interaction of the charges) as we approach a perfect vacuum. These limiting properties (the limiting

value of the force) will evidently characterize the field in a perfect vacuum. It must be noted, as experiments proved, that when the density of the air diminishes from the normal one to the limit achieved experimentally, the properties of the field change so insignificantly that we can completely disregard these changes in the majority of cases if there is no need of special accuracy. Hence, we can consider that the properties of the field in air are identical to those in a vacuum.

2. That the force of interaction of charges is inversely proportional to the square of the distance between them, which is expressed by formula (1.1), can be directly verified experimentally. As regards the dependence of this force on the magnitude of the charges, here matters are somewhat more complicated because the magnitude of the charges itself, in turn, can be determined only by measuring the force of their interaction. If the number of charges is at least four, however, the required relationship can nevertheless be verified by sequentially measuring the forces of interaction between pairs of charges.

Let us assume for simplicity's sake that when these measurements are made the charges being studied are always placed at the same distance from each other (the other charges are eliminated). With this condition observed, expression (1.1) gives us the relationships

$$F_{23} : F_{13} = q_2 : q_1 \quad \text{and} \quad F_{24} : F_{14} = q_2 : q_1$$

where F_{ik} is the force of interaction of the charges q_i and q_k .

Thus, the ratio q_2/q_1 (and also the ratios q_3/q_1 , q_4/q_1 , etc.) can be determined from two *independent* series of measurements (of the forces F_{13} and F_{23} on one hand and the forces F_{14} and F_{24} on the other). It is the coincidence of the results of these independent measurements that gives us the right to assert that each charge can be characterized by a constant number q_i so that the force F_{ik} will be proportional to the product $q_i q_k$.

By measuring the forces of interaction, we can naturally determine only the *ratio* of the magnitudes of the charges, q_i/q_k , while the unit of charge may be chosen arbitrarily. The unit of charge in the absolute system of units is selected so that when measuring the forces and distances in the cgs system the proportionality constant between F and $q_1 q_2 / R_{12}^2$ will equal unity, i.e. we get the *equality*

$$F = \frac{q_1 q_2}{R_{12}^2} \quad (1.2)$$

Hence, the absolute unit of electric charge is such a charge that acts on an equal one at a distance of 1 cm from it with a force of 1 dyne.

In the practical system of units, the unit of electric charge is the coulomb:

$$1 \text{ coulomb (C)} = 3 \times 10^9 \text{ abs. units of electricity}^*$$

As regards the sign of charges, it is purely conventionally considered that the charges which appear on glass when it is rubbed with silk or flannel are positive. Hence, the charges that are repelled by these charges on the glass are also positive.

It follows from Eq. (1.2) that in the absolute system of units the *dimension* $[q]$ of an electric charge q , i.e. the relationship between a unit of charge and a unit of length (L), time (T), and mass (M) is

$$[q] = [R\sqrt{F}] = L\sqrt{MLT^{-2}} = M^{1/2}L^{3/2}T^{-1}$$

3. It is very important that Coulomb's law (1.1) or (1.2) holds only for the interaction of *point* charges, i.e. charged particles of infinitely small dimensions. Only in this case does the very concept of the distance between the charges have a quite definite unambiguous meaning, and only in this case is the interaction of charged particles independent of their shape.

As is the rule in physics, the expression "infinitely small" should naturally not be understood here in its strictly mathematical sense. In physics, the expression "infinitely small" (or "infinitely great") quantity is always understood in the sense of "sufficiently small" (or "sufficiently great") quantity—sufficiently small with respect to another *quite definite* physical quantity. When encountering the term "infinite", we must always clearly understand what quantity has been taken in each separate case as the measure.

In the formulation of Coulomb's law, *the infinitely small (point) value of the dimensions of charged bodies is understood in the sense that they are sufficiently small relative to the distance between these bodies*, sufficiently small in the sense that with the given distance between the bodies the force of their interaction no longer changes within the limits of the *preset accuracy of measurements* upon a further reduction of their dimensions and an arbitrary change in their shape. Since we are meanwhile limiting ourselves to a macroscopic consideration of phenomena, we must remember that a physically infinitely small, or "point" charge may actually contain an

* It must be noted that apart from the absolute (Gaussian) system of units which we shall use in this book under the name "absolute system", two other systems are often used in physics and engineering—the electrostatic and electromagnetic ones. The relationship between these systems will be considered in Chapter 4. It is meanwhile sufficient to note that the absolute (Gaussian) units of the *electrical* quantities (charge, field intensity, potential, current, etc.) coincide with the electrostatic units, and the absolute units of the *magnetic* quantities (magnetic field intensity, self-inductance, magnetic moment) coincide with the electromagnetic units.

Further, with respect to the practical units, it is also necessary to distinguish the so-called "absolute" and "international" practical units (see Sec. 4.18).

exceedingly great number of separate electrons and protons. For example if it became necessary to determine the force of electric interaction between two charged stars, then notwithstanding their tremendous dimensions we would be right in considering them as point charges because in view of the enormous distance between the stars their dimensions and shape could have no appreciable affect on the force of their interaction. On the other hand, two charged pith balls having a radius of 0.1 cm and at a distance of 0.5 cm from each other cannot be considered as point charges, and Coulomb's law may not be directly applied to them. To determine the force of their interaction, we must mentally divide these balls into infinitely small (i.e. sufficiently small in comparison with the distance of 0.5 cm) elements of volume and determine according to Coulomb's law the interaction between the charges of each pair of these elements of volume. The force of interaction between the balls will equal the resultant of these elementary forces.*

4. When determining the resultant of electric forces, we must naturally take account of the circumstance that these forces are vectors and apply the rules of vector calculus to them. We shall represent vectors by bold face letters, and their numerical value by light face italic letters. For instance, \mathbf{R}_{12} stands for a radius-vector drawn from point 1 to point 2, and R_{12} for the numerical value of the distance between points 1 and 2. It is obvious that $\mathbf{R}_{21} = -\mathbf{R}_{12}$, and $R_{21} = R_{12}$ ** . When writing Coulomb's law (1.2) in the vector form, we must distinguish the force \mathbf{F}_{12} with which the charge q_1 acts on q_2 from the force \mathbf{F}_{21} with which the charge q_2 acts on the charge q_1 because these forces are equal in value, but opposite in direction ($\mathbf{F}_{21} = -\mathbf{F}_{12}$). If the charges q_1 and q_2 have the same sign (they repel each other), then the directions of the vectors \mathbf{F}_{12} and \mathbf{R}_{12} coincide, and

$$\mathbf{F}_{12} = \frac{q_1 q_2}{R_{12}^2} \frac{\mathbf{R}_{12}}{R_{12}} = \frac{q_1 q_2}{R_{12}^3} \mathbf{R}_{12} \quad (1.3)$$

because the numerical value of the vector \mathbf{R}_{12}/R_{12} equals unity, and the product $q_1 q_2$ is positive. This equation obviously also holds for unlike charges because here the product $q_1 q_2$ is negative, and the force \mathbf{F}_{12} is directed oppositely to the vector \mathbf{R}_{12} (attraction). Finally,

* This calculation is based on the assumption that the force of interaction between two (point) charges does not depend on whether or not these charges are subjected to the action of other ones. This assumption together with the statement that the resultant of electric forces equals the vector sum of these forces forms the content of the principle of *superposition* of electric fields. This principle summarizes experimental data and is the basis of the modern theory of electricity.

** For purposes of brevity, we shall sometimes call the vector \mathbf{R}_{12} simply the distance from point 1 to point 2, and the vector \mathbf{R}_{21} the distance from point 2 to point 1.

it is evident that

$$\mathbf{F}_{21} = -\mathbf{F}_{12} = \frac{q_2 q_1}{R_{21}^3} \mathbf{R}_{21}$$

5. We shall note in conclusion that the decisive vote in the question of the correctness of Coulomb's law, as in general of any law on which the relevant branch of theoretical physics is based, belongs not only to the direct experimental verification of this law. It also belongs, and this is much more significant, to the agreement with experimental data of the entire complex of theoretical conclusions having this law as one of their cornerstones.

1.2 Electric Field

1. If we put a "test" charge q' at point P at the distance R from the charge q , we can detect a force acting on this test charge. It is due to the presence of the charge q . This force is *detected* only when the second (test) charge is present, and we can consider that it *appears* only when *both* charges q' and q are present. The studying of electrical phenomena will be very greatly facilitated, however, if we proceed from the notion that both at point P and at all points of space surrounding the charge q there *always* exists an electric force due to the presence of this charge. And it is no matter whether the existence of this force manifests itself in action on another (test) charge (if such a charge is available), or does not manifest itself in any way (when other charges are absent).

If a space contains electric forces that are detected when we introduce electric charges into it, we say that there is an *electric field* in it. In the following, we shall assume that any charge induces an electric field in the surrounding space. As long as we are within the confines of electrostatics, the concept of a field can be considered as a purely conditional one introduced only for the convenience of describing electrical phenomena. When passing over to a discussion of a varying electromagnetic field, however, in particular to a discussion of electromagnetic waves, we shall see that the concept of a field has a deep physical meaning, and that an electromagnetic field is an objective reality.

2. According to Coulomb's law, the force acting on a "test" charge q' when it is introduced into the field of other charges is proportional to the magnitude of this test charge q' . Therefore, the forces of an electric field will be determined if we determine at each point of this field the force acting on a unit positive charge placed in it. This force acting on a charge of $q' = 1$ is called the *intensity of the field*, or *tension*, or *force of the field* or simply the *electric vector* and is usually denoted by the letter \mathbf{E} . We shall use the first and last terms.

It follows from Coulomb's law (1.3) that the intensity of the field of a point charge q at the distance \mathbf{R} from this charge is

$$E = \frac{q}{R^2} \quad \text{and} \quad \mathbf{E} = \frac{q}{R^3} \mathbf{R} \quad (1.4)$$

where \mathbf{R} is a radius-vector extending from the charge q to the point of the field being considered.

The intensity of the field of two or more charges equals the vector sum of the intensities of the field of each of these charges taken separately.

3. It must be remembered that the intensity or force of a field \mathbf{E} is not a force in the conventional meaning of the word as used in mechanics. The mechanical or, as it is sometimes called, the *ponderomotive* force \mathbf{F} (from the Latin "pondus"—weight, hence "ponderomotive" means "moving ponderable bodies") acting on a charged body is determined by the *product* of the force of the field \mathbf{E} and the charge q :

$$\mathbf{F} = q\mathbf{E} \quad (1.5)$$

Accordingly, in the absolute system of units, the force or intensity of a field \mathbf{E} has the dimension (see p. 25)

$$[E] = \frac{[F]}{[q]} = \frac{MLT^{-2}}{M^{1/2}L^{3/2}T^{-1}} = M^{1/2}L^{-1/2}T^{-1}$$

4. On the basis of Eq. (1.5), one of the main tasks of electrostatics—determining the mechanical forces of interaction of a given system of charges—consists in determining the *field* \mathbf{E} of these charges. It should be noted that to determine the field of the electrical vector \mathbf{E} , as in general *to determine the field* of any vector, *means to determine the value and direction of the vector at each point of this field*. On a later page, we shall treat methods of the theoretical calculation of electric fields. As regards the experimental studying of these fields, we must note the following.

An electric field can be studied experimentally by introducing a test charge of a known magnitude into it and measuring the ponderomotive forces \mathbf{F} acting on this charge at different points of the field. Generally speaking, however, the very fact that a test charge has been introduced into a field changes its nature because the forces of the test charge field call forth redistribution of the charges on the conductors in the field (electric induction), shifting of these conductors, etc. To avoid this distortion of the initial nature of a field, we must perform our measurements with the aid of *infinitely small test charges*, i.e. charges that are so small that the change in the distribution of the charges due to their presence within the limits of the preset accuracy of observations cannot affect the results of the measure-

ments. Speaking of test charges, we shall always assume in the following that this condition has been observed (unless otherwise indicated).

1.3 Gauss's Law

1. If we know the arrangement of charges, then the field \mathbf{E} of these charges can be determined by the summation of Coulomb fields of type (1.4) induced by each of the elements of these charges separately. Such direct summation, generally speaking, requires quite complicated calculations in each separate case. This problem may often be made much simpler by using certain theorems and laws dealing with the general properties of an electric field. We shall now pass over to their consideration.

For this purpose, we shall calculate the flux of the vector \mathbf{E} through an infinitely small element of area dS of a surface*. We shall first assume that the field \mathbf{E} is induced by the point charge q at point O .

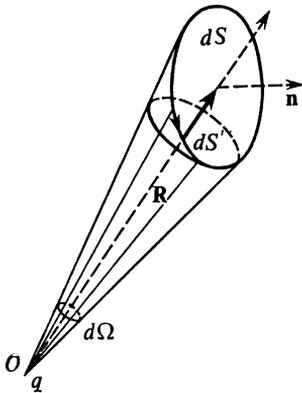


Fig. 1

If \mathbf{R} is a radius-vector extending from the charge to the element of area dS of the surface (Fig. 1), then on the basis of Eqs. (1.4) and (A.12) the flux $d\Phi$ of the vector \mathbf{E} through this element of area will be

$$d\Phi = \mathbf{E}\mathbf{n} dS = \frac{q}{R^3} \mathbf{R}\mathbf{n} dS = \frac{q}{R^2} \cos(\mathbf{R}, \mathbf{n}) dS \quad (1.6)$$

where \mathbf{n} is a unit normal to the surface.

The product $\cos(\mathbf{R}, \mathbf{n}) dS$ is numerically equal to the projection of the element of area dS on a surface perpendicular to \mathbf{R} . This product is positive if we see the inner side of the element of area

* The fundamentals of vector analysis are set out in the Appendix. The formulas of the Appendix are denoted by the letter A, for example (A.21).

dS from point O [the angle (\mathbf{R}, \mathbf{n}) is acute], and negative if we see its outer side:

$$\cos(\mathbf{R}, \mathbf{n}) dS = \pm dS'$$

where dS' is the absolute value of the area of the projection of the element of area dS of the surface onto a plane perpendicular to \mathbf{R} .

The element of area dS' of the surface perpendicular to the radius-vector \mathbf{R} coincides with an element of a spherical surface having the radius R with its centre at point O . If we denote by $d\Omega$ the solid angle subtending the surface element dS' from point O , then, as is known from solid geometry,

$$d\Omega = \frac{dS'}{R^2} = \pm \frac{\cos(\mathbf{R}, \mathbf{n}) dS}{R^2} \quad (1.7)$$

and, consequently,

$$d\Phi = \pm q d\Omega$$

The element of area dS will evidently be subtended from point O by the same solid angle $d\Omega$. If we agree to consider the angle $d\Omega$ to be positive when we see the inner side of dS from point O , and negative when we see its outer side, then we can write:

$$d\Phi = E_n dS = q d\Omega \quad (1.8)$$

where E_n is the projection of the vector \mathbf{E} onto the normal \mathbf{n} .

Thus, the flux of the electric vector \mathbf{E} through an element of area dS of an arbitrarily oriented surface in the field of a point charge q depends, apart from the magnitude of this charge, only on the positive or negative solid angle at which this element of area can be seen from point O occupied by the charge*.

2. Reverting to the flux Φ of the vector \mathbf{E} through a finite surface with the area S , we get**:

$$\Phi = \int_S E_n dS = q \int_S d\Omega = q\Omega \quad (1.9)$$

where Ω is the positive or negative solid angle at which the entire surface S can be seen from the charge q , i.e. the solid angle formed by radius-vectors extending from q to the boundary of the surface.

* As can be seen from the derivation of Eq. (1.8), it follows from the fact that the field intensity \mathbf{E} is directed radially and with an increase in the distance from the charge diminishes according to the same law (inversely proportional to R^2) as the solid angle $d\Omega$ corresponding to the given element of area dS does. Consequently, for all central fields in general that change inversely proportional to the square of the distance (for example the fields of gravitation and the fields of magnetic poles), formulas similar to Eq. (1.8) and all the equations ensuing directly from it will also hold.

** Multiple integrals will be denoted in the book by a single integral sign. See the list of symbols at the beginning of the book.

(Fig. 2). It is extremely significant that when the surface S is closed, this angle can have only one of two values: 4π and 0.

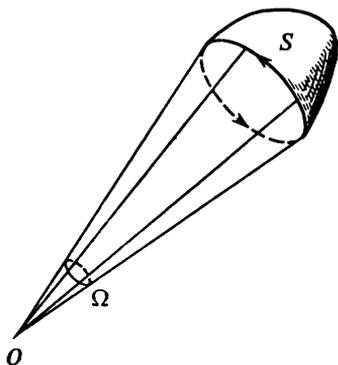


Fig. 2

Indeed, the point charge may be arranged either inside the closed surface or outside of it. Consideration of a point charge on a surface itself is deprived of a physical meaning because we can take advantage of our notion of a point charge only provided that the actual dimensions of the charge are small in comparison with the distance from it to the points of the field being considered. The notion of a point charge on a surface through which we are determining the flux of an electric vector does not obviously comply with this condition.

If the charge is within the closed surface S , then the surface surrounds it on all sides and, consequently, is seen from it at the angle $\Omega = 4\pi$. Hence, in this case

$$\Phi = \oint_S E_n dS = 4\pi q$$

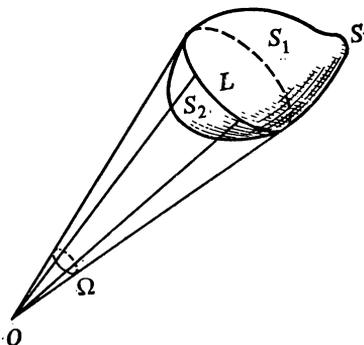


Fig. 3

If the charge q is at point O outside of the closed surface S , then from O we can draw a pencil of tangents to the surface S (Fig. 3).

These tangents form a cone contacting S along a certain closed line L that divides the surface S into two parts S_1 and S_2 . Both parts of the surface S will be seen from point O at the same solid angle corresponding to the tangent cone angle. One of the parts will be seen from the inner side of the surface, and the other from the outer side. Thus, the angles Ω_1 and Ω_2 equal in magnitude and opposite in sign will correspond to the parts S_1 and S_2 of the surface. Consequently, the fluxes of the electric vector through S_1 and S_2 will be equal in value but opposite in sign, and their sum will be zero. Thus, the flux of the vector \mathbf{E} through any closed surface not enclosing the charge q equals zero:

$$\Phi = \oint_S E_n dS = 0$$

Both these possible cases (the charge is inside or outside of a surface) can be covered by a single formula

$$\Phi = \oint_S E_n dS = 4\pi q \quad (1.10)$$

if only we agree to understand in this formula by q the magnitude of the charge *inside* the surface S and, therefore, assume that q equals zero if the charge is outside this surface.

3. We have derived Eq. (1.10), which is exceedingly simple, for a field induced by a single point charge. This equation also holds for a field of an arbitrary system of electric charges. Indeed, any system of charges can be resolved into a combination of elementary (point) charges. Let \mathbf{E} be the intensity of the resultant field of the entire system of charges, and \mathbf{E}_i the intensity of the field of the i -th elementary charge q_i . Hence

$$\mathbf{E} = \sum_i \mathbf{E}_i$$

and

$$E_n = \sum_i E_{in}$$

Therefore,

$$\Phi = \oint_S E_n dS = \sum_i \oint_S E_{in} dS = 4\pi \sum_i q_i \quad (1.11)$$

the last sum covering only the charges that are inside the surface S . This formula expresses a fundamental law called Gauss's law*:

* This law should not be confused with Gauss's theorem known from vector analysis (see A.17) and establishing the relationship between the flux of an arbitrary vector through a closed surface and the volume integral of divergence of this vector.

In an arbitrary electrostatic field (in a vacuum), the flux of the total electric flux vector \mathbf{E} through an arbitrary closed surface equals the total charge inside this surface (the algebraic sum) multiplied by 4π .

1.4 Electric Field of Charged Surfaces

1. The application of Gauss's law extremely simplifies the solution of a number of problems in electrostatics. In this section, we shall apply it to a consideration of some of the properties of the field of charged surfaces.

It is quite obvious, strictly speaking, that a charge always occupies a certain volume and cannot be concentrated on an infinitely thin (geometrical) surface. However, a layer of a charge whose thickness is sufficiently small in comparison with its distance from the points of the field being studied can be considered as a surface charge with the same right with which we consider point charges.

The *surface charge density* σ is the charge per unit area of the given surface. If a charge is unevenly distributed over a surface, its density at the given point of the surface is the limit of the ratio of the charge Δq of an element of area ΔS of the surface to this element:

$$\sigma = \lim_{\Delta S \rightarrow 0} \frac{\Delta q}{\Delta S} = \frac{dq}{dS} \quad (1.12)$$

2. *Jump of the normal component of vector \mathbf{E} on a charged surface.* Let us consider an arbitrary charged surface S . We shall arbitrarily select the direction of an outward normal \mathbf{n} to this surface and use the subscripts 1 and 2 to indicate quantities relating to the inner and outer (relative to the normal \mathbf{n}) sides of the surface, respectively.

We shall mentally separate a right prism around the point of the charged surface being considered. Its generatrices dl are perpendicular to the surface. Assume that this prism cuts out from the surface an element S' which is so small that we can consider it to be flat and uniformly charged (Fig. 4).

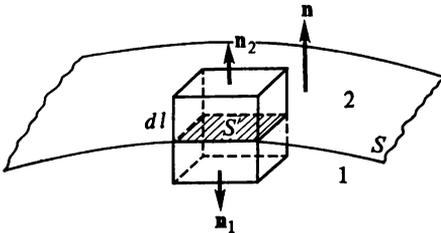


Fig. 4

The prism will contain the charge $\sigma S'$ on the element cut out by the prism from the charged surface S . Hence, the electric flux vector

through the surface of a prism according to Gauss's law should equal

$$\Phi = \oint E_n dS = 4\pi\sigma S'$$

On the other hand, this flux can be calculated directly. The flux through the bottom base of the prism is $E_1 \cos(\mathbf{E}_1, \mathbf{n}_1)S'$, and through the top base is $E_2 \cos(\mathbf{E}_2, \mathbf{n}_2)S'$, where E_1 and E_2 are the values of the vector \mathbf{E} at the corresponding bases of the prism, and \mathbf{n}_1 and \mathbf{n}_2 are the outward normals to these bases. We shall denote the flux of the vector \mathbf{E} through the lateral faces of the prism by Φ' ; hence

$$\Phi = \{E_1 \cos(\mathbf{E}_1, \mathbf{n}_1) + E_2 \cos(\mathbf{E}_2, \mathbf{n}_2)\} S' + \Phi' \quad (1.13)$$

The direction of the normal \mathbf{n}_2 coincides with that of the normal \mathbf{n} , while the direction of the normal \mathbf{n}_1 is opposite.

Consequently,

$$E_1 \cos(\mathbf{E}_1, \mathbf{n}_1) = -E_{1n} \quad \text{and} \quad E_2 \cos(\mathbf{E}_2, \mathbf{n}_2) = E_{2n}$$

where E_{1n} and E_{2n} are the projections of the vectors \mathbf{E}_1 and \mathbf{E}_2 onto the normal \mathbf{n} . Thus,

$$\Phi = (E_{2n} - E_{1n}) S' + \Phi' = 4\pi\sigma S'$$

We shall now reduce the height of the prism, dl , without changing the area of its base S' . The magnitude of the flux Φ' through the infinitely diminishing lateral face of the prism will tend to zero like a second-order infinitesimal (in comparison with Φ), so that the total flux vector through the surface of the prism will consist in the limit in the flux through its base:

$$\Phi = (E_{2n} - E_{1n}) S' = 4\pi\sigma S'$$

whence

$$E_{2n} - E_{1n} = 4\pi\sigma \quad (1.14)$$

Hence, the normal components of vectors \mathbf{E} at two adjacent points of a field separated by a charged surface S differ from each other by $4\pi\sigma^*$.

In other words, the normal component of the vector \mathbf{E} experiences a jump of $4\pi\sigma$ when passing through any charged surface regardless of the shape of this surface and of whether or not there are charges

* It must be noted that the value of the difference $E_{2n} - E_{1n}$ does not depend on the choice of the direction of the normal to the discontinuity surface. Although when the direction of \mathbf{n} is changed, the sign of the projections of the vector \mathbf{E} onto the direction of \mathbf{n} will be reversed, at the same time the side of the surface that was formerly considered the first (i.e. the inner one with respect to \mathbf{n}), must now be named the second, and vice versa.

outside of it. The explanation is that the field of the surface charges at the opposite sides of the surface are directed oppositely, namely, away from the surface if it is charged positively, and toward the surface if charged negatively.

3. *Field of a uniformly charged infinite plane.* The intensity of the field of a uniformly charged infinite plane P , from considerations of symmetry, should be at right angles to the plane and should be directed oppositely at both sides of it. It is directed away from the plane P if its charge is positive (Fig. 5) and toward the plane if it is negative.

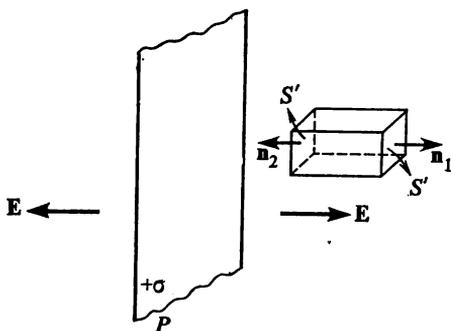


Fig. 5

The numerical value of the intensity E at different points of the field can depend only on their distance from the plane P and should be the same at all the points of any plane parallel to P .

Let us mentally separate a right prism having a base area of S' and generatrices perpendicular to the plane P in its field. We shall first assume that the prism is not intersected by the plane P , i.e. is completely at one side of this plane.

Calculation of the electric flux vector through the surface of this prism will again lead us to Eq. (1.13). Taking into account that the flux Φ' through a lateral face of the prism in this case equals zero (because E is parallel to this surface), we get

$$\Phi = \{E_1 \cos(\mathbf{E}_1, \mathbf{n}_1) + E_2 \cos(\mathbf{E}_2, \mathbf{n}_2)\} S'$$

Since our conditions stipulate that the prism does not intersect the charged plane P , there are no charges in it. Hence, according to Gauss's law (1.11), the flux Φ should equal zero, i.e.

$$E_1 \cos(\mathbf{E}_1, \mathbf{n}_1) + E_2 \cos(\mathbf{E}_2, \mathbf{n}_2) = 0$$

The vectors E_1 and E_2 have identical directions, while the normals \mathbf{n}_1 and \mathbf{n}_2 have opposite directions. Thus, $\cos(\mathbf{E}_1, \mathbf{n}_1) = -\cos(\mathbf{E}_2, \mathbf{n}_2)$, which gives us $E_1 = E_2$.

Since the position of the prism bases can be selected arbitrarily, it follows from this equation that at all points of the half-space limited by the plane P the vector E is constant in magnitude. Seeing that it is also constant in direction, then $E = \text{const}$. In the other

half-space, the vector \mathbf{E} will evidently have the same magnitude and the opposite direction.

We can use Eq. (1.14) to determine the numerical value of the vector \mathbf{E} . In the case being considered, the components of this vector in the direction of a normal \mathbf{n} to the plane obviously have opposite signs at different sides of the plane and are numerically equal to the vector \mathbf{E} itself. Therefore,

$$E_{2n} - E_{1n} = 2E_{2n} = \pm 2E = 4\pi\sigma$$

The plus sign should be taken in the term $\pm 2E$ when the plane is charged positively ($\sigma > 0$), and the minus sign when it is charged negatively ($\sigma < 0$). Consequently, at all the points of the field of an infinite plane we have

$$E = 2\pi|\sigma| \quad (1.15)$$

where $|\sigma|$ is the absolute value of the charge density of this plane.

4. We shall now try to find out why the intensity of the field of a charged infinite plane remains constant instead of diminishing with an increasing distance from the plane. Let the points A , A' , and A'' be on one perpendicular to the plane P (Fig. 6). We shall consider the intensity of the field induced at these points by the charges of two elements of area of the plane, dS_1 and dS_2 , that are symmetrical

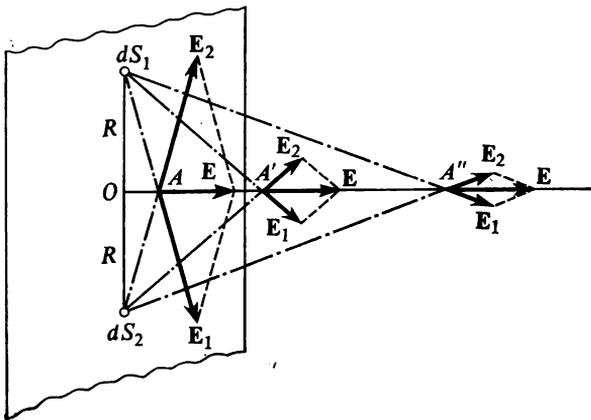


Fig. 6

relative to the perpendicular OAA' and at the distance R from its base O . At points that are farther from the plane, the intensity of the fields \mathbf{E}_1 and \mathbf{E}_2 of each of these charges taken separately will be smaller than at closer ones, but at the same time the angle between \mathbf{E}_1 and \mathbf{E}_2 diminishes with an increasing distance from the plane. Hence, notwithstanding the reduction of the components \mathbf{E}_1 and \mathbf{E}_2 with an increasing distance from the plane, the magnitude

of their resultant \mathbf{E} may not only decrease, but also grow owing to the diminishing of the angle between them. This depends on the ratio between the distances R and OA or OA' . It is clear that if the point A is close to O , the resultant of all the charges of the plane at this point is determined almost exclusively by the ones close to O because the intensities of the fields of remote charges are almost parallel to the plane P and their sum is zero or almost zero. As the distance from O to A grows, this parallelity is violated, the resultant of the remote charges increases, and that of the close ones decreases. Direct calculation shows that as a consequence the intensity of the resultant field of all the charges of an infinite plane does not change at all with an increasing distance from the plane. We shall not give this calculation because we have already found its result [Eq. (1.15)] by applying Gauss's law.

Problem 1. The surface of an infinitely long circular cylinder is uniformly charged with electricity so that the charge per unit of its length is q' . Prove that the field inside the cylinder \mathbf{E}_i and the external one \mathbf{E}_e are expressed by the following formulas:

$$\mathbf{E}_i = 0 \quad \text{and} \quad \mathbf{E}_e = \frac{2q'r}{r^2} \quad (1.16)$$

where \mathbf{r} is the vector of the shortest distance from the point of the field being considered to the axis of the cylinder. Show that the jump of the electric vector when passing through the charged surface of the cylinder equals $4\pi\sigma$.

Problem 2. A charge q is uniformly distributed on a spherical surface having an arbitrary radius. Prove that the field inside and outside the sphere is expressed, respectively, by the formulas

$$\mathbf{E}_i = 0 \quad \text{and} \quad \mathbf{E}_e = \frac{q\mathbf{R}}{R^3} \quad (1.17)$$

where \mathbf{R} is a radius-vector extending from the centre of the sphere to the point of the field being considered, i.e. prove that the field outside a uniformly charged spherical surface is such as if its entire charge were concentrated at its centre. Show that the jump of the electric vector \mathbf{E} when passing through the charged surface of the sphere equals $4\pi\sigma$.

Problem 3. A charge q is uniformly distributed with the density ρ over a spherical volume having the radius a . Show that the field \mathbf{E}_e outside the sphere is such as if the entire charge of the sphere were concentrated at its centre, and that the field \mathbf{E}_i inside the sphere is directly proportional to the distance from the centre

of the sphere:

$$\left. \begin{array}{l} R \geq a, \quad \mathbf{E}_e = \frac{q\mathbf{R}}{R^3} \\ R \leq a, \quad \mathbf{E}_i = \frac{q\mathbf{R}}{a^3} \end{array} \right\} \quad (1.18)$$

or

$$\left. \begin{array}{l} R \geq a, \quad \mathbf{E}_e = \frac{4}{3} \pi \rho \left(\frac{a}{R} \right)^3 \mathbf{R} \\ R \leq a, \quad \mathbf{E}_i = \frac{4}{3} \pi \rho \mathbf{R} \end{array} \right\} \quad (1.19)$$

It should be noted that in this case the vector \mathbf{E} is continuous everywhere.

1.5 Conductors in an Electric Field

The physical purport of studying the field of surface charges is that with electrostatic equilibrium the *charges of the conductors are concentrated in a very thin surface layer* of them. In the overwhelming majority of cases, this layer with sufficient accuracy can be considered as infinitely thin. It is immediately clear that if we impart, for example, a negative charge (a surplus of electrons) to a metal body, then owing to the mutual repelling of the elements of this charge (electrons) they concentrate on its surface. A strict proof of this statement can be given on the basis of the fact that in electrostatic equilibrium, the *electric field inside conductors equals zero**.

Indeed, a conductor is a body distinguished by the following property: if the intensity of the electric field \mathbf{E} differs from zero at a point inside a conductor, an electric current, i.e. a flow of charges, appears in it. This property can be considered as a *definition* of the term “*conductor of electricity*”.

From the standpoint of the electron theory of the most important class of conductors—metals—this fact is explained by the circumstance that if a metal is in the solid (or liquid) state of aggregation, then part of the electrons in its atoms detach themselves from the latter. The positive ions of the metal remaining after the detachment of these “free” electrons form its solid skeleton (a crystal lattice). The free electrons are in the spaces between the ions in the form of a sort of “electron gas”. An external electric field, no matter how weak it is, causes these free electrons to move in the direction of

* Also see Sec. 3.4, in particular Eq. (3.31).

the forces acting on them, i.e. induces an electric current*. This current continues until the field intensity inside the conductor becomes equal to zero, i.e. until the field of the charges that have redistributed themselves in the volume of the conductor compensates for the external field.

Since the electric vector inside a conductor equals zero, then the flux of this vector through any closed surface inside the conductor will also equal zero. Hence, according to Gauss's law, the charge inside any such surface also equals zero. And this means that with electrostatic equilibrium there are no charges inside a conductor (it is more correct to say that the positive and negative charges inside them are mutually neutralized), and that all the charges are on their surface.

Thus, the field of a charged metal cylinder or sphere is determined by the same formulas (1.16) and (1.17) as the field of the relevant charged surfaces. In accordance with the above general law, it follows from these formulas, particularly, that $\mathbf{E} = 0$ inside a metal charged sphere or cylinder. As regards Eq. (1.14), for charges on the surface of conductors, this equation becomes much simpler. Since $\mathbf{E} = 0$ inside conductors, this equation acquires the form

$$E_n = 4\pi\sigma \quad (1.20)$$

where \mathbf{E} = value of the electric vector in direct proximity to the surface of the conductor

\mathbf{n} = direction of an *outward* normal to its surface.

Thus, with electrostatic equilibrium, the normal component of the field intensity near a conductor is determined *only by the density* σ of the charge on the adjacent element of its surface and does not at all depend on the distribution of the charges at other sections of the field, etc.

Example. Determine the field between the plates of an infinite parallel-plate capacitor. The field between the plates of such a capacitor from considerations of symmetry should be perpendicular to its plates. It should also be the same at all points of any plane parallel to the plates. It follows from this circumstance, as for the field of an infinite plane already considered, that the field between the plates of a capacitor is *homogeneous*, i.e. that the vector \mathbf{E} is constant in magnitude and direction *at all* the points of this field. According to Eq. (1.20), however, the numerical value of the vector \mathbf{E} at the surface of the plates should be

$$E = 4\pi|\sigma| \quad (1.21)$$

(since $E_n = \pm E$). Consequently, this will also be the value of E throughout the entire space between the plates.

* The mechanism of a current will be considered in greater detail in Secs. 3.6 and 3.7.

It is obvious that in Eq. (1.21) we can understand σ to be the density of either the positive or the negative plate of the capacitor. Since, on the other hand, the vector \mathbf{E} has one quite definite value in the space between the plates, then, consequently, with electrostatic equilibrium the absolute value of the density of the charge on both plates should be the same.

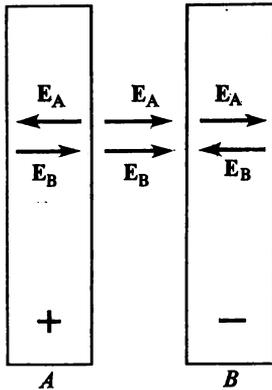


Fig. 7

All these results can also be directly obtained from Eq. (1.15). The field \mathbf{E} of a capacitor is the resultant of the fields \mathbf{E}_A and \mathbf{E}_B of two charged infinite planes, namely, the *internal surfaces* of the capacitor plates A and B . Hence, $\mathbf{E} = \mathbf{E}_A + \mathbf{E}_B$, while $E_A = 2\pi|\sigma_A|$ and $E_B = 2\pi|\sigma_B|$. As follows from Fig. 7, the vectors \mathbf{E}_A and \mathbf{E}_B inside each of the metal plates are directed oppositely, while between the plates they have the same direction. Since inside the plates E should equal zero (a conductor!), then $E_A = E_B$ and, consequently, $\sigma_B = -\sigma_A = \sigma$. Therefore in the space between the plates, $E = 4\pi|\sigma|$, Q.E.D.

The field between the plates of a parallel-plate capacitor of *finite dimensions* will be the same if only the distance between the plates is small in comparison with their dimensions. Distortion of the field and violation of its homogeneity will become noticeable only at the edges of a finite capacitor at distances from them compatible with the distance between the plates.

For a capacitor with finite dimensions, the total charge of each plate is finite, and the (mean) density of their surface charge is

$$\sigma = \frac{q}{S}$$

where S is the surface area of a plate. Thus, Eq. (1.21) becomes

$$E = \frac{4\pi q}{S} \quad (1.22)$$

Problem 4. Show that the field intensity in the space of a cylindrical capacitor (two coaxial cylinders) is determined by equation (1.16):

$$\mathbf{E} = \frac{2q'}{r^2} \mathbf{r}$$

where q' is the charge of a unit length of the inner cylinder; the charge of a unit length of the outer cylinder is $-q'$.

Problem 5. Show that the field in the space of a spherical capacitor (two concentric spheres) is determined by Eq. (1.17):

$$\mathbf{E} = \frac{q}{R^3} \mathbf{R}$$

where q is the charge of the inner sphere; the charge of the outer sphere is $-q$.

1.6 Sources of an Electric Field. Surface Divergence

1. The surface integral in Eq. (1.11) of Gauss's law can be transformed with the aid of Gauss's general theorem (A.17) into an integral over the volume V bounded by the surface S :

$$\oint_S E_n dS = \int_V \operatorname{div} \mathbf{E} dV = 4\pi \sum_i q_i \quad (1.23)$$

Such a transformation is possible, however, only if $\operatorname{div} \mathbf{E}$ has a definite finite value at all the points of the volume V confined within the surface S , i.e. if the vector \mathbf{E} is finite and continuous in this volume. Particularly, inside the surface S there should be neither point charges of a finite magnitude nor surface charges of a finite surface density because the field intensity of a point charge at $R \rightarrow 0$ tends to infinity [Eq. (1.4)], and, in addition, the direction itself of the vector \mathbf{E} with $R = 0$ becomes indefinite; on charged surfaces the continuity of the vector \mathbf{E} is violated: its normal component undergoes a jump of $4\pi\sigma$ [Eq. (1.14)].

It must be noted that the very concepts of a point and a surface charges have only an auxiliary significance, and we introduced them to facilitate our treatment of the field of charges at distances sufficiently great in comparison with the dimensions of the charges themselves. In studying a field near or inside charges, we must return to our notion of the volume distribution of charges. Let us assume, for instance, that the charge q , which we considered to be a point one, is actually uniformly distributed over the volume of a sphere having an arbitrarily small but finite radius a . In this case, the field outside and inside the sphere is determined by Eqs. (1.18). The latter

show that the vector \mathbf{E} is finite and continuous at all points of a field [in particular when $R = a$, i.e. on the surface of a spherical charge, both equations (1.18) give the same value for E , namely, $E_e = E_i = q/a^2$].

This result has a general significance: for all cases of the *volume distribution* of a charge with a finite density, the electric vector \mathbf{E} is *finite and continuous everywhere*. Indeed, in this case from each point P inside the charge serving as the centre, we can circumscribe a sphere of a sufficiently small but nevertheless finite radius a so that this sphere can be considered as uniformly charged. At all points of this sphere, the field of the charges of the sphere itself is finite and continuous according to Eqs. (1.18). The field of the charges outside the sphere is finite and continuous because these charges are at a finite distance from the inner points of the sphere. Hence, the resultant field of all the charges is finite and continuous.

2. Thus, with the volume distribution of the charges within the surface S , the surface integral in Eq. (1.23) may always be transformed into a volume one. It should be remembered that in the general case of non-uniform distribution of charges, the *volume (space) density* of a charge at a given point is the limit of the ratio of the charge Δq in the volume ΔV surrounding this point to the magnitude of this volume [cf. Eq. (1.12)]:

$$\rho = \lim_{\Delta V \rightarrow 0} \frac{\Delta q}{\Delta V} = \frac{dq}{dV} \quad (1.24)$$

where the Greek letter ρ , as everywhere in the following, stands for the volume density of a charge, or, as it is more often called, the charge density. Therefore, the charge dq of an element of volume dV equals

$$dq = \rho dV \quad (1.25)$$

while the total charge in a finite volume V equals

$$\sum_V q = \int_V \rho dV$$

Introducing this expression into Eq. (1.23), we get

$$\int \operatorname{div} \mathbf{E} dV = \int 4\pi\rho dV \quad (1.26)$$

These integrals should equal each other regardless of our choice of the domain of integration of V . This is possible only if their integrands equal each other at each point of space. Hence,

$$\operatorname{div} \mathbf{E} = 4\pi\rho \quad (1.27)$$

or in Cartesian coordinates

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 4\pi\rho \quad (1.28)$$

This differential equation is one of the fundamental equations of electrostatics and of all electrodynamics in general. It allows us to determine the divergence of the electric vector at each point of a field according to the value of the charge density *at the same point** regardless of the distribution of the charges at other regions of the field. Conversely, to determine the charge density at a given point of a field, it is sufficient to know the value of the divergence $\text{div } \mathbf{E}$ at this point of the field.

By analogy with hydrodynamics, the points of the field of an arbitrary vector \mathbf{a} at which $\text{div } \mathbf{a} \neq 0$ are generally called the *sources* of the field. The numerical value of $\text{div } \mathbf{a}$ is called the *strength* or *abundance* of the field sources (see Sec. A.4 of the Appendix, *Vector Analysis*). Using this terminology, we can say that the *sources of an electric field* are only at the points of the field where there are *electric charges*, and the strength or abundance of these sources (with volume distribution of the charges) equals $4\pi\rho$.

3. From the viewpoint of the macroscopic theory being treated, all charges are continuously distributed volume charges. When the thickness of a layer occupied by a charge is small in comparison with distances that lend themselves to measurement, however, it is convenient to retain our notion of surface charges. This relates first of all to the surface charges of conductors. Since the vector \mathbf{E} changes in a jump when it passes through charged surfaces [Eq. (1.14)], these surfaces are called *surfaces of discontinuity* of an electric vector. It is evident that differential equation (1.27) may not be applied for surfaces of discontinuity (which also follows from our reservations at the beginning of this section) and must be replaced by Eq. (1.14):

$$E_{2n} - E_{1n} = 4\pi\sigma$$

This equation is called the *boundary condition* for the vector \mathbf{E} and in essence is the limiting form of Eq. (1.27) for charges arranged in an infinitely thin layer.

Since in the following we shall repeatedly have to encounter such relationships, we shall prove the following general theorem here. Let us assume that a vector \mathbf{a} is continuous and finite *everywhere* and *everywhere* complies with the equation

$$\text{div } \mathbf{a} = 4\pi\rho \quad (\text{A})$$

* That is, in essence, according to the magnitude of the charge in an infinitely small volume surrounding this point [see Eq. (1.24)].

where ρ is everywhere the finite density of a certain "charge" q [for example of an electric charge determined by an equation of type (6.2)]. We shall consider a certain charged layer having the finite thickness dl inside which \mathbf{a} remains continuous according to the initial condition (Fig. 8). If we reduce the thickness dl of the

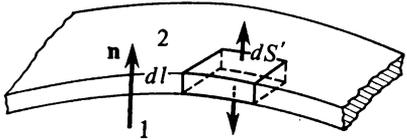


Fig. 8

layer to zero without changing its charge, then the continuity of the vector \mathbf{a} will be violated, and Eq. (A) in the limit on a charged surface will acquire the form

$$a_{2n} - a_{1n} = 4\pi\sigma \quad (\text{B})$$

where σ = surface density of the charge determined by an equation of the type of (1.12)

a_{1n} and a_{2n} = values of the normal components of the vector \mathbf{a} at different sides of the charged surface.

To prove the correctness of this statement, we shall consider a cylindrical section of the charged layer with the base dS' . Multiplying Eq. (A) by dV and integrating with respect to the volume of this section, we get the following expression on the basis of Eq. (1.25) and Gauss's theorem (A.17):

$$\int_V \operatorname{div} \mathbf{a} dV = \oint_S a_n dS = 4\pi \int_V \rho dV = 4\pi dq$$

where dq = total charge of the section being considered
 S = surface confining it.

By repeating the reasoning which in Sec. 1.4 resulted in Eq. (1.13), we see that

$$\oint a_n dS = (a_{2n} - a_{1n}) dS' + \Phi' = 4\pi dq$$

where Φ' is the flux of the vector \mathbf{a} through the lateral surface of the section of the layer being considered. Upon transition to the limit $dl \rightarrow 0$, the value of Φ' becomes equal to zero. Hence, dividing this equation by dS' , we get

$$a_{2n} - a_{1n} = 4\pi \frac{dq}{dS'} = 4\pi\sigma$$

i.e. Eq. (B), Q.E.D.

Thus, Eq. (B) is the limiting form of Eq. (A). This is why the jump of the normal component of the arbitrary vector \mathbf{a} on a surface of discontinuity is often called the *surface divergence* of this vector.) Unlike the *volume divergence* determined by Eq. (A. 18),

$$\operatorname{div} \mathbf{a} = \lim_{\Delta V \rightarrow 0} \frac{\oint a_n dS}{\Delta V}$$

the surface divergence is designated by Div with a *capital* (and not a small) letter D^* :

$$a_{2n} - a_{1n} = \operatorname{Div} \mathbf{a} \quad (1.29)$$

Consequently, the theorem we have proved can be symbolically written as follows:

$$\operatorname{div} \mathbf{a} = 4\pi\rho \rightarrow \operatorname{Div} \mathbf{a} = 4\pi\sigma \quad (1.30)$$

Finally, using the terminology mentioned above, we can call surfaces of discontinuity of the normal component of the vector \mathbf{a} the *surface sources* of this vector.

4. Equations (1.27) and (1.14) are quite sufficient for solving the so-called “reverse” problem of electrostatics: finding the distribution of (volume and surface) charges given the field of the electric vector \mathbf{E} . Particularly, the arrangement of surface charges is determined by the arrangement of the surfaces of discontinuity of the vector \mathbf{E} . To solve the “direct” problem, however, i.e. to find the electric field given the distribution of the charges, these equations are not sufficient because *one* differential equation (1.27) is not sufficient to determine the *three* components E_x , E_y , and E_z of the vector \mathbf{E} . To solve the “direct” problem of electrostatics, we must also use some other properties of an electrostatic field. We shall now pass over to their consideration.

1.7 Work of Electric Forces.

Its Independence of the Shape of the Path.

Continuity of the Tangential Components of the Vector \mathbf{E}

1. The work done by the forces of an electric field when the charge q is carried over the distance ds is

$$qE \cos(\mathbf{E}, ds) ds = q\mathbf{E} ds$$

* It must be noted that the value of $\operatorname{Div} \mathbf{a}$ does not depend on the choice of the direction of the normal \mathbf{n} to the surface of discontinuity (see the footnote on p. 34).

Particularly, the work W done when a *unit positive charge* is carried over the distance ds equals

$$W = \mathbf{E} ds \quad (1.31)$$

Finally, the work done when a unit positive charge is carried over the finite path L is

$$W = \int_L \mathbf{E} ds \quad (1.32)$$

where the symbol L at the integral sign indicates that the sum of the values of the integrand must be calculated for all the elements of the line L . This operation is called *integration over the line L* .

2. The work of electric forces along a given path L , generally speaking, may depend both on the position of the initial and terminal points of the path and on its shape. As we shall now show, however, the electric field of *fixed charges* has the exceedingly important property that *the work of the forces of this field along the path between two arbitrary points depends only on the position of these points and does not at all depend on the shape of the path*. Force fields having this feature* are called conservative or *potential fields*.

It should be noted that in a potential field, the *work* of the forces of the field along any *closed path* must equal zero. Indeed, any closed path $MNPQM$ can be arbitrarily divided into two parts MNP and PQM (Fig. 9). The work done along the path PQM is obviously

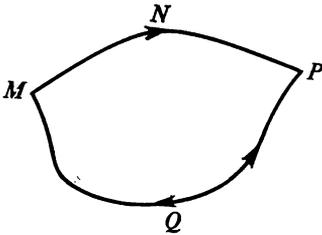


Fig. 9

equal to the work along the same path when it is traversed in the reverse direction from M to P , but with the opposite sign. The latter work in the potential field, according to definition, equals that done along the path MNP . Consequently, the total work done along the closed path $MNPQM$ equals zero, Q.E.D.

Conversely, if the work done by the forces of a field along a closed path equals zero, then the work done by these forces along the path

* With the condition observed for an electrostatic field that the field is fixed (constant in time), and that the forces acting on a body placed in the field depend only on the position of the body and not on its velocity.

between two arbitrary points M and P does not depend on the shape of this path, i.e. the field is a potential one. Indeed, let us consider two arbitrary paths MNP and MQP leading from M to P (see Fig. 9). We shall compose the closed path $MNPQM$ from them. The work done along this closed path, according to our condition, is zero. Hence, the work done along the section MNP of the path equals that done along PQM with the opposite sign or, in other words, equals the work done along the path MQP , Q.E.D.

Thus, the equality to zero of the work done along an arbitrary closed path is a necessary and sufficient condition for the work to be independent of the shape of the path and can be considered as a distinguishing feature of a potential field. In general, the field of an arbitrary vector \mathbf{E} regardless of its physical meaning (force, velocity, etc.) is a potential field only when with any choice of the closed path of integration L we have

$$\oint_L \mathbf{E}_s ds = 0 \quad (1.33)$$

The line integral of an arbitrary vector \mathbf{E} around an arbitrary closed path L is called *circulation* of this vector around the path L (see Appendix, Sec. 5). Thus, condition (1.33) consists in the requirement that the circulation of the vector \mathbf{E} around any closed path be equal to zero.

3. Passing over to a proof of the potential nature of an electrostatic field, we shall first consider the work of electric forces in the field of an elementary (point) charge q . The work of these forces with an infinitely small displacement ds of a "test" unit positive charge is (Fig. 10)

$$W = \mathbf{E} ds = \frac{q}{R^3} \mathbf{R} ds = \frac{q}{R^2} ds \cos(\mathbf{R}, ds) = \frac{q}{R^2} dR$$

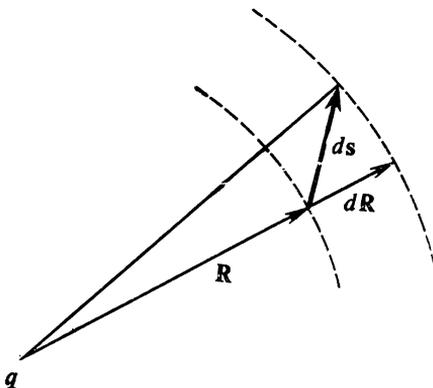


Fig. 10

where dR is the projection of the displacement of the test charge, ds , onto the radius-vector \mathbf{R} passing from the charge q inducing the field. Examination of Fig. 10 shows that dR is also the increment of the *numerical* value of the radius-vector \mathbf{R} , i.e. the increase in the distance from the charge q to the test charge. Consequently, the work W can be represented in the form of the total differential of a *scalar* function of the point, namely, q/R :

$$W = \frac{q}{R^2} dR = d\left(-\frac{q}{R}\right) = -d\left(\frac{q}{R}\right) \quad (1.34)$$

where R is the *numerical* value of the radius-vector \mathbf{R} . Hence, the work done when moving a unit positive charge from the point P_1 to the point P_2 along the finite path L is (Fig. 11):

$$W = \int_L \mathbf{E} ds = -\int_L d\left(\frac{q}{R}\right) = -\left(\frac{q}{R_2} - \frac{q}{R_1}\right) \quad (1.35)$$

where R_1 and R_2 are the distances to the initial and final points of the path from the charge q . Thus, the work of electric forces along an arbitrary path in the field of a fixed elementary (point) charge does indeed depend only on the positions of the initial and final points of this path and, consequently, does not at all depend on the shape of the path. For example, the work of electric forces along the path L' (Fig. 11) equals the work along the path L —the surplus

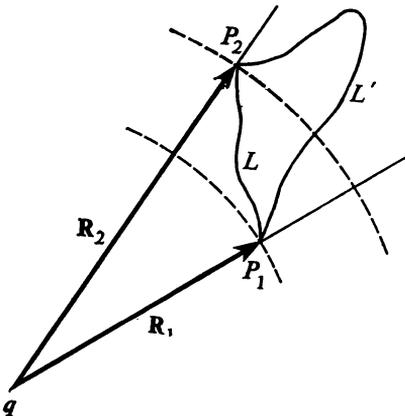


Fig. 11

work done along the path L' when moving the test charge beyond the limits of a sphere having the radius R_2 is compensated by the negative work done upon the following approaching of the test charge to the charge q along the last portion of the path L' .

Thus, the field of a fixed elementary (point) charge is a potential field. The sum of potential fields* is obviously also a potential field because if the work done by the component forces does not depend on the shape of the path, then that done by the resultant force also does not depend on it. Since the field of an arbitrary system of charges can be considered as the sum of the fields of each of the elements of these charges, then, consequently, any *electrostatic field is a potential one* and complies with condition (1.33).

4. The integral condition (1.33) can be transformed into a differential one. According to the well-known theorem of vector analysis (A.29), the component of the curl of a vector \mathbf{E} along an arbitrary direction \mathbf{n} at an arbitrary point of the field P is

$$\text{curl}_{\mathbf{n}}\mathbf{E} = \lim_{dS \rightarrow 0} \frac{\oint E_s ds}{dS}$$

where dS stands for an infinitely small area containing the point P and perpendicular to the vector \mathbf{n} . The numerator in the right-hand side contains the circulation of the vector \mathbf{E} along the contour of this area dS which according to Eq. (1.33) becomes equal to zero. Therefore, $\text{curl}_{\mathbf{n}}\mathbf{E} = 0$. Owing to the arbitrary nature of the direction \mathbf{n} , this means that the curl of the vector \mathbf{E} at all points of an electrostatic field equals zero:

$$\text{curl } \mathbf{E} = 0 \quad (1.36)$$

Conversely, from Eq. (1.36) on the basis of Stokes's theorem (A.27) well known from vector analysis, we get Eq. (1.33), so that the integral and differential conditions (1.33) and (1.36) are equivalent to each other.

5. It follows from condition (1.33), particularly, that *the tangential components of the field intensity are continuous* [unlike the normal components; see Eq. (1.14)], i.e. it follows that the components of the intensity that are tangent to an arbitrary surface at any point on it have identical values at both sides of the surface.

Indeed, let us assume that the opposite is true and that the continuity of the tangential components of the vector \mathbf{E} along the surface S_1 (Fig. 12) is violated. This means that if P_1, P_2 and P'_1, P'_2 are two pairs of points separated by the surface S_1 but infinitely close to one another, then the work of the electric forces $\int E_s ds$ along the path $P_1P'_1$ differs by a finite value from the work done by these forces along the infinitely close path $P_2P'_2$. On the other hand, owing

* By the sum of vector fields we shall understand a field at each of whose points the vector of the field equals the sum of the vectors of the constituent fields.

to the finite nature of the field forces, their work along the infinitely small paths P_1P_2 and $P'_1P'_2$ is infinitely small. Therefore, the work of the field forces along the closed path $P_1P'_1P'_2P_2P_1$ should differ from zero. In other words, it follows from our assumption that the work of a field done when a test charge is carried along a closed path differs from zero, which is impossible in an electrostatic field.

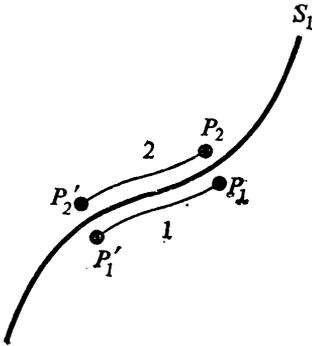


Fig. 12

Thus, with any selection of the surface S , the components of the electric vector \mathbf{E} tangent to it remain continuous. In other words, if \mathbf{t} is a unit vector in a plane tangent to an arbitrary surface S , then

$$E_{2t} = E_{1t} \quad (1.37)$$

where E_{2t} and E_{1t} are the components of the vector \mathbf{E} in the direction \mathbf{t} at both sides of the surface S .

Particularly, since the field intensity inside conductors equals zero, then the tangential component of \mathbf{E} at their outer surface must also equal zero. Consequently, *at the surface of conductors, the electric vector is directed normally to their surface* ($E = \pm E_n$), and Eq. (1.20) can be written in the form

$$\mathbf{E} = 4\pi\sigma\mathbf{n} \quad (1.38)$$

1.8 Potential of an Electrostatic Field

1. The circumstance that the work of the forces of an electrostatic field along a given path depends only on the position of the initial and final points of the path makes it possible to introduce the exceedingly important concept of the potential of an electrostatic field. Its definition is as follows: *the potential difference between two points of an electrostatic field equals the work done by the forces of the field in moving a unit positive charge from the first point to the second taken with the opposite sign.* It is assumed that upon

movement of the test unit charge all the charges including the field remain fixed*.

The potential difference $d\varphi$ between two infinitely close points separated by the distance ds is therefore equal to:

$$d\varphi = -W = -\mathbf{E} ds \quad (1.39)$$

while the potential difference $\varphi - \varphi_0$ between the two points P and P_0 at a finite distance from each other is determined by the integral

$$\varphi - \varphi_0 = - \int_{P_0}^P \mathbf{E} ds \quad (1.40)$$

This integral can be taken *over any path* connecting the points P_0 and P . The concept of potential difference obviously has a definite unambiguous meaning only because of the independence of the work of electric forces on the shape of the path which we have proved above, or, which is the same, because of the fact that the intensity \mathbf{E} of an *electrostatic* field complies with condition (1.33).

The definition given for the concept of the potential of the field of the vector \mathbf{E} and contained in Eqs. (1.39) and (1.40) can evidently be applied to the field of an arbitrary vector complying with condition (1.33) regardless of the physical meaning of the vector (force, velocity, etc.) and of the physical meaning of the relevant potential (the potential of forces, the potential of velocities, etc.).

2. It is obvious that any value selected beforehand may always be assigned to the potential φ_0 of an arbitrary point P_0 of a field. This corresponds to the circumstance that measurement of the work allows us to determine only the *difference* between the potentials of two points of a field, but not the absolute value of the potential. As soon as the value of the potential at any *single* point of a field is fixed, however, its value *at all* the remaining points of the field is unambiguously determined by Eq. (1.40).

Generally, the additive constant in the expression of the potential is selected so that the potential φ_∞ of infinitely remote points equals zero (in practical electrical measurements the Earth's potential is often assumed to equal zero). When this condition is observed, the potential φ of an arbitrary point P of a field will be determined by the expression:

$$\varphi = \varphi_\infty - \int_{\infty}^P \mathbf{E} ds = \int_P^{\infty} \mathbf{E} ds \quad (\varphi_\infty = 0) \quad (1.41)$$

* If the movement of a *unit* test charge may cause displacement of the charges inducing the field (for instance by electrostatic induction on conductors), it is necessary to measure directly only the work done upon the movement of an infinitely (i.e. sufficiently) small test charge. The work done upon the movement of a unit charge is calculated from the results of these measurements.

Thus, the potential of the point P will equal the work done by the forces of a field when a unit positive charge is removed from this point to infinity.

The potential difference between the points P and P_0 in the field of an elementary (point) charge q , according to Eqs. (1.35) and (1.40), is:

$$\varphi - \varphi_0 = \frac{q}{R} - \frac{q}{R_0} \quad (1.42)$$

In this case, to comply with the condition that $\varphi_\infty = 0$, it is evidently sufficient to assume that $\varphi_0 = q/R_0$; therefore the potential of the field of a point charge q at a distance R from it will be ($\varphi_\infty = 0$)

$$\varphi = \frac{q}{R} \quad (1.43)$$

3. The potential of the field of an arbitrary system of point charges q_1, q_2, \dots, q_n obviously equals the sum of the potentials of the fields of each of these charges taken separately:

$$\varphi = \sum_{i=1}^n \frac{q_i}{R_i} \quad (1.44)$$

where R_i is the distance from the point of a field having the potential φ to the charge q_i . Naturally, both this and the preceding formulas have a meaning only for points of a field whose distances from the "point" charges q_i are great in comparison with the sizes of these charges.

For surface charges, the charge of each surface can be resolved into a combination of elementary charges of infinitely small surface elements dS :

$$dq = \sigma dS$$

Substituting dq for q_i in Eq. (1.44) and passing over from summation to integration over all the elements of all the charged surfaces, we get the potential of the field of surface charges:

$$\varphi = \int \frac{\sigma dS}{R} \quad (1.45)$$

In the field of space charges, the part of the elementary charges will be played by the charges $dq = \rho dV$ of infinitely small elements of volume dV , and Eq. (1.44) for the potential will acquire the form

$$\varphi = \int \frac{\rho dV}{R} \quad (1.46)$$

where R is the distance from a point of the field having the potential φ to the element of volume dV .

It must be noted that although R is in the denominator of the integrands in Eqs. (1.45) and (1.46), these expressions nevertheless remain finite at all points of the field of space and surface charges. Let us consider, for example, Eq. (1.46) and introduce a system of spherical coordinates R , θ , and α with its centre at the point of the field being investigated (θ is the polar angle and α is the azimuth or longitude, see Fig. 101 in the Appendix). The element of volume dV is expressed in these coordinates as follows:

$$dV = R^2 \sin \theta \, d\alpha \, d\theta \, dR$$

and Eq. (1.46) will become

$$\varphi = \iiint \rho R \sin \theta \, d\alpha \, d\theta \, dR \quad (1.47)$$

The integrand of Eq. (1.47) remains constant at the value of $R = 0$.

In view of the circumstance, however, that Eqs. (1.45) and (1.46) have been derived from Eqs. (1.43) and (1.44) having sense only for finite values of R (because when $R \rightarrow 0$ the value of $\varphi = q/R$ tends to infinity), and also in view of the special importance of Eqs. (1.45) and (1.46), we shall derive them below (Sec. 1.12) again in a different way not depending on the one we have just treated and show that they can be applied to all the points of the field of surface and space charges.

4. The potential nature of an electrostatic field can also be proved without applying Coulomb's law by reasoning based on the law of conservation of energy and the impossibility of a perpetual-motion machine. Indeed, assume that when a test charge is moved along a closed path L in the field of fixed charges (see the footnote on p. 51), the forces of this field perform the positive work W . Since when the test charge returns to its initial position the entire system also returns to it, then by passing along the path L an arbitrary number of times, we would get the work W each time, i.e. would have a perpetual-motion machine. If when passing along the path L the forces of the field do negative work, then it is only necessary to reverse the direction of motion in order to get positive work. Thus, the work of the forces of a field along any closed path must equal zero, whence follows, as we have seen, the existence of an unambiguous field potential*.

* Such reasoning cannot be applied, for example, to the magnetic field of steady (constant) currents for whose maintenance a continuous expenditure of energy of the current sources is needed. The work done upon the movement of a magnetic charge (pole) along a closed path in the field of a steady current may differ from zero and can be performed at the expense of an additional consumption of energy by the sources of current. Indeed, the movement of a magnetic charge excites electromotive forces of induction, and a corresponding change in the e.m.f. of the current sources is needed to keep the current constant during movement of the charge.

5. The unit of potential in the absolute system of units is determined as follows: the difference between the potentials of two points of a field equals unity if upon the displacement of an absolute unit of charge from one point to the other the forces of the field do a unit of work, i.e. work amounting to one erg. The dimension of potential will therefore be

$$[\varphi] = \frac{\text{work}}{\text{charge}} = \frac{ML^2T^{-2}}{M^{1/2}L^{3/2}T^{-1}} = M^{1/2}L^{1/2}T^{-1}$$

The absolute unit of potential is too great in comparison with the potential differences which we usually have to deal with in practice. This is why a different unit has been chosen for practical use. The practical unit of potential is the volt: $1 \text{ V} = \frac{1}{300}$ absolute unit of potential.

Example. Determine the potential of the field of a dipole. Let us assume that two equal point charges of opposite signs, $+q$ and $-q$, are at a distance l from each other, the vector \mathbf{l}

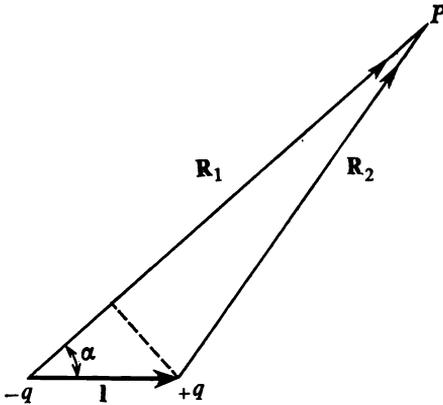


Fig. 13

being directed from the negative charge to the positive one (Fig. 13). The vector \mathbf{p} :

$$\mathbf{p} = q\mathbf{l} \quad (1.48)$$

is called the *electric moment* of these charges. The potential of the two charges at an arbitrary point P of the field is

$$\varphi = q \left(\frac{1}{R_2} - \frac{1}{R_1} \right) = \frac{q(R_1 - R_2)}{R_1 R_2}$$

If the distance l between the charges $+q$ and $-q$ is small in comparison with the distance from them to the points of the field being investigated, then the combination of the charges $+q$ and $-q$ is called a *dipole*, which means "double pole". In this case we can assume approximately that

$$R_1 R_2 = R^2 \text{ and } R_1 - R_2 = l \cos \alpha$$

where α is the angle between the direction of the dipole moment and the radius-vector \mathbf{R} extending from the dipole to the "point of observation" P (Fig. 13). Owing to the smallness of the distance l , it does not matter from what point of the dipole this radius-vector \mathbf{R} has been extended. Thus, the potential of a dipole acquires the form

$$\varphi = \frac{ql \cos \alpha}{R^2} = \frac{p \cos \alpha}{R^2} = \frac{\mathbf{p}\mathbf{R}}{R^3} \quad (1.49)$$

This expression can also be presented in a somewhat different form with the aid of the well-known formula of vector analysis (A.10):

$$\text{grad}_a \left(\frac{1}{R} \right) = -\text{grad}_q \left(\frac{1}{R} \right) = -\frac{\mathbf{R}}{R^3}$$

where the subscript a stands for the space derivative of $1/R$ with respect to the coordinates of the "point of observation", i.e. the final point of the vector \mathbf{R} , and the subscript q for differentiation with respect to the coordinates of the "source point", i.e. the initial point of the vector \mathbf{R} (see the Appendix, Sec. 2). On the basis of these relationships, Eq. (1.49) can be written as follows:

$$\varphi = \mathbf{p} \text{grad}_q \left(\frac{1}{R} \right) = -\mathbf{p} \text{grad}_a \left(\frac{1}{R} \right) \quad (1.50)$$

This formula can also be easily obtained directly. Indeed,

$$\frac{\varphi}{q} = \frac{1}{R_2} - \frac{1}{R_1}$$

equals the increment of the scalar $1/R$ upon movement over the length l of the point of the source of the radius-vector \mathbf{R} extending from the dipole (field source) to the point of observation P . Restricting ourselves to the first derivative of $1/R$ with sufficiently small values of l , we get:

$$\frac{1}{R_2} - \frac{1}{R_1} = l \text{grad}_q \left(\frac{1}{R} \right) = -l \text{grad}_a \left(\frac{1}{R} \right)$$

whence formula (1.50) directly follows.

Problem 6. Use Eq. (1.40) as the basis to show that the potential of the field of a charged infinite plane and a charged hollow cylinder is determined by the following formulas, respectively:

$$\begin{aligned} &\text{for a plane } \varphi = \varphi_0 - 2\pi\sigma x, \\ &\text{for a cylinder } \begin{cases} \varphi_i = \varphi_0, \\ \varphi_e = \varphi_0 - 2\kappa \ln \frac{r}{r_1} \end{cases} \end{aligned}$$

where φ_0 = value of the potential on the relevant charged surface

x = coordinate perpendicular to the plane

r = distance from the cylinder axis

r_1 = radius of the cylinder.

It should be noted that the condition $\varphi_\infty = 0$ cannot be complied with in both of these cases.

Problem 7. Show that the potential of the field of a sphere having the radius a that is uniformly charged with electricity throughout its entire volume with a charge density ρ provided that $\varphi_\infty = 0$ is

$$\left. \begin{aligned} \varphi_e &= \frac{e}{R} \quad (R \geq a) \\ \varphi_i &= 2\pi\rho \left(a^2 - \frac{R^2}{3} \right) \quad (R \leq a) \end{aligned} \right\} \quad (1.51)$$

where $q = \frac{4\pi a^3}{3} \rho$ = total charge of the sphere

R = distance from its centre.

Show that $\varphi_e = \varphi_i$ when $R = a$.

1.9 Capacitance. Capacitors

1. One of the most characteristic features of an electrostatic field is that with electrostatic equilibrium *the potential of the field is constant over the entire length of each separate conductor*.* Indeed, with electrostatic equilibrium, the field intensity inside a conductor equals zero (Sec. 1.5). Since any two points of a conductor can be connected by a line completely within the conductor, then, consequently, the difference between the potentials of these points determined by the line integral of the vector \mathbf{E} [Eq. (1.40)] equals zero, Q.E.D.

This circumstance for an electrostatic field makes it possible to speak simply of the *potential of a conductor* (i.e. the potential of each of its points).

* Naturally in the absence of extraneous e.m.f.'s of a chemical, thermal, etc. origin (see Sec. 3.4).

2. The *capacitance of an isolated conductor*, i.e. a conductor infinitely removed from all other conductors, is the magnitude of the charge needed to impart a unit potential to this conductor. It is assumed that the additive constant in the expression for the potential is selected so that the potential equals zero at infinity. The symbol C is customarily used to denote capacitance.

3. It should be noted that the capacitance of an isolated sphere (in absolute units) numerically equals its radius. Indeed, the external potential of the field of a sphere having the radius a and the charge q is $\varphi = q/R$. On the surface of the sphere, $R = a$ and $\varphi = q/a$. The value of the potential is the same inside the entire sphere. Hence, the potential of the sphere will equal unity when $q = a$, and this means that the capacitance C of the sphere equals a :

$$C = a$$

This equation shows that in absolute units the capacity should have the dimension of length. Indeed:

$$[C] = \frac{[q]}{[\varphi]} = \frac{M^{1/2} L^{3/2} T^{-1}}{M^{1/2} L^{1/2} T^{-1}} = L$$

Thus, *in the absolute system of units, the capacitance is measured in units of length*, a capacitance of 1 cm being equal to the capacitance of an isolated sphere with a radius of 1 cm.

Since the practical unit of charge (the coulomb) is 3×10^9 times greater, and the practical unit of potential (volt) is $1/300$ of the corresponding absolute units, then the practical unit of capacitance — the farad (F) — is 9×10^{11} times greater than the absolute unit:

$$1 \text{ farad} = \frac{1 \text{ coulomb}}{1 \text{ volt}} = 9 \times 10^{11} \text{ cm (abs. units of capacitance)}$$

This unit of capacitance is so great that customary practice is to express the capacitance either in centimetres or in microfarads (millionths of a farad):

$$1 \mu\text{F} = 9 \times 10^5 \text{ cm}$$

4. If a conductor is not isolated, the potential it acquires when a definite charge is imparted to it depends noticeably on the shape and arrangement of other conductors. The reason is that the field of a charged conductor causes redistribution of the charges on all the adjacent conductors, including, naturally, uncharged ones (*electrostatic induction*). When equilibrium is reached, the charges of the conductors are arranged on them so that inside each conductor the resultant field of the induced charges and of the inducing charge equals zero (the condition of electrostatic equilibrium, see Sec. 1.5). This process, of course, is connected with redistribution of the charge on the inducing conductor itself. Thus, the potential of the charged

conductor equals the sum of the potentials of its own redistributed charge and of the charges induced by it on the other conductors. The determination of how this resulting potential φ and the capacitance of a conductor $C = q/\varphi$ depend on the shape and arrangement of adjacent conductors involves, generally speaking, considerable mathematical difficulties.

5. It is possible, however, to achieve *complete* independence of a system of conductors of the arrangement and electrical state of all foreign conductors by their *electrostatic shielding*, i.e. by enclosing them within a metal shell. The charges outside the shell do not affect the electrical state of the space within it because the field of these external charges in the internal space is compensated by the field of the charges they induce on the outer surface of the shell. Indeed, if the shell were completely filled with metal, i.e. formed a solid conductor, then the intensity of the field of the external charges and of the charges induced by them on the outer surface of the shell should equal zero at all the internal points of the conductor. It is evident that it remains equal to zero if the inner portions are removed from the conductor.

Thus, a metal shell completely eliminates the dependence of the electrical state of the inner space on the external space because these spaces are separated by a layer of metal in which the field equals zero. This is why electrostatic shielding is always used in accurate measurements to eliminate external influence on electrical measuring instruments, apparatuses, and the like.

It must also be noted that the charges of the conductors inside the shell induce a charge on its inner surface that equals them in magnitude and is opposite in sign. This can be seen by applying Gauss's law (1.11) to the closed surface S inside the shell (Fig. 14):

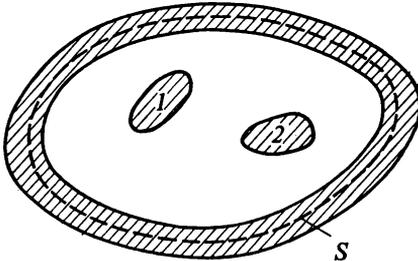


Fig. 14

the electric flux through this surface will obviously equal zero. A charge will concentrate on the outer surface of the shell that is equal (in magnitude and sign) to the charge of the conductors inside the shell.

6. Let us now turn to a capacitor consisting of two (or several) insulated conductors ("plates") of which one is completely confined inside the other as, for example, in a spherical and an infinite cylindrical capacitors. Owing to the field between the plates of such a capacitor being independent of the arrangement of the conductors surrounding it, as indicated above, the capacitance of such a capacitor also does not depend on external circumstances. By the *capacitance of a capacitor*, unlike that of an isolated conductor, one must understand the ratio of the charge of the capacitor q to the potential difference $\varphi_2 - \varphi_1$ of these plates (and not to the potential of one of the plates):

$$C = \frac{q}{\varphi_2 - \varphi_1} \quad (1.52)$$

Further, by the charge q of a capacitor is meant the absolute value of the charges of each of the capacitor plates that are equal in magnitude and opposite in sign.

We should understand that the charge of capacitor plates means only the charges on the inner surfaces of the plates facing each other*.

If none of two (or more) conductors forms a closed system confining the remaining conductors, then the capacitance of this system of conductors (of this capacitor) will practically be independent of the arrangement of the surrounding bodies only if the dimensions of the conductors are great in comparison with the distance between them. Only in this condition will the space between the capacitor plates be shielded to a considerable extent, if not completely, by the plates themselves from the action of an external field.

Problem 8. Show that the capacitance C of a parallel-plate, cylindrical and spherical capacitors is determined by the following formulas:

parallel-plate capacitor:

$$C = \frac{S}{4\pi d} \quad (1.53)$$

cylindrical capacitor:

$$C = \frac{L}{2 \ln \frac{r_2}{r_1}} \quad (1.54)$$

* Thus, by the charge q of a spherical capacitor consisting of a metal sphere A surrounded by a hollow metal sphere B we should understand the charge of the internal sphere A or the charge equal to it in absolute value and arranged on the *inner* surface of the external sphere B . Apart from this charge q , there may be an arbitrary charge q' on the outer surface of sphere B that has no influence on the potential difference between A and B .

spherical capacitor:

$$C = \frac{R_1 R_2}{R_2 - R_1} \quad (1.55)$$

where S = surface area of the capacitor plates

d = distance between them

L = length of the cylinder

r_1 and r_2 = radiuses of the internal and external cylinders

R_1 and R_2 = radiuses of the internal and external spherical surfaces.

Note. Since the capacitance of a cylindrical capacitor is proportional to its length L , we can speak of the "capacitance of a unit length of a capacitor" C' equal to

$$C' = \frac{C}{L} = \frac{1}{2 \ln \frac{r_2}{r_1}} \quad (1.56)$$

It is obvious that

$$C' = \frac{q}{L(\varphi_2 - \varphi_1)} = \frac{q'}{\varphi_2 - \varphi_1} \quad (1.57)$$

where q' is the charge per unit of capacitor length. The concept of the capacitance of a unit length (like the above expression for C') can be used only when the length of a cylindrical capacitor is so great in comparison with the distance between its plates that the end portions of the capacitor in which its field is appreciably distorted (the so-called "fringe effects") may be completely disregarded. Only in this case is C proportional to L .

1.10 Gradient of Electrostatic Potential. Lines of Force

1. It follows from Eq. (1.39),

$$d\varphi = -\mathbf{E} ds = -E_s ds$$

that

$$\frac{\partial \varphi}{\partial s} = -E_s \quad (1.58)$$

where $\partial\varphi/\partial s$ stands for the derivative of φ along the direction s of the vector ds (see Appendix, Sec. A.2). In accordance with the definition of gradient, this space derivative of the scalar φ coincides with the component of the gradient $\text{grad } \varphi$ along the direction s

[see Eq. (A.4)]:

$$\frac{\partial \varphi}{\partial s} = \text{grad}_s \varphi$$

Thus,

$$E_s = -\text{grad}_s \varphi$$

Since this equality of the projections of the vectors \mathbf{E} and $-\text{grad} \varphi$ must occur with any choice of the direction of \mathbf{s} , then the vectors themselves should be equal to each other:

$$\mathbf{E} = -\text{grad} \varphi \quad (1.59)$$

Hence, *the intensity of an electrostatic field \mathbf{E} equals the gradient of the electrostatic potential φ taken with the opposite sign.*

Since the gradient of a potential is directed toward its growth, and the numerical value of the gradient is a measure of the rate of this growth, we can say that the intensity of an electric field is a measure of the rate of the potential drop, or, simply, that it equals the *potential drop*. The direction of the field intensity coincides with that of the orthogonal trajectories of equipotential surfaces (see Appendix, Sec. A.2). Consequently, these orthogonal trajectories (gradient lines) coincide with the *lines of the electric forces*.

2. An electric line of force is a line the tangent to which at any point coincides in its direction with the vector of the electric field intensity \mathbf{E} at the same point (in other words, with the direction of the electric force acting on a unit positive charge). It is obvious that we can draw one and only one line of force through each point of a field in which $\mathbf{E} \neq 0$. At each such point P , the vector \mathbf{E} has a quite definite direction. Laying off from P an arbitrarily small length along the direction of \mathbf{E} , we arrive at the point P' at which the vector \mathbf{E} may have a different direction than at P . Laying off from P' an arbitrarily small length in the corresponding direction, we arrive at a new point P'' where we can repeat the same operation, etc. In the limit, with infinite reduction of the lengths forming it, the broken line obtained coincides with the required line of force.

To obtain an analytic equation of lines of force, it is sufficient to take account of the fact that an element of length ds of a line of force is parallel to the field intensity \mathbf{E} , i.e. that its components along the coordinate axes dx , dy , and dz are proportional to the components E_x , E_y , and E_z of the vector \mathbf{E} :

$$\frac{dx}{E_x} = \frac{dy}{E_y} = \frac{dz}{E_z} \quad (1.60)$$

Equations (1.60) are equivalent to a system of two ordinary differential equations, for example $dx/dz = E_x/E_z$ and $dy/dz = E_y/E_z$ whose integrals have the form $f_1(x, y, z) = C_1$ and $f_2(x, y, z) = C_2$.

Here C_1 and C_2 are integration constants. A combination of the last two equations is the equation of a line of force. The arbitrary nature of the choice of the constants C_1 and C_2 corresponds to the possibility of arbitrarily choosing the coordinates x_0 , y_0 and z_0 of the point of a field through which we want to draw the given line of force.

Physicists in the nineteenth century for a long time tried to explain electromagnetic phenomena by deformations and circulatory movements of a special all-penetrating hypothetic medium—ether, and assumed that the lines of force coincide with the axes of the deformation (or the axes of the torsion) which the ether is subjected to in an electric field. By the beginning of the twentieth century, however, the complete untenability of the mechanistic theory of ether was established*, and at present, when using the concept of “lines of force”, it must be remembered that this concept has a conditional and auxiliary significance and that lines of force serve only for a graphical illustration of the direction of an electric vector.

3. It should be mentioned in passing that just like the density of arrangement of the lines depicting equipotential surfaces when drawing them in the proper way serves as a measure of the value of the potential gradient, i.e. a measure of the field intensity (see p. 614), the lines of force can be used in a similar way for the same purpose.

It is naturally impossible to show in a drawing all the lines of force passing through all the points of a field and filling up all the space occupied by the field. Usually, lines of force are drawn so that at any portion of a field the number of lines intersecting a unit area of its surface perpendicular to them is proportional to the numerical value of the field intensity on this area. Thus, the density of arrangement of the lines of force can serve as a measure of the field intensity. The number of lines intersecting an arbitrary element of the surface dS will evidently be proportional to the product of the numerical value of the vector \mathbf{E} and the projection of the element of area dS onto a plane perpendicular to \mathbf{E} . This product $E dS \cos(\mathbf{E}\mathbf{n})$ equals the flux of the vector \mathbf{E} through the element dS . Therefore,

* Space (including a vacuum) has not only length, but also a number of other physical properties characterized by an electromagnetic and a gravitational fields, and may be in different physical states; these states of space are due to the bodies in it and act, in turn, on these bodies. There can be no objections in essence to using the term “ether” in the meaning of the carrier of these physical properties of “empty” space; there can only be a doubt as to the expediency of using such terminology corresponding to the division of the single concept of physical space into the concept of ether and that of geometrical space, i.e. pure length deprived of all other physical properties. Very often, however, a different, mechanistic meaning is introduced into the term “ether”. The notion of ether as of a continuous liquid or of a complex of very minute atoms is undoubtedly false, as is in general any notion of the movements of the elements of ether in space.

the expression “the number of lines of force intersecting a given surface” is sometimes used instead of “the flux of a vector through a given surface”. This number of lines is considered to be positive or negative depending on whether the lines of force intersect the given surface in the direction of a positive (external) or negative (internal) normal to it.

It must be indicated that with the above method of drawing lines of force, the total number of these lines intersecting *any* closed surface S should be proportional to the algebraic sum of the charges contained within S because according to Gauss’s law (1.11), the sum of these charges is proportional to the flux of the vector \mathbf{E} through S . It is quite natural that when determining the number of lines intersecting S we must take each of them with the appropriate sign (+ or —).

In particular, the number of lines of force intersecting any closed surface containing no charges equals zero. In other words, the (positive) number of lines emerging from the volume confined within this surface equals the (negative) number of lines entering it. Hence it follows that lines of force can neither begin nor terminate in sections of a field that contain no charges*. On the other hand, these

lines cannot also be closed. Otherwise, the line integral $\oint_L \mathbf{E}_s \cdot d\mathbf{s}$ with

respect to each of the closed lines of force L would differ from zero (because the elements $d\mathbf{s}$ of the lines of force are parallel to \mathbf{E} and, consequently, the integrand is essentially positive), which contradicts Eq. (1.33). Hence, *in an electrostatic field the lines of force either*

* The only exception to this rule are points (or lines) of indeterminacy at which the field intensity E vanishes, and the direction of the lines of force becomes indeterminate. For example, the middle of the distance between two equal charges of the same sign is such a point of indeterminacy.

Let us assume, for instance, that two equal positive charges are at the points A and B and the point O is at the middle of section AB . The straight line AO is a line of force along which the field intensity E diminishes with an increasing distance from A and becomes equal to zero at the point O . The same relates to the line of force BO : both oppositely directed lines of force AO and BO terminate at the point O . At the same time, a pencil of straight lines of force emerges from this point. The lines are in a plane of symmetry at right angles to the length AB . The flux through the closed surface encircling the point O is naturally equal to zero.

This example also clearly shows the conditional nature of the concept of field tubes of force, i.e. filaments having a finite section whose surfaces are formed by lines of force. Indeed, a field tube having a section however small and enclosing the line AO expands unlimitedly with an increasing distance from the charge A . As it approaches the plane of symmetry passing through O , the section of this tube tends to infinity and coincides in the limit with the plane of symmetry.

begin and terminate on electric charges, or one of their ends* extends to infinity.

Thus, to obtain a proper picture of a field, it is evidently sufficient to draw lines from each element of a charge whose number is proportional to the magnitude of this charge.

There is a third possibility for unclosed lines apart from those considered above: upon infinite continuation they may closely fill a certain limited section of space when unlimitedly extended without intersecting or closing. We shall acquaint ourselves with such *magnetic* lines of force in Chapter 4. For the lines of force of an *electrostatic* field, however, this possibility is excluded because a line filling a certain section of space must approach points that it has passed previously as close as desired when extended sufficiently. If P and P' are two such infinitely close points on such a line of force L , then

the integral $\int_P^{P'} E_s ds$ over this line will be substantially positive and

will have a *finite* value. At the same time, if only the vector \mathbf{E} is finite, then this integral should differ by only an *infinitely small*

amount from the integral $\oint E_s ds$ over the *closed* contour formed by

the length PP' of the line of force and an infinitely small length of the straight line connecting P' and P . But the latter integral, according

to Eq. (1.33), equals zero, i.e. differs by a *finite* amount from $\int_P^{P'} E_s ds$.

This contradiction proves the impossibility of the existence of lines of force of the indicated type.

Problem 9. Show on the basis of Eq. (1.49) that the field intensity of a dipole having the moment \mathbf{p} (the lines of force of the field are shown in Fig. 15), is

$$\mathbf{E} = \frac{3(\mathbf{p}\mathbf{R})\mathbf{R}}{R^5} - \frac{\mathbf{p}}{R^3} \quad (1.61)$$

and that in a spherical system of coordinates R, θ, α with its centre at the dipole and a polar axis parallel to \mathbf{p} , the components of the

* Both ends of a line of force cannot be at infinity if all the charges are in a finite region of space. To prove this, it is sufficient to apply Eq. (1.33) to the closed path obtained when the line being considered is closed at infinity by the arc of a circle having an infinite radius and take into account that the

integral $\int E_s ds$ along this arc tends to zero when the radius R of the arc grows (because when R is sufficiently great $E \sim \frac{1}{R^2}$, while $\int ds \sim R$).

vector \mathbf{E} are

$$\left. \begin{aligned} E_R &= \frac{2p \cos \theta}{R^3} \\ E_\theta &= \frac{p \sin \theta}{R^3} \\ E_\alpha &= 0. \end{aligned} \right\} \quad (1.62)$$

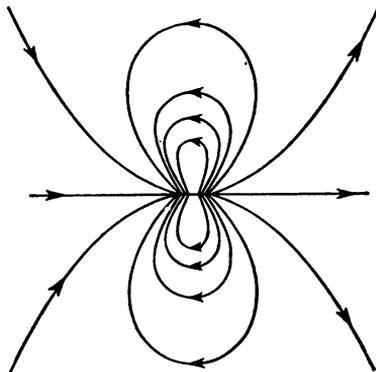


Fig. 15

Thus, the angle β between a line of force and the radius-vector \mathbf{R} is determined by the relationship

$$\tan \beta = \frac{E_\theta}{E_R} = \frac{1}{2} \tan \theta$$

At identical distances R from a dipole, the field along its axis ($\theta = 0$ or $\theta = \pi$) is twice as strong as in the equatorial plane ($\theta = \pi/2$).

1.11 Poisson and Laplace Equations

Equation (1.59) establishes the relationship between the potential of an electrostatic field and the intensity of this field. This equation allows us to obtain the relationship between the potential and the charge density. For this end, we must form the divergence of both parts of this equation and then use Eq. (1.27):

$$\operatorname{div} \operatorname{grad} \varphi = -\operatorname{div} \mathbf{E} = -4\pi\rho \quad (1.63)$$

According to the rules of vector analysis [see Eq. (A.40)]

$$\operatorname{div} \operatorname{grad} \varphi = \nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} \quad (1.64)$$

so that Eq. (1.63) can be written as follows*:

$$\nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = -4\pi\rho \quad (1.65)$$

This differential equation is called the *Poisson equation*. At sections of a field where there are no electric charges, this equation becomes ($\rho = 0$)

$$\nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0 \quad (1.66)$$

This particular form of the Poisson equation is called the *Laplace equation***.

The Poisson equation makes it possible to determine the potential of the field of space charges if we know how they are arranged. The solution (integral) of this differential equation (in definite boundary conditions) should obviously coincide with Eq. (1.46) which we have derived earlier:

$$\varphi = \int \frac{\rho dV}{R}$$

We shall prove this somewhat later by direct calculation. Meanwhile, we shall note that it is more convenient to proceed from the differential equation (1.65) instead of from the integral (1.46) for solving some problems.

Example. Determine the density of the thermionic current between two infinite plate electrodes in a vacuum. This example of the application of the Poisson equation has been taken not from electrostatics, but from the science of a current and is of great significance for the theory of cathode (amplifying) valves.

It is general knowledge that heated metals emit from their surface into the surrounding space a beam of free electrons. If we apply a definite potential difference to two metal electrodes and heat the negative electrode (cathode), then the electrons continuously emitted by the glowing cathode will be attracted to the surface of the positive electrode (anode). The beam of electrons travelling from the cathode to the anode is equivalent to an electric current. This current is called *thermionic*.

* The quantity $\nabla^2\varphi$ is sometimes designated by $\Delta\varphi$ and is called the Laplacian of the scalar φ [cf. the Laplace equation (1.66)].

** Equations (1.65) and (1.66) were studied by P. Laplace and S. Poisson mainly in connection with their investigations of the gravitational field of ponderable masses. It can easily be seen that equations of this kind determine the potential of any force field induced by the centres of forces in it (ponderable masses, electric charges, magnetic poles) according to the law of inverse proportion to the square of the distance.

Let us select the axes of Cartesian coordinates so that their origin is on the cathode and the x -axis is perpendicular to the plane of the electrodes and is directed toward the anode. Let us assume that the cathode potential equals zero, and the anode potential equals φ_a . It follows from considerations of symmetry that equipotential surfaces are parallel to the electrodes; hence $\partial\varphi/\partial y = \partial\varphi/\partial z = 0$, and the Poisson equation for the space between the electrodes becomes

$$\frac{\partial^2\varphi}{\partial x^2} = -4\pi\rho \quad (1.67)$$

If we denote by $n(x)$ the number of electrons per unit of volume in the space between the electrodes at the distance x from the cathode, and by e the *absolute* value of the charge of an electron, then the charge density at this distance will be:

$$\rho = -n(x)e$$

Let us assume for simplicity's sake that the electrons emitted by the cathode have no initial velocity when leaving its surface*. Along the path from the cathode to the anode, the forces of the electric field will perform the work $-(-e)\varphi_a = e\varphi_a$ on electrons having the charge $-e$. This work will obviously transform into the kinetic energy of motion of the electrons. Denoting by $v(x)$ the velocity of an electron at the distance x from the cathode, and by $\varphi(x)$ the potential at the same distance, we get

$$\frac{mv^2(x)}{2} = e\varphi(x) \quad (1.68)$$

where m is the mass of an electron. Finally, the density j of the electric current, i.e. the charge flowing per unit time through an area of 1 cm^2 perpendicular to the current (i.e. perpendicular to the x -axis), evidently equals

$$j = en(x)v(x) \quad (1.69)$$

because nv is the number of electrons passing through this area in unit time. Unlike ρ , n , and v , the quantity j is a constant that does not depend on x , since when a stationary state is achieved an identical number of electrons passes through any plane parallel to the electrodes.

Let us exclude from Eq. (1.67) all the unknown functions of x except φ . First of all,

$$\frac{\partial^2\varphi}{\partial x^2} = -4\pi\rho = 4\pi ne = \frac{4\pi j}{v}$$

* Actually, they have initial velocities whose mean value, however, is small in comparison with the velocity acquired by the electrons along the path to the anode under the action of the field of the electrodes (in the usual conditions of an experiment).

But it follows from Eq. (1.68) that

$$v = \sqrt{\frac{2e}{m}} \varphi$$

hence,

$$\frac{\partial^2 \varphi}{\partial x^2} = 4\pi j \varphi^{-1/2} \sqrt{\frac{m}{2e}}$$

Introducing the notation

$$A = 9\pi \sqrt{\frac{m}{2e}} \quad (1.70)$$

we get

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{4}{9} A j \varphi^{-1/2} \quad (1.71)$$

It is easy to see by substitution that by solving differential equation (1.71) which, according to the condition of the problem, becomes equal to zero at the cathode and, in addition, complies with the condition that*

$$\frac{\partial \varphi}{\partial x} = 0 \text{ when } x = 0$$

the potential is

$$\varphi = (Aj)^{2/3} x^{4/3} \quad (1.72)$$

If we denote the distance between the anode and the cathode by L , then when $x = L$ the potential φ should become equal to φ_a . Hence,

$$\varphi_a = (Aj)^{2/3} L^{4/3}$$

whence

$$j = \frac{1}{A} \frac{\varphi_a^{3/2}}{L^2} = \frac{1}{9\pi} \sqrt{\frac{2e}{m}} \frac{\varphi_a^{3/2}}{L^2} \quad (1.73)$$

* With small potential differences between the electrodes, the current is far from saturation, and the number of electrons arriving at the anode in a unit time is small in comparison with their number emitted from the cathode. The surplus electrons form a layer, or "cloud", on the outer surface of the cathode ($x = 0$) from which separate electrons partly diffuse back into the cathode, and partly into the space between the electrodes from where they are transported by the field to the anode. This layer itself as a whole is stationary, which can occur only when

$$E = \frac{\partial \varphi}{\partial x} = 0$$

Thus, the density of a thermionic current does not obey Ohm's law, but grows proportional to the power $3/2$ of the voltage φ_a applied to the electrodes and inversely proportional to the square of the distance between them. This distinction of the laws of a thermionic current from those of a current in metals is due to two reasons. First, electrons in metals collide with the positive ions forming the solid skeleton of the metal and as a result encounter a resistance to their motion that is absent upon motion in a vacuum*. Second, with a thermionic current only free electrons are in the space between the electrodes. Their charge is not compensated by that of positive ions as occurs in metals, consequently the field of this so-called "space charge" distorts the field of the electrodes.

It should be noted that Eq. (1.73) stops being true at great current densities**. When the anode potential is increased, a moment sets in after which all the electrons emitted by the cathode are immediately carried along to the anode. A further increase in the anode potential can obviously no longer result in an increase in the current density which thus acquires a constant value (the saturation current).

Problem 10. Let R stand for the distance from a given point of space to an arbitrarily chosen initial point P . Show that the scalar $\psi = 1/R$ complies with the Laplace equation

$$\nabla^2 \left(\frac{1}{R} \right) = 0 \quad (1.74)$$

The point $R = 0$ is not considered.

Problem 11. An infinite plane plate with a thickness of $2a$ is uniformly charged with electricity with a volume density of ρ . The x -axis is perpendicular to the plate, and the origin of coordinates is in the middle plane at equal distances from both surfaces of the plate. Show that the potential of the field inside and outside the plate equals, respectively,

$$\varphi_i = -2\pi\rho x^2 \quad \text{and} \quad \varphi_e = -2\pi\rho a(2x - a)$$

while the vector \mathbf{E} is directed along the x -axis from the middle plane $x = 0$ (if $\rho > 0$) and numerically equals $E = 4\pi\rho|x|$ when $|x| \leq a$ or $E = 4\pi\rho a$ when $|x| \geq a$.

Compare this case with the limiting one of an infinite charged plane (Sec. 1.4).

* A collision of electrons with each other does not change their total momentum.

** The boundary condition $d\varphi/dx=0$ at $x = 0$ that determines the solution of Eq. (1.72) stops corresponding to the physical conditions of the experiment.

Problem 12. Find the potential of the field of a sphere uniformly charged throughout its volume [Eq. (1.51)] on the basis of the Poisson equation in spherical coordinates.

1.12 Potential of Space and Surface Charges

1. In this section, we shall prove expressions (1.45) and (1.46) for the potential of surface and space charges. Our proof will be deprived of the shortcomings of our previous derivation of these expressions from the formula for the potential of a point charge (see pp. 51f)*. We shall proceed from the Poisson equation (1.65) for the potential of an electrostatic field and use *Green's theorem* [see Eq. (A.53)] according to which

$$\int_V (\psi \nabla^2 \varphi - \varphi \nabla^2 \psi) dV = \oint_S \left(\psi \frac{\partial \varphi}{\partial n} - \varphi \frac{\partial \psi}{\partial n} \right) dS. \quad (1.75)$$

Here S is the area of the surface confining the volume V , and φ and ψ are two arbitrary scalar functions that are continuous within the volume V and have finite derivatives of both the first and the second orders.

Let us find the value of the electric potential φ at a certain point P of a field. Let R be the distance from an arbitrary point of the field to the point P . We shall assume in Green's theorem (1.75) that $\psi = 1/R$ and take into consideration Eq. (1.74):

$$\nabla^2 \frac{1}{R} = 0$$

and the Poisson equation (1.65):

$$\nabla^2 \varphi = -4\pi\rho.$$

Introducing these values into Green's formula (1.75) and dividing by -4π , we get

$$\int_V \frac{\rho}{R} dV = \frac{1}{4\pi} \oint_S \left\{ \varphi \frac{\partial}{\partial n} \left(\frac{1}{R} \right) - \frac{1}{R} \frac{\partial \varphi}{\partial n} \right\} dS \quad (1.76)$$

2. Let us first assume that throughout the entire volume V we are considering which includes the point P and is confined within the surface S , the potential φ and its derivatives are continuous functions

* The delta-function is often used in modern books on electrodynamics to describe point charges. This improper function makes it quite convenient to obtain a solution of the Poisson and d'Alembert equations. Applications of the delta-function in electrodynamics and in particular for solving the above-mentioned equations with its aid can be found, for example, in: Panofsky, W.K.H. and Phillips, M. *Classical Electricity and Magnetism*. Reading, Mass., Addison-Wesley (1955), Secs. 1.1, 1.5, and 13.2.

of the point. The scalar $\psi = 1/R$ and its derivatives, on the other hand, are continuous and finite throughout the space except for the point P . Since Green's theorem may be applied only to portions of space in which both scalars φ and ψ and their derivatives are continuous, then the point P must be excluded from the region of integration of V . For this purpose, we shall surround the point P with the sphere S_0 (Fig. 16) having an arbitrary small radius R_0

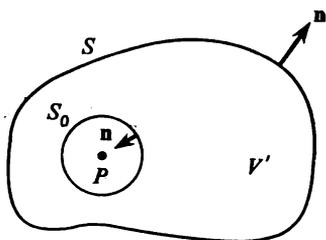


Fig. 16

and apply Eq. (1.76) to the volume V' contained between the outer surface S and that of the sphere S_0 (Fig. 16):

$$\int_{V'} \frac{\rho}{R} dV = \frac{1}{4\pi} \oint_{S+S_0} \left\{ \varphi \frac{\partial}{\partial n} \left(\frac{1}{R} \right) - \frac{1}{R} \frac{\partial \varphi}{\partial n} \right\} dS \quad (1.77)$$

where the subscript $S + S_0$ at the sign of the surface integral means that the integral should be taken over the surfaces S and S_0 . Let us consider in greater detail the integral over the surface S_0 .

The outer normal to the surface of the sphere S_0 relative to the volume of integration V' is directed toward its centre and is opposite to the radius-vector \mathbf{R} . Hence, for the surface S_0 , we have

$$\frac{\partial}{\partial n} \frac{1}{R} = - \frac{\partial}{\partial R} \left(\frac{1}{R} \right) = \frac{1}{R^2} = \frac{1}{R_0^2}$$

and

$$\frac{\partial \varphi}{\partial n} = - \frac{\partial \varphi}{\partial R}$$

Let us introduce these values into the surface integral of Eq. (1.77) and then apply the so-called "mean-value theorem" of integral calculus:

$$\begin{aligned} & \frac{1}{4\pi} \oint_{S_0} \left\{ \varphi \frac{\partial}{\partial n} \left(\frac{1}{R} \right) - \frac{1}{R} \frac{\partial \varphi}{\partial n} \right\} dS = \\ & = \frac{1}{4\pi} \oint_{S_0} \left\{ \frac{\varphi}{R_0^2} + \frac{1}{R_0} \frac{\partial \varphi}{\partial R} \right\} dS = \\ & = \frac{1}{4\pi} \left\{ \frac{1}{R_0^2} \bar{\varphi} + \frac{1}{R_0} \left(\frac{\partial \bar{\varphi}}{\partial R} \right) \right\} \oint_{S_0} dS \end{aligned}$$

where $\left(\frac{\partial \bar{\varphi}}{\partial R}\right)$ and $\bar{\varphi}$ are certain mean values of the quantities $\frac{\partial \varphi}{\partial R}$ and φ on the surface S_0 of the sphere. Since $\oint dS$ equals the total surface area of the sphere $4\pi R_0^2$, then the right-hand side of the equation equals

$$\bar{\varphi} + R_0 \left(\frac{\partial \bar{\varphi}}{\partial R}\right)$$

We shall now reduce the radius R_0 to zero, contracting the sphere S_0 to the point P . The last term of the above expression will vanish, and the mean value of the potential φ on the surface of an infinitely small sphere can be taken equal to the value of the potential φ_P at its centre P . Thus,

$$\lim_{R_0 \rightarrow 0} \frac{1}{4\pi} \oint_{S_0} \left\{ \varphi \frac{\partial}{\partial n} \left(\frac{1}{R} \right) - \frac{1}{R} \frac{\partial \varphi}{\partial n} \right\} dS = \varphi_P \quad (1.78)$$

Consequently, in the limit with $R_0 \rightarrow 0$, Eq. (1.77) becomes

$$\int_V \frac{\rho}{R} dV = \varphi_P + \frac{1}{4\pi} \oint_S \left\{ \varphi \frac{\partial}{\partial n} \left(\frac{1}{R} \right) - \frac{1}{R} \frac{\partial \varphi}{\partial n} \right\} dS$$

or

$$\varphi_P = \int_V \frac{\rho}{R} dV + \frac{1}{4\pi} \oint_S \left\{ \frac{1}{R} \frac{\partial \varphi}{\partial n} - \varphi \frac{\partial}{\partial n} \left(\frac{1}{R} \right) \right\} dS \quad (1.79)$$

where the volume integral may be extended over the entire volume V confined within the surface S because when $R_0 \rightarrow 0$ the volume V' tends to V , while the integrand also remains finite when $R = 0$ (p. 53).

Thus, the potential φ at the point P within the volume V in which φ and its first and second derivatives are finite and continuous is determined by the potential of the space charges in V [cf. Eq. (1.46)] and the values of φ and $\partial\varphi/\partial n$ on the surface S confining the volume V .

3. Let us now assume that although the potential φ remains finite and continuous in the volume V , the continuity of the gradient of φ may be violated on certain "surfaces of discontinuity", i.e. that the values of $\text{grad } \varphi$ at different sides on the surface of discontinuity may differ by a finite value (a change in $\text{grad } \varphi$ in a jump). We shall see that the physical meaning of this assumption consists in presuming that there are charged surfaces in the volume V .

Let us first assume that there is only one open surface of discontinuity S_1 in the volume V . We arbitrarily choose the direction of

a positive normal to it and temporarily denote it by \mathbf{N} (Fig. 17) instead of \mathbf{n} , which we customarily use. We then draw a closed surface S'_1 around S_1 (S'_1 is depicted by a dash line in Fig. 17). Now Eq. (1.79) can obviously be applied to the volume V' confined between the surfaces S and S'_1 . The surface integral in Eq. (1.79) will divide into two integrals: over the surface S and over S'_1 .

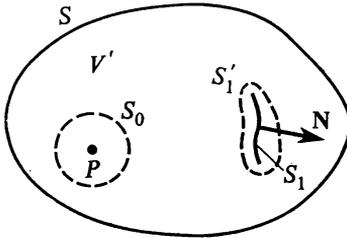


Fig. 17

Now we contract the surface S'_1 so that it gets closer and closer to S_1 . In the limit, S'_1 coincides with S_1 , and integration over S'_1 consists in *double* integration over the surface of discontinuity S_1 : once over the inner side (relative to the normal \mathbf{N}) and once over the outer side of this surface:

$$\begin{aligned} \lim_{s'_1 \rightarrow s_1} \frac{1}{4\pi} \oint_{S'_1} \left\{ \frac{1}{R} \frac{\partial \varphi}{\partial n} - \varphi \frac{\partial}{\partial n} \left(\frac{1}{R} \right) \right\} dS = \\ = \frac{1}{4\pi} \int_{S_1} \left\{ \frac{1}{R} \frac{\partial \varphi}{\partial n} - \varphi \frac{\partial}{\partial n} \left(\frac{1}{R} \right) \right\}_1 dS + \\ + \frac{1}{4\pi} \int_{S_1} \left\{ \frac{1}{R} \frac{\partial \varphi}{\partial n} - \varphi \frac{\partial}{\partial n} \left(\frac{1}{R} \right) \right\}_2 dS \end{aligned}$$

where the subscripts 1 and 2 stand for the values of the integrands on the inner (relative to the normal \mathbf{N}) and on the outer sides of the surface of discontinuity S_1 . In this equation, \mathbf{n} obviously stands for the outward normal with respect to the volume of integration V' , i.e. directed from V' to S_1 . In other words, on the inner side from S_1 by \mathbf{n} we should understand the direction \mathbf{n}_1 , and on the outer side, the direction \mathbf{n}_2 (Fig. 18). Since \mathbf{N} is parallel to \mathbf{n}_1 and antiparallel to \mathbf{n}_2 , then

$$\begin{aligned} \left\{ \frac{\partial}{\partial n} \left(\frac{1}{R} \right) \right\}_1 &= \frac{\partial}{\partial N} \left(\frac{1}{R} \right), \\ \left\{ \frac{\partial}{\partial n} \left(\frac{1}{R} \right) \right\}_2 &= - \frac{\partial}{\partial N} \left(\frac{1}{R} \right) \end{aligned}$$

and

$$\left(\frac{\partial \varphi}{\partial n}\right)_1 = \left(\frac{\partial \varphi}{\partial N}\right)_1, \quad \left(\frac{\partial \varphi}{\partial n}\right)_2 = -\left(\frac{\partial \varphi}{\partial N}\right)_2$$

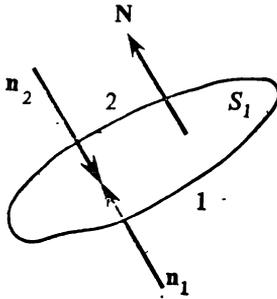


Fig. 18

where $(\partial\varphi/\partial N)_1$ and $(\partial\varphi/\partial N)_2$ are the values of the derivative of φ with respect to the normal \mathbf{N} on the inner and outer sides of the surface S_1 . Denoting the corresponding values of the potential φ by φ_1 and φ_2 and using the expressions obtained in the preceding equation, we get after transformations

$$\begin{aligned} & \lim_{S'_1 \rightarrow S_1} \frac{1}{4\pi} \oint_{S'_1} \left\{ \frac{1}{R} \frac{\partial \varphi}{\partial n} - \varphi \frac{\partial}{\partial n} \left(\frac{1}{R} \right) \right\} dS = \\ & = \frac{1}{4\pi} \int_{S_1} (\varphi_2 - \varphi_1) \frac{\partial}{\partial N} \left(\frac{1}{R} \right) dS - \\ & - \frac{1}{4\pi} \int_{S_1} \left\{ \left(\frac{\partial \varphi}{\partial N} \right)_2 - \left(\frac{\partial \varphi}{\partial N} \right)_1 \right\} \frac{1}{R} dS \end{aligned} \quad (1.80)$$

Since according to our assumption, the potential φ is continuous everywhere, its values φ_1 and φ_2 at both sides of the surface are the same, and the first term of the right-hand side of Eq. (1.80) vanishes*. Hence, if we introduce the notation

$$\left(\frac{\partial \varphi}{\partial N} \right)_2 - \left(\frac{\partial \varphi}{\partial N} \right)_1 = -4\pi\sigma \quad (1.81)$$

then the right-hand side of Eq. (1.80) becomes

$$\int_{S_1} \frac{\sigma}{R} dS$$

* The case $\varphi_1 \neq \varphi_2$ will be treated in Sec. 1.14.

At the same time, when S'_1 coincides with S_1 , the volume V' evidently coincides with the volume V confined by the surface so that Eq. (1.79) becomes

$$\begin{aligned} \varphi_P = & \int_V \frac{\rho dV}{R} + \int_{S_1} \frac{\sigma dS}{R} + \frac{1}{4\pi} \oint_S \left\{ \frac{1}{R} \frac{\partial \varphi}{\partial n} - \right. \\ & \left. - \varphi \frac{\partial}{\partial n} \left(\frac{1}{R} \right) \right\} dS \end{aligned} \quad (1.82)$$

This is the expression for the potential if inside the volume V enclosed by the surface S there is an open surface S_1 disrupting the continuity of the gradient of φ . If there are several such surfaces, the above reasoning can be applied to each of them, so that in this case by the second term in the right-hand side of Eq. (1.82) we should understand the *sum* of the integrals over *all* the surfaces of discontinuity inside of V . Finally, this equation can also be applied to closed surfaces of discontinuity because any closed surface can be broken up into two open ones.

4. The first term of Eq. (1.82) is the potential of the space charges in the volume V , while its second term should evidently be interpreted as the potential of surface charges distributed with the density σ over the surface of discontinuity S_1 [cf. Eq. 1.45]. This interpretation matches quite well the results obtained previously. Indeed, if we denote, as previously, an arbitrarily selected positive normal to S_1 by \mathbf{n} instead of by \mathbf{N} , then Eq. (1.81) will become

$$\left(\frac{\partial \varphi}{\partial n} \right)_2 - \left(\frac{\partial \varphi}{\partial n} \right)_1 = -4\pi\sigma \quad (1.83)$$

But according to Eq. (1.58),

$$E_n = - \frac{\partial \varphi}{\partial n}$$

where E_n is the normal component of the field intensity. Hence, Eq. (1.83) can be written as follows:

$$E_{2n} - E_{1n} = 4\pi\sigma \quad (1.84)$$

Comparing Eq. (1.84) with Eq. (1.14), we see that the quantity σ determined by Eq. (1.83) is indeed equal to the density of the electric charge on the surface S_1 . We thus again arrive at the conclusion that the surfaces of discontinuity of the *normal component* of the potential gradient (i.e. the surfaces of discontinuity of E_n) are physically equivalent to charged surfaces, the jump of this component $\partial\varphi/\partial n$ being proportional to the density of the surface charge.

5. Let us finally turn to the last term of Eq. (1.82) which is an integral over the boundary surface S of the volume V and expresses the dependence of the values of the potential φ in the volume V on the values of this potential and its first derivatives on the boundary surface of this volume.

This term will completely drop out of the expression for the potential if by the volume of integration V we understand the entire infinite space (i.e. remove the surface S confining V to infinity) and impose the following boundary conditions on φ and its derivatives: at infinity φ tends to zero not slower than $1/R$, and its first derivatives with respect to the coordinates not slower than $1/R^2$, i.e.

$$R\varphi \text{ and } R^2 \text{ grad } \varphi \text{ at } R \rightarrow \infty \text{ remain finite} \quad (1.85)$$

We have already used the first of these conditions in Sec. 1.8 by equating to zero the value of the potential at infinity [Eq. (1.41)]; the second condition is directly connected with the first one. Physically, it means that at infinity the intensity \mathbf{E} of an electric field equals zero, i.e. that all the charges are in the finite region of space. In the entire following text, we shall call only that solution of the Poisson equation the potential of an electrostatic field which complies with conditions (1.85)*.

6. We shall now show that when conditions (1.85) are imposed, and the boundary surface S is removed to infinity, the last term of expression (1.82) does indeed vanish. Let us choose a sphere having the radius R with its centre at point P as the surface S . The outward normal to this sphere coincides with the radius-vector \mathbf{R} so that

$$\frac{\partial}{\partial n} \left(\frac{1}{R} \right) = \frac{\partial}{\partial R} \left(\frac{1}{R} \right) = -\frac{1}{R^2}$$

It follows from conditions (1.85), consequently, that the integrand of the integral interesting us tends to zero when $R \rightarrow \infty$ not slower than $1/R^3$ whereas the surface of integration grows in proportion only to R^2 . Therefore, this integral at $R \rightarrow \infty$ tends to zero, and Eq. (1.82) acquires the form (if we omit the subscript P of φ)

$$\varphi = \int \frac{\rho dV}{R} + \int \frac{\sigma dS}{R} \quad (1.86)$$

where R is the distance from an element of the space charge ρdV or the surface charge σdS to the point of the field having the potential φ , and where integration must be extended over the entire space occupied by the charges [cf. Eq. (1.86) with the previous formulas (1.45) and (1.46)].

* With the exception, naturally, of the cases when charges extending to infinity, such as infinitely charged plates and cylinders, are conditionally taken into consideration.

It follows from the above that the surface integral in Eq. (1.82) takes into consideration the field of charges outside of the volume of integration V (and also the possibility of adding an arbitrary additive constant to φ).

7. It must be remembered that throughout the preceding text we assumed that both the potential itself and its first derivatives (gradient) are *finite everywhere*. Infinity of the gradient of φ (i.e. a jump of the potential) would mean an infinite intensity of the electric field, which is physically meaningless. A finite value of $\text{grad } \varphi$ means that φ is *continuous*, which we have assumed up to now. It should be mentioned here that in Sec. 1.14 we shall return to the question of surfaces of discontinuity of the potential from a somewhat different viewpoint.

8. If the size of the charge q occupying the volume V_1 is so small in comparison with its distance R to the point P of the field being considered that the distance from all the elements of the charge to P may be considered the same, then the potential of this charge at P will be

$$\int_{V_1} \frac{\rho dV}{R} = \frac{1}{R} \int_{V_1} \rho dV = \frac{q}{R}$$

which coincides with Eq. (1.43) for the potential of a point charge.

1.13 Typical Problems of Electrostatics

1. The concept of potential introduced above considerably facilitates the solution of electrostatic problems because the determining of the electric intensity *vector field* \mathbf{E} consists in finding the *field of the scalar* φ ; in other words, the determination of *three* functions of a point (the components of the vector \mathbf{E}) is replaced by the determination of only *one* function φ .

Knowing the density of the space and surface charges, we can determine the potential of a field [Eq. (1.86)]. Conversely, knowing the gradient of the potential φ , according to the divergence of this gradient $\nabla^2 \varphi$ and according to the magnitude of the jumps of its normal component on the surfaces of discontinuity, we can unambiguously determine the distribution of the charges [Eqs. (1.65) and (1.83)].

It is naturally impossible practically to measure the density of charges or the potential gradient at all the points of a field. For this reason, an experimenter actually has to do with problems of a different kind, namely, given the arrangement and shape of all the conductors in a field, he has to *determine the field* of these conductors and the *distribution of the charges* over their surface *if either the*

potential of each conductor is known (problem A), or if the total charge of each conductor is known (problem B). We assume that the space charges are absent because the charges of conductors are concentrated on their surface (Sec. 1.5), while dielectrics are not treated in this chapter.

2. We shall show first of all that even with these conditions, which at first appearances are formulated quite generally, an electrostatic field and, consequently, the distribution of the charges are determined *unambiguously*.

Let us assume that the opposite is true and let φ and φ' be two different functions of a point complying with the conditions of problem A or B. Owing to the absence of space charges, both φ and φ' must comply with the Laplace equation (1.66) throughout the entire space; hence, their difference $\varphi'' = \varphi - \varphi'$ complies with the same equation

$$\nabla^2 \varphi'' = 0$$

Assuming in Green's formula (A.52) that $\psi = \varphi = \varphi''$, we thus obtain

$$\int_V (\nabla \varphi'')^2 dV = \oint_S \varphi'' \frac{\partial \varphi''}{\partial n} dS \quad (1.87)$$

where it is assumed that integration extends over the entire space V outside of the conductors, so that by S we must understand the total surface area of all the conductors.

Since when solving problem A, the quantities φ and φ' must take on preset values on S , then φ'' on all the surfaces S equals zero. Consequently,

$$\int_V (\nabla \varphi'')^2 dV = 0 \quad (1.88)$$

Seeing that the integrand is positive, it follows from this equation that $\nabla \varphi'' = \text{grad } \varphi''$ equals zero in the entire space, i.e. that $\varphi'' = \text{const}$. Since in addition, φ'' vanishes on the surfaces of conductors, then it equals zero everywhere. Hence,

$$\varphi = \varphi'$$

which proves the unambiguous solution of problem A.

Turning to problem B, we must note that the potentials φ and φ' , and, therefore, also φ'' should have a constant value on the surface of each conductor. Hence, for the surface of each conductor we can write

$$\oint \varphi'' \frac{\partial \varphi''}{\partial n} dS = \varphi'' \oint \frac{\partial \varphi''}{\partial n} dS$$

But on the surface of conductors, according to Eq. (1.20), we have

$$E_n = -\frac{\partial \varphi}{\partial n} = 4\pi\sigma \quad \text{and} \quad E'_n = -\frac{\partial \varphi'}{\partial n} = 4\pi\sigma'$$

where σ and σ' are the charge densities of the conductors corresponding to the solutions of φ and φ' . Consequently,

$$\frac{\partial \varphi''}{\partial n} = \frac{\partial \varphi}{\partial n} - \frac{\partial \varphi'}{\partial n} = 4\pi\sigma' - 4\pi\sigma$$

and

$$\begin{aligned} \oint \varphi'' \frac{\partial \varphi''}{\partial n} dS &= 4\pi\varphi'' \left\{ \oint \sigma' dS - \oint \sigma dS \right\} = \\ &= 4\pi\varphi'' (q' - q) \end{aligned}$$

where q and q' are the values of the total charge of a conductor corresponding to the solutions of φ and φ' . According to the condition B, the charges q and q' must equal the same preset value. Since the above reasoning can be applied to the surface of *each* conductor, then the entire right-hand part of Eq. (1.87) vanishes whence, as in problem A, it follows that

$$\varphi'' = \varphi - \varphi' = \text{const}$$

Thus, the different solutions of problem B may differ only in an insignificant additive constant in the expression for the potential which, by the way, will vanish if we impose conditions (1.85) at infinity. It is not difficult, finally, to see that if conditions of type A are given for part of the conductors and conditions of type B for the remaining ones, the solution of the problem remains unambiguous.

3. Thus, we have proved that the solution of problems A and B of electrostatics is unambiguous. It must be noted that considerable mathematical difficulties are involved, generally speaking, in finding the solution. If we succeed in some way or other, however, to find an expression for φ that satisfies the conditions A or B, then the unambiguity theorem permits us to conclude that the found expression is the only one and therefore the true solution of the problem. The skilful use of this circumstance will appreciably facilitate the consideration of a number of problems of electrostatics. We can illustrate this here, unfortunately, only by a single example, using for purposes of simplification the concept of a point charge. This example is a particular case of application of the general *method of images* (it will also be used in solving the last example in Sec. 2.4).

Example. The point charge q is at the distance d from an infinite conductor occupying the left-hand half-space (Fig. 19). Determine the field in the right-hand half-space and the density of the charges induced by the charge q on the surface of the conductor. This example

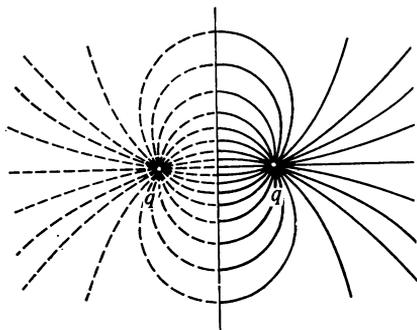


Fig. 19

relates to the type of problems considered above because the total charge q of the conductor carrying the “point” charge is given; the constant potential of the infinite conductor, on the other hand, can be taken conditionally equal to zero. Hence, these conditions determine the unambiguous solution of the problem.

To find this solution, let us assume that the charge $q' = -q$ is on the continuation of the perpendicular dropped from q onto the surface of the conductor at a distance of d from this surface and then that the conductor is slowly removed. Therefore, the plane that coincided earlier with the surface of the conductor will have zero as its required potential because all the points of this field will be at equal distances from the charges $+q$ and $-q$ that are equal in magnitude and opposite in sign. Hence, the field of the combination of these charges in the right-hand half-space complies with the conditions of the problem, from which on the basis of the unambiguity theorem it follows that this field in the right-hand half-space is identical with the required field of the charge q and of the charges induced by it on the surface of the infinite conductor.

Thus, our problem has been reduced to the very simple one of determining the field of two point charges $+q$ and $-q$. The field inside the conductor naturally equals zero so that in the left-hand half-space the field of the charges $+q$ and $-q$ (the dash lines in Fig. 19) does not coincide with that of the charge q and of the conductor.

Problem 13. Complete the solution of the example considered above and show that the density of the charges induced by the charge q on the surface of an infinite conductor equals

$$\sigma = -\frac{qd}{2\pi R^3}$$

where R is the distance from an element of surface area of the conductor to the charge q , and that the total charge induced on the conductor equals $-q$.

1.14 Electrical Double Layer

We shall need the results of this section only in connection with our treatment of *magnetic* double layers, or sheets, in Chapter 4, so that reading of this section may be postponed until required.

1. Assume that two very close surfaces S_1 and S'_1 (Fig. 20) that are parallel to each other are charged with electricity of opposite signs so that the densities of the charges σ and σ' on elements of the two surfaces opposite each other are equal in magnitude and opposite in sign ($\sigma = -\sigma'$); let us further assume for definiteness that $\sigma > 0$.

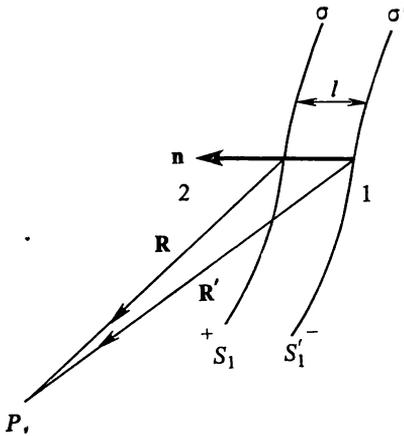


Fig. 20

If the distance between S_1 and S'_1 is vanishingly small in comparison with the distance from these surfaces to the points of the field being considered, then the complex of the surfaces S_1 and S'_1 is called an *electrical double layer*. The potential of this layer at a certain point P according to Eq. (1.86) will be

$$\varphi = \int_{S_1} \frac{\sigma dS}{R} + \int_{S'_1} \frac{\sigma' dS}{R'} = \int_{S_1} \sigma \left(\frac{1}{R} - \frac{1}{R'} \right) dS \quad (1.89)$$

where R and R' are, respectively, the distances from point P to the element dS of the positively charged surface S and from the element of the negative surface S_1' opposite to it. But $(1/R) - (1/R')$ is the increment of the reciprocal of the numerical value of the radius-vector \mathbf{R} drawn from the element of the double layer being considered to the point P when the origin of the vector moves from the negative surface to the positive one. Assume that \mathbf{n} is the direction of a normal to the double layer running from its negative to its positive side, and assume that l is the width of the layer (the distance between S_1 and S_1'). Repeating the reasoning which in Sec. 1.8 led us to Eq. (1.50), we shall see that when $l \ll R$, we have

$$\frac{1}{R} - \frac{1}{R'} = \mathbf{n}l \operatorname{grad}_a \left(\frac{1}{R} \right) = -\mathbf{n}l \operatorname{grad}_a \left(\frac{1}{R} \right) \quad (1.90)$$

Using Eq. (1.90) in Eq. (1.89) and introducing the notation

$$\tau = \sigma l, \quad (1.91)$$

we get:

$$\varphi = - \int_{S_1} \tau \mathbf{n} \operatorname{grad}_a \left(\frac{1}{R} \right) dS \quad (1.92)$$

This is the final expression for the potential of a double layer* in which it is assumed that the radius-vector \mathbf{R} has been drawn from the layer to point P of the field being investigated. The quantity τ equal to the product of the density of a charge on the surface of a layer and the width of the layer is called the *strength* or the *moment of the layer*. We shall use the first term, reserving the second one to designate a different concept (see Chap. 4).

2. On the basis of Eq. (A. 10), the integrand in Eq. (1.92) can be written in the form

$$-\mathbf{n} \operatorname{grad}_a \left(\frac{1}{R} \right) dS = \frac{\mathbf{n}\mathbf{R}}{R^3} dS = \frac{1}{R^2} \cos(\mathbf{R}, \mathbf{n}) dS$$

On the other hand, according to Eq. (1.7), we have

$$\frac{1}{R^2} \cos(\mathbf{R}, \mathbf{n}) dS = \pm d\Omega$$

where $d\Omega$ is the solid angle at which the element dS of the double layer is subtended from point P . As regards the sign, in the given case unlike Sec. 1.3, we consider \mathbf{R} to be directed from the element dS toward the point P . Consequently, $\cos(\mathbf{R}, \mathbf{n})$ will be positive if

* An electrical double layer can obviously be considered as a complex of dipoles having the length l and parallel to the normal \mathbf{n} , the charges of the dipoles being arranged along the surface of the layer with the density σ [cf. Eq. (1.50)].

from the point P we see the positive side of the element dS of the double layer, and negative in the opposite case (because according to our conditions, \mathbf{n} is directed from the negative to the positive surface of the layer). Let us agree to consider the solid angle $d\Omega$ positive if we see the positive side of the element dS from P , and negative in the opposite case. Therefore, Eq. (1.92) can be written in a simpler form:

$$\varphi = \int_{S_1} \tau d\Omega \quad (1.93)$$

3. If the strength of the layer τ is constant along its entire length—such a layer is called *homogeneous*—then its potential φ acquires the form

$$\varphi = \tau \int_{S_1} d\Omega = \tau\Omega \quad (1.94)$$

where Ω stands for the *algebraic* sum of the solid angles at which the elements of the surface of the double layer are subtended from point P . If all these surface elements are seen from P from the same, for instance positive, side, then the absolute value of Ω obviously equals the solid angle at which the entire double layer is subtended from P or, which is the same, at which the *contour* of this layer is subtended from P . If the entire layer does not comply with this condition, it can always be decomposed into several parts that do comply with it. Hence the import of Eq. (1.94) can be expressed as follows: *the potential of a homogeneous double layer at point P equals the product of the strength of the layer τ and the solid angle Ω , taken with the appropriate sign, at which the contour of this layer is subtended from P .*

4. The potential of a closed double layer (for example a layer on the surface of a sphere) is expressed especially simply. As was shown in Sec. 1.3, any closed surface is subtended at an angle of $\pm 4\pi$ from any point inside this surface and at an angle of 0 from any external point (see, particularly, Fig. 3). Hence, the potential of a closed double layer equals zero in the entire external space and equals $\pm 4\pi\tau$ at all points enclosed by the layer. The sign of the potential depends on whether the positive or the negative side of the layer is facing inward. Thus, the field intensity of a closed layer equals zero (because $\text{grad } \varphi = 0$), the potential of the field, however, undergoes a jump of $4\pi\tau$ when passing through the surface of the layer.

It is significant that the potential of any *open* layer undergoes the same jump of $4\pi\tau$ when passing through its surface. To verify this, let us mentally add a second layer of the same strength to the layer being considered in order to close it. Upon an infinitely small displacement of the point of observation from one side of the initial layer to the other, the potential of the additional layer at a finite distance remains virtually constant, whereas the potential of the

closed layer as a whole (the initial layer plus the additional one) changes by $4\pi\tau$. Hence, this jump equals that of the potential of the *initial* layer.

Thus, the *potential* of any (either closed or open) *double layer* undergoes a jump of $4\pi\tau$ at its surface. This jump is evidently directed from the negative side of the layer to the positive one, i.e. the potential of the layer evidently grows when passing through the layer in the direction of the positive normal \mathbf{n} . In other words, a double layer is a *surface of discontinuity of the potential*, so that if φ_1 is the value of the potential at the negative side of a layer and φ_2 its value at the positive side, then

$$\varphi_2 - \varphi_1 = 4\pi\tau \quad (1.95)$$

5. Strictly speaking, we can talk about a jump of the potential on a surface of discontinuity only with respect to infinitely thin double layers; the thickness of real electrical layers, however, may be disregarded only at sufficiently great distances from them. But if the thickness of a layer is small in comparison with the required accuracy of measuring distances, it is often convenient to use the notion of an infinitely thin layer, notwithstanding the fact that, as indicated on page 77, the intensity of the field on surfaces of discontinuity of the potential becomes equal to infinity, i.e. loses its physical meaning. Using the results of the following section, we can show, in addition (which we shall let our reader do), that the electrical energy of a double layer of a finite thickness in the limit, when its thickness is infinitely small, also tends to infinity.

It must be noted, however, that we shall not use the notion of an *electrical* double layer in the following and, accordingly, *we shall always assume that the continuity of the electrical potential φ is not violated anywhere*. We have only treated the theory of the electrical double layer here in order to prepare for studying *magnetic* double layers in Chapter 4.

It must be further noted that on each of the charged surfaces combining to form a double layer, the derivative $\partial\varphi/\partial n$, in accordance with Eq. (1.83), undergoes a jump of $\pm 4\pi\sigma$. These jumps are equal in magnitude and opposite in sign, however, so that when passing from one side of a layer to the other the quantity $\partial\varphi/\partial n$, and also E_n , remain continuous.

6. The potential of a double layer can be directly determined from Eq. (1.80). If S_1 is the surface of discontinuity of the potential, then the first term of the right-hand side of this equation

$$\begin{aligned} & \frac{1}{4\pi} \int_{S_1} (\varphi_2 - \varphi_1) \frac{\partial}{\partial N} \left(\frac{1}{R} \right) dS = \\ & = \frac{1}{4\pi} \int_{S_1} (\varphi_2 - \varphi_1) \mathbf{n} \operatorname{grad}_q \left(\frac{1}{R} \right) dS \end{aligned}$$

does not vanish. Denoting, in accordance with Eq. (1.95), the potential jump through $4\pi\tau$ and taking into account that

$$\text{grad}_q \left(\frac{1}{R} \right) = - \text{grad}_a \left(\frac{1}{R} \right)$$

we directly arrive at Eq. (1.92).

7. In conclusion, we shall mention as an example a case when the use of the concept of an electrical double layer may be convenient.

When a current passes through an electrolyte under certain conditions (depending on the material of the electrode, the chemical nature of the electrolyte, etc.), the so-called *polarization of the electrodes* is observed (do not confuse this with polarization of dielectrics which we shall treat in the following chapter!): the intensity of the current flowing through the electrolyte with a constant potential difference applied from outside to the electrodes diminishes with time and may virtually drop to zero.

This phenomenon can be interpreted as follows. Let us assume that the ions, which carry the current in an electrolyte, for example the anions (negative ions), upon approaching the positive electrode attracting them do not give up their charge to it, as usually happens, but only arrange themselves in a layer at the surface of this electrode. A layer of positive charges on the surface of the positive electrode will oppose this layer of charged negative ions (Fig. 21). Thus, an electrical double layer will be formed at this surface whose charge

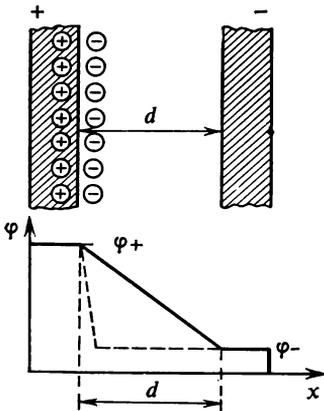


Fig. 21

and thickness will grow until the potential jump $\varphi_2 - \varphi_1 = 4\pi\tau$ in this layer becomes equal to the potential difference applied from outside. The current will stop flowing through the solution because the entire change in the potential will be concentrated only in the thin double layer at the positive electrode. Throughout the entire remaining part of the solution, the potential will take on a constant

value, and the field intensity \mathbf{E} will vanish (electrostatic equilibrium). The graph in the lower portion of Fig. 21 shows the values of the potential at different distances from the electrodes before the formation of the double layer (the ordinates of the solid curve) and after its formation (dashed curve).

Thus, the formation of a double layer may be the cause of polarization of electrodes. This phenomenon, however, may also be due to a number of causes of another kind.

1.15. Energy of Interaction of Electric Charges

1. Upon the movement of electric charges, the forces of Coulomb interaction between them perform the definite work W . It is obvious that we must ascribe a definite energy of interaction to any system of charges. The work W is performed at the expense of the reduction in this energy:

$$W = -dU \quad (1.96)$$

We shall often call the energy of interaction of charges U simply the electrical energy.

2. Let us use Eq. (1.96) to calculate first of all the energy of two point charges q_1 and q_2 at a distance of R_{12} from each other. Any change in the distance between the charges is attended by the work of electric forces. Let us assume, for instance, that the charge q_2 remains stationary whereas the charge q_1 moves in the field of the charge q_2 from point P_1 to point P'_1 . If $\varphi_1 = q_2/R_{12}$ is the value of the potential of the field of the charge q_2 at point P_1 , and $\varphi_1 + d\varphi_1$ is its value at point P'_1 , then the work W of the electric forces during this motion will be

$$W = -q_1 d\varphi_1$$

whence

$$W = -dU = -q_1 d\varphi_1$$

and, consequently,

$$U = q_1\varphi_1 = \frac{q_1q_2}{R_{12}} \quad (1.97)$$

Since only changes in energy can be observed and not its absolute value, we have omitted the additive integration constant here for purposes of simplicity because it does not depend on the mutual arrangement of the charges (compare with what is said about the intrinsic energy of the charges at the end of the next section). In this connection, the only variable part of the energy U that we take into consideration can have negative values (if the charges q_1 and q_2 have opposite signs).

We would naturally arrive at the same expression for U by considering the motion of the charge q_2 in the field of the stationary charge q_1 or, finally, the simultaneous motion of both charges.

Letting φ_2 stand for the potential of the charge q_1 at the point where the charge q_2 is ($\varphi_2 = q_1/R_{12}$), we can write the following equation instead of Eq. (1.97):

$$U = \frac{q_1 q_2}{R_{12}} = q_2 \varphi_2$$

It is more convenient to write the mutual electrical energy of the charges q_1 and q_2 in the symmetrical form:

$$U = \frac{1}{2} (q_1 \varphi_1 + q_2 \varphi_2) \quad (1.98)$$

To determine the energy of a system of n point charges q_i ($i = 1, 2, \dots, n$), we must evidently write equations similar to (1.97) or (1.98) for each pair of these charges and summate all the expressions obtained. After next collecting all the terms of the sum containing q_k as a multiplier, we shall see that the coefficient of q_k , which we shall denote by $\varphi_k/2$, will equal

$$\begin{aligned} \frac{1}{2} \varphi_k &= \frac{1}{2} (\varphi_{k,1} + \varphi_{k,2} + \dots + \varphi_{k,k-1} + \\ &+ \varphi_{k,k+1} + \dots + \varphi_{k,n}) \end{aligned}$$

where $\varphi_{k,i}$ is the potential of the charge q_i at the point where the charge q_k is.

The expression in parentheses is the value of the potential of the field of the entire system of charges at the point where the charge q_k is, or, more exactly, the potential of the entire system of charges except for the charge q_k itself (the potential φ_{kk} of the charge q_k at the point of the field where it is itself is not contained in the expression for φ_k and in general is deprived of a physical meaning because it becomes equal to infinity). Thus, the mutual energy of a system of n charges is

$$U = \frac{1}{2} \sum_{k=1}^{k=n} q_k \varphi_k \quad (1.99)$$

where φ_k is the potential of the field at the point where the charge q_k is.

To determine the dependence of U on the mutual distance between the charges, we shall use Eq. (1.44), which when employing our present notation will be written as follows:

$$\varphi_k = \sum_{i=1}^n \frac{q_i}{R_{ki}} \quad (i \neq k)$$

where summation should be extended over all the subscripts i except for $i = k$. Introducing this condition in Eq. (1.99), we get

$$U = \frac{1}{2} \sum_{i,k} \frac{q_k q_i}{R_{ki}} \quad (i \neq k) \quad (1.100)$$

This formula can also be obtained directly from Eq. (1.97) for the mutual energy of a pair of charges. The appearance of the factor $\frac{1}{2}$ before the summation sign is explained by the fact that the energy of each pair of charges is included in this sum twice; for example it contains both the term $q_1 q_2 / R_{12}$ and the term $q_2 q_1 / R_{21}$ equal to it. 3. As always, when using the notation of point charges, we must remember that the above equations may be employed only when the charges of a system are at distances from one another that are sufficiently great in comparison with the dimensions of the charges themselves. To eliminate this restriction, let us pass over to a consideration of space and surface charges. Decomposing a system of these charges into a complex of elementary charges ρdV and σdS , applying Eq. (1.99) to the latter, and passing over from summation to integration, we obtain

$$U = \frac{1}{2} \int \rho \varphi dV + \frac{1}{2} \int \sigma \varphi dS \quad (1.101)$$

where φ is the value of the potential of the field of all the space and surface charges in the volume element dV or on the surface area element dS .

Although Eq. (1.101) may seem to be only an altered form of Eq. (1.99) corresponding to the substitution of the notion of space and surface charges for that of point charges, these equations, however, are actually different in their *content*. Namely, we shall show in the following section that Eq. (1.101) expresses the total energy of a system of electric charges, whereas Eq. (1.99) does not take into account the so-called intrinsic energy of the charges q_k .

Example 1. *Energy of a point charge and a dipole in an external electric field.* We often have to do with the work of electric forces upon the movements of a charge q in a given "external" field of other charges remaining stationary. The mutual energy of these "external" charges (and also their intrinsic energy and that of the charge q , see Sec. 1.16) remains constant. The variable part of the energy of the field at whose expense the work of the electric forces is performed is called "the energy of the charge q in the external field". It obviously equals

$$U = q\varphi \quad (1.102)$$

where φ is the potential of the external field at the point where the charge q is. Equation (1.97) is a particular case of Eq. (1.102). If an external field contains two charges, $q > 0$ and $q' = -q$, forming a dipole having an infinitely small length l (p. 55), then the energy of these charges in the external field is

$$U = q\varphi + q'\varphi' = q(\varphi - \varphi')$$

where φ and φ' are the potentials of the external field at the poles of the dipole. But within values of the second order of smallness, we have

$$\varphi = \varphi' + \frac{\partial \varphi}{\partial l} l = \varphi' + \mathbf{l} \text{ grad } \varphi = \varphi' - \mathbf{IE}$$

Hence,

$$U = -q \mathbf{IE} = -\mathbf{pE} \quad (1.103)$$

where \mathbf{p} = dipole moment [see Eq. (1.48)]

\mathbf{E} = intensity of the external field at the place where the dipole is.

This expression, naturally, does not take into account the mutual energy of the dipole charges whose magnitude changes only when the length l of the dipole changes.

Example 2. *The direct calculation of the electrical energy of a charged capacitor.* Before a capacitor is charged, each of its plates is electrically neutral, i.e. contains equal quantities of positive and negative electricity. We shall charge a capacitor by removing electricity of a definite sign from one of its plates and transferring it to the other one. This is accomplished in practice by connecting the capacitor plates with a conductor and including an e.m.f., for example a galvanic cell, in the circuit. This cell pumps electricity from one plate to the other until the potential difference across these plates reaches a known value. Assume that at a certain moment of this process the potentials of the plates reach the values φ_1 and φ_2 , and $\varphi_2 > \varphi_1$. The transfer of the following portion of electricity dq from the first plate to the second one is attended by *negative* work of the forces of the electric field equal to $dW = dq(\varphi_1 - \varphi_2)$ (see Sec. 1.8). It is obvious that the e.m.f. of the cell overcoming the forces of the capacitor field, which is "external" with respect to this field, performs the positive work:

$$dW_e = -dW = dq(\varphi_2 - \varphi_1)$$

Using Eq. (1.52), we get

$$dW_e = \frac{q dq}{C}$$

where C is the capacitance of the capacitor. The total work done to increase the charge of the plates from zero to, let us say, q' will be

$$W_e = \int_{q=0}^{q=q'} \frac{q dq}{C} = \frac{q'^2}{2C}$$

This work is done at the expense of a reduction in the chemical energy of the galvanic cell and transforms into the energy of the electric field of the charged capacitor. Denoting this latter energy by U and taking Eq. (1.52) into consideration, we get the following expressions for U :

$$U = W_e = \frac{q^2}{2C} = \frac{1}{2} C(\varphi_2 - \varphi_1)^2 = \frac{q(\varphi_2 - \varphi_1)}{2} \quad (1.104)$$

where we have omitted the prime of q .

1.16 Energy of an Electric Field

1. Equation (1.101) of electrical energy can be written in a different mathematical form, and this transformation reveals possibilities for an absolutely new *physical interpretation* of the relationships it expresses.

To prepare this transformation, let us assume in Green's theorem (A. 52) that $\psi = \varphi$. Taking into account that

$$\nabla\varphi = \text{grad } \varphi = -\mathbf{E}$$

and that

$$\nabla^2\varphi = -4\pi\rho$$

we get

$$\int_V (-\varphi \cdot 4\pi\rho + E^2) dV = \int_{S+S'_1} \varphi \frac{\partial\varphi}{\partial n} dS$$

where the surface integral should be considered to cover, first, the surface S confining from outside the volume of integration V , and, second, the surfaces S'_1 which separate from this volume the charged surfaces S_1 that may be in it, i.e. the surfaces of discontinuity of the gradient of φ (cf. Sec. 1.12). As regards the potential φ , we shall consider it to be continuous everywhere, i.e. we shall not consider electrical double layers.

As in deriving Eq. (1.80), we shall contract the surfaces S'_1 until they coincide with the discontinuity surfaces S_1 . Repeating our previous reasoning and using the notation of Sec. 1.12, we obtain

assuming that $\varphi_1 = \varphi_2 = \varphi$:

$$\begin{aligned} \lim_{S'_1 \rightarrow S_1} \oint_{S'_1} \varphi \frac{\partial \varphi}{\partial n} dS &= \int_{S_1} \left\{ \left(\varphi \frac{\partial \varphi}{\partial n} \right)_1 + \left(\varphi \frac{\partial \varphi}{\partial n} \right)_2 \right\} dS = \\ &= - \int_{S_1} \varphi \left\{ \left(\frac{\partial \varphi}{\partial N} \right)_2 - \left(\frac{\partial \varphi}{\partial N} \right)_1 \right\} dS \end{aligned} \quad (1.105)$$

or in view of Eq. (1.81)

$$\lim_{S'_1 \rightarrow S_1} \oint_{S'_1} \varphi \frac{\partial \varphi}{\partial n} dS = 4\pi \int_{S_1} \varphi \sigma dS \quad (1.106)$$

Using Eq. (1.106) in (1.105), dividing both sides by 8π and switching over terms, we get

$$\begin{aligned} \frac{1}{8\pi} \int_V E^2 dV &= \frac{1}{2} \int_V \rho \varphi dV + \\ &+ \frac{1}{2} \int_{S_1} \sigma \varphi dS + \frac{1}{8\pi} \oint_{S_1} \varphi \frac{\partial \varphi}{\partial n} dS \end{aligned} \quad (1.107)$$

The first two terms of the right-hand side of this equation are similar to Eq. (1.101) for the energy U , but in the given case integration is performed not over all the charges in the field, but only over those of them that are inside the volume V . The sum of these terms does not coincide with the mutual energy of the charges inside of V because the value of the potential also depends on the arrangement of the charges outside of V .

2. We shall assume, however, that integration is performed over the *entire field*. This means that the region of integration of V covers, first, all the interacting charges, and, second, the entire field of these charges.

This definition requires no explanations if there is a closed surface S of finite dimensions confining the entire system of interacting charges at all of whose points the field intensity E vanishes. This surface can be considered as the boundary of the total field. For example, the boundary of the total field of the charges confined in a metal shell (Fig. 14) is inside this shell.

In the majority of cases, however, no such closed surface of finite dimensions exists, and the boundary of the total field is spread to infinity. In these cases, the concept of a total field should be defined more precisely as follows. In each concrete case of using this concept, we are interested in the values of integrals of quite definite physical quantities [for example the electric or magnetic field intensity, the product $\varphi \frac{\partial \varphi}{\partial n}$ in Eq. (1.107), etc.]. The volume integrals are taken

over the volume V of the total field, and the surface integrals over its boundary S . The term *total field* is applied to an *infinite volume* V only if upon a limit transition from a finite volume V to an infinitely great one, *the integrals of all the quantities of interest to us over the surface S of this volume tend to zero*. Since upon this limiting transition the area of the surface S grows in proportion to R^2 , where R is the distance from the surface S to a finite section of space in which the sources of the field are concentrated (the electric charges and currents), the integrands in the surface integrals we are interested in should diminish at a faster rate than $1/R^2$ when $R \rightarrow \infty$. This is exactly the condition for applying the concept of a total field (if the latter is not confined by a closed surface having finite dimensions).

In the following, we shall simply say for brevity's sake that *according to the definition of the total field, the integrals over the surface S confining the total field become equal to zero*, i.e. may be eliminated, and that, consequently, only the integrals over the *volume* of the total field are to be considered.

3. The integrand in the last term of Eq. (1.107), in accordance with condition (1.85), diminishes at infinity not slower than $1/R^3$. Therefore, when extending integration in Eq. (1.107) over the total field, this term vanishes, and the sum of the first two terms, according to Eq. (1.101), will be equal to the total energy of the field U :

$$U = \frac{1}{2} \int \rho \varphi \, dV + \int \sigma \varphi \, dS = \frac{1}{8\pi} \int E^2 \, dV$$

4. Thus, the electrical energy of a total field is

$$U = \frac{1}{8\pi} \int E^2 \, dV \quad (1.108)$$

From the *mathematical* viewpoint, Eq. (1.108) is only a transformed form of Eq. (1.101) and is absolutely equivalent to it. As indicated above, however, a formal mathematical transformation of equations very often reveals the possibility of an absolutely new physical *interpretation* of the relationships they express. Equation (1.108) expresses electrical energy in the form of an infinite sum of addends each of which equals $(1/8\pi)E^2 \, dV$ and relates to a definite element of the volume dV . Therefore, the equation may be assumed to have the following physical meaning: electrical energy is carried by an electric field, and the energy of the field is *localized* in space so that each unit of volume contains an amount of energy u equal to

$$u = \frac{1}{8\pi} E^2 \quad (1.109)$$

where E is the intensity of the electric field in the given element of the volume. The quantity u can be called the *volume density of electrical energy*.

Conversely, Eqs. (1.99) and (1.100) can be formally interpreted to have the meaning that electrical energy is the energy of *interaction* of electric charges, and this is *action at a distance* (long-range interaction). For example, Eq. (1.100) expresses the total energy of a system of charges in the form of the sum of the energies of interaction of each pair of them. It is quite obvious that with such an interpretation the possibility of localizing the energy in definite sections of space is eliminated.

The mechanical theory of electromagnetic phenomena considered Eqs. (1.108) and (1.109) to have the following physical content. From the viewpoint of this theory, the induction of an electric field consists in the appearance of deformations of a hypothetic elastic medium—ether. The electric vector \mathbf{E} is a measure of this deformation, while the energy of an electric field is the elastic energy of the deformed ether. It is known from the theory of elasticity that each element of volume of a deformed body contains a definite amount of elastic energy proportional to the square of the deformation of the element. Hence, the volume density of the elastic energy of the ether in an electric field should be proportional to the square of the field intensity \mathbf{E} , which agrees quite well with Eq. (1.109).

It can be considered as an established fact at present that such a mechanical interpretation of electrical phenomena does not withstand the criticism of experimental data*. The idea itself of the localization of electrical energy in space with a volume density of u , in other words the notion that electrical energy is the energy of an electric field has become an established property of science. It is quite natural that owing to the complete mathematical equivalence of Eqs. (1.101) and (1.108) both of these equations and, consequently, both of their above interpretations well agree with experimental data. The equivalence of these equations, however, relates only to a *constant* electric field. When passing over to studying varying electromagnetic fields and particularly to studying electromagnetic waves, we shall acquaint ourselves with phenomena that can be interpreted only on the basis of the assumption on the localization of energy in an electromagnetic field.

5. Let us now turn to the question of the different content of Eqs. (1.99) and (1.100), on one hand, and of Eqs. (1.101) and (1.108) on the other. That these equations differ in their content follows if only from the circumstance that the energy U determined by Eq. (1.108) cannot take on negative values (because $E^2 > 0$), whereas according to Eq. (1.100) the energy of interaction of two point charges q_1q_2/R_{12} is negative if these charges are of opposite signs. The explanation is that Eqs. (1.99) and (1.100) take into account only the

* Moreover, not only do electric forces not consist in the elasticity of a hypothetic medium, but, as we now know, the elastic properties themselves of material bodies are explained by the electrical interaction of the atoms of these bodies.

interaction of a number of "point" charges, but not the *interaction of separate elements* of each such charge *between one another*. Indeed, if we have to do, for example, only with a single "point" charge q_1 , then expressions (1.99) and (1.100) will become equal to zero, whereas Eqs. (1.101) and (1.108) will have a value that differs from zero and is positive, equal to the so-called *intrinsic energy* of the charge q_1 . We can see that this intrinsic energy is always positive either directly from Eq. (1.108), or from the fact that by ascribing a certain volume to the charge q , dividing it into elements dq_i and calculating the energy of interaction of these elements, we obtain the sum of *positive* expressions of the kind $dq_i dq_k / R_{ik}$ (because all the elements of the same charge have the same sign). The intrinsic energy of a charge naturally depends on its dimensions and equals the work that the forces of mutual repulsion between the elements of the charge would do if these elements fled off in different directions to infinity.

Let us consider in conclusion the total (i.e. intrinsic and mutual) energy of two charges q_1 and q_2 . Assume that each of these charges separately induces the field \mathbf{E}_1 and \mathbf{E}_2 , respectively, so that the resultant field \mathbf{E} of the two charges equals

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$$

and

$$\mathbf{E}^2 = (\mathbf{E}_1 + \mathbf{E}_2)^2 = E_1^2 + E_2^2 + 2(\mathbf{E}_1 \mathbf{E}_2)$$

The total energy of the charges q_1 and q_2 , according to Eq. (1.108), will be

$$U = \frac{1}{8\pi} \int E^2 dV = \frac{1}{8\pi} \int E_1^2 dV + \\ + \frac{1}{8\pi} \int E_2^2 dV + \frac{1}{8\pi} \int 2(\mathbf{E}_1 \mathbf{E}_2) dV$$

or

$$U = U_{11} + U_{22} + U_{12} \tag{1.110}$$

where

$$U_{11} = \frac{1}{8\pi} \int E_1^2 dV \quad \text{and} \quad U_{22} = \frac{1}{8\pi} \int E_2^2 dV \tag{1.111}$$

are the intrinsic energies of the charges q_1 and q_2 , and

$$U_{12} = \frac{1}{8\pi} \int 2(\mathbf{E}_1 \mathbf{E}_2) dV \tag{1.112}$$

is their mutual energy. From

$$(\mathbf{E}_1 - \mathbf{E}_2)^2 \geq 0$$

it follows that

$$E_1^2 + E_2^2 \geq 2(\mathbf{E}_1 \mathbf{E}_2)$$

so that

$$U_{11} + U_{22} \geq U_{12}$$

Thus, the positive intrinsic energy of charges is always greater than (or at least equal to) their mutual energy, which may have either positive or negative values.

The correctness of this statement also follows directly from the circumstance that the intrinsic energy of a charge corresponds to the interaction between its own elements, the average distance between which is smaller than their distance from the elements of another charge.

The intrinsic energy of a charge remains constant upon any possible motion of the charge that does not change its shape or dimensions. Therefore, upon such motion, the terms U_{11} and U_{22} can be considered as additive constants in the expression for the total energy U whose changes are entirely determined by a change in the mutual energy of the charges U_{12} . With sufficiently great distances between the charges, Eq. (1.112) for U_{12} will evidently become the same as Eq. (1.97) (see Problem 15 at the end of the section).

6. It is extremely important to note that the *energy* of an electric field *does not have the property of additivity*, i.e. that the energy of the field \mathbf{E} , which is the sum of two fields \mathbf{E}_1 and \mathbf{E}_2 , generally speaking, does not equal the sum of the energies of the components of the fields (if only U_{12} differs from zero). Particularly, when the field intensity grows n times, the energy of the field grows n^2 times.

Example. *The total electrical energy of a system of charged conductors.* Let a field contain n conductors. We shall denote by S_i , φ_i , and q_i the surface area, potential, and total charge of the i -th conductor. Taking into consideration that all the charges of the conductors are on their surface so that $\rho = 0$, and that the potential of each conductor is constant along its entire length, we get from Eq. (1.101)

$$U = \frac{1}{2} \sum_{i=1}^n \oint_{S_i} \sigma \varphi \, dS = \sum_{i=1}^n \varphi_i \oint_{S_i} \sigma \, dS$$

The integral of σ over the surface of a conductor equals its total charge q_i , hence

$$U = \frac{1}{2} \sum_{i=1}^n e_i \varphi_i \quad (1.113)$$

This equation, expressing the *total energy of charged conductors*, should not be confused with the quite similar equation (1.99) expressing the *mutual energy of point charges*. Unlike Eq. (1.113), φ_i in Eq. (1.99) is not the total potential of the field at the place where the charge q_i is (see Sec. 1.15). It should be noted that Eq. (1.104) for the energy of a capacitor is a particular case of Eq. (1.113)*.

Problem 14. Show that the (intrinsic) electrical energy of a charged sphere having the radius a is $U = q^2/2a$ if the charge q is distributed over the surface of the sphere (a conductor) and is $U = 3q^2/5a$ if the charge is uniformly distributed over the entire volume of the sphere.

Problem 15. Show that if the distance between the charges q_1 and q_2 is sufficiently great in comparison with their dimensions (point charges), then Eq. (1.112) for the mutual energy of these charges becomes identical with Eqs. (1.97) and (1.98).

1.17 Ponderomotive Forces

1. According to the definition given in Sec. 1.2, the basic quantity characterizing an electric field—its intensity \mathbf{E} —equals the ponderomotive (mechanical) force acting at a given point of the field on a test charge (charged body) calculated for a unit charge. It must be remembered here that the force acting on a charge is determined by the intensity of the field into which the charge is placed, and not of the field which the charge itself induces. Consequently, when saying, for instance, that the force acting on a point charge q equals

$$\mathbf{F} = q\mathbf{E} \quad (1.114)$$

[cf. Eq. (1.5)], we must understand by \mathbf{E} the intensity of the field induced by all the charges of the system *except* the charge q itself.

2. The problem of the force acting on *surface charges* requires special investigation because the field intensity \mathbf{E} has different values at the two sides of a charged surface [Eq. (1.14)], and, consequently, the value of the vector \mathbf{E} on the surface itself remains indefinite.

For a single isolated conductor, all the electric forces consist in the mutual repulsion of the elements of the charge of the conductor. Since the charge elements mutually repulsing one another cannot escape from the conductor, ponderomotive forces will be applied to

* We invite our reader to convince himself, as an exercise, taking the concrete examples of a plane, cylindrical, and spherical capacitors, that the values of the energy obtained by Eq. (1.104) on the one hand and by Eq. (1.108) on the other are identical, using direct calculation.

its surface that tend to extend it*. Similar forces will evidently also be applied to the surface of a non-isolated conductor placed in an arbitrary electrostatic field. To determine the magnitude of these forces, let us consider a certain element dS of the surface area of the conductor. The intensity of the field at the *outer side* of the element dS , according to Eq. (1.38), equals

$$\mathbf{E} = 4\pi\sigma\mathbf{n}$$

(σ is the density of its charge) and is directed normally to its surface; inside the conductor $\mathbf{E} = 0$.

But \mathbf{E} is the intensity of the resultant field of all the charges present including that of the element dS itself:

$$\mathbf{E} = \mathbf{E}' + \mathbf{E}''$$

where \mathbf{E}' = field of the element dS

\mathbf{E}'' = field of the remaining charges.

At two adjacent points at different sides of the element dS , the field \mathbf{E}'' of these charges will obviously be the same, while the field \mathbf{E}' will have the same magnitude E' , but an opposite direction (Fig. 22). Hence, at the outer side of the element dS , we have $E = E' + E'' = 4\pi\sigma$, and at the inner side we have $E = E' - E'' = 0$, whence

$$E' = E'' = 2\pi\sigma$$

The force which the charge σdS of the element dS is subjected to is determined by the field \mathbf{E}'' of the charges outside of this element and, therefore, is equal to

$$E''\sigma dS = 2\pi\sigma^2 dS = \frac{1}{2} E\sigma dS$$

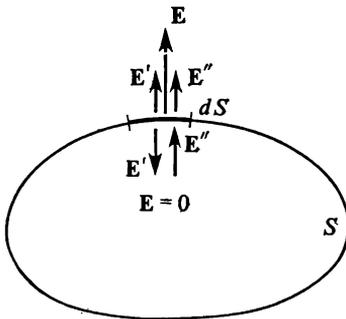


Fig. 22

* If we impart an electric charge to a soap bubble, then under the influence of these forces of repulsion the bubble will expand until they are balanced by the forces of surface tension and by the difference between the pressure of the air inside and outside of the bubble.

Thus, the ponderomotive force acting on a unit surface area of a conductor is

$$f = 2\pi\sigma^2 = \frac{1}{2} E\sigma = \frac{1}{8\pi} E^2 \quad (1.115)$$

This force is directed, as can easily be seen, along an outward normal to the surface. The quantity f can be called the *surface density of the ponderomotive forces*.

3. Let us also consider for completeness the volume distribution of a charge having the density ρ , although this case is virtually unrealizable in an electrostatic field in the absence of dielectrics*. If electric charges are inseparably connected with the elements dV of the volume of a body so that motion of the charge ρdV is possible only upon the corresponding motion of an element dV of the body and vice versa, then each element of this body will be acted upon by the force

$$\mathbf{F} = \mathbf{f} dV = \mathbf{E}\rho dV \quad (1.116)$$

where \mathbf{E} is the intensity of the field in the element dV . In this case, it is not necessary to subtract the field of the charge of the element dV itself from the total field \mathbf{E} because the intensity of the field of an infinitely small volume charge ρdV is infinitely small even inside the charge itself and tends to zero with an infinite reduction of its dimensions. It is the simplest to verify this by assuming that the element dV has the form of a sphere so that its field is determined by Eq. (1.19).

It follows from Eq. (1.116) that the *volume density* of the ponderomotive forces equals**

$$\mathbf{f} = \rho\mathbf{E} \quad (1.117)$$

Example. *Forces acting on a dipole.*

Let \mathbf{E} and \mathbf{E}' be the intensities of the external (with respect to the dipole) field at the points P and P' where its negative and positive charges are (Fig. 23). Hence, the resultant \mathbf{F} of the forces of the

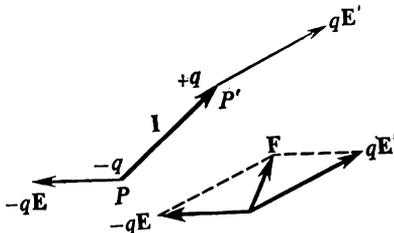


Fig. 23

* Except for non-homogeneous conductors in which extraneous e.m.f.s are acting (Sec. 3.4). We shall not stop to consider this case.

** In the following, we shall always denote the density of forces by the lowercase \mathbf{f} , retaining the capital \mathbf{F} for designating the forces acting on a separate charge, dipole, conductor, etc.

field applied to these charges will be

$$\mathbf{F} = q\mathbf{E}' - q\mathbf{E} = q(\mathbf{E}' - \mathbf{E})$$

The difference $\mathbf{E}' - \mathbf{E}$ is the increment of the vector \mathbf{E} on segment PP' equal to the length of the dipole \mathbf{l} . Owing to the smallness of this segment, the increment can be expressed, according to Eq. (A.51), by the formula

$$\mathbf{E}' - \mathbf{E} = l \frac{\partial \mathbf{E}}{\partial l} = \mathbf{l} \nabla \cdot \mathbf{E}$$

whence

$$\mathbf{F} = q\mathbf{l} \nabla \cdot \mathbf{E} = \mathbf{p} \nabla \cdot \mathbf{E} \quad (1.118)$$

Thus, the magnitude of the force \mathbf{F} acting in an electric field on a dipole depends on the rate of the change in this field in the direction of the dipole axis; in a homogeneous field the forces acting on the poles of a dipole are equal in magnitude and opposite in direction and, consequently, mutually balance each other. Apart from the resultant \mathbf{F} , for the complete determination of the forces acting on a dipole, we must determine the moment \mathbf{N} of these forces relative to the centre of the dipole. Their moment relative to the point P at which the charge $-q$ is obviously equals

$$\mathbf{N} = [\mathbf{l} \cdot q\mathbf{E}'] = [\mathbf{p}\mathbf{E}']$$

In the limit with a sufficient smallness of \mathbf{l} , the point P coincides with the midpoint of the dipole, and \mathbf{E}' coincides with \mathbf{E} , so that finally

$$\mathbf{N} = [\mathbf{p}\mathbf{E}] \quad (1.119)$$

It follows from this expression that a dipole tends to turn in an electric field so that its moment \mathbf{p} will be parallel to the field. When \mathbf{p} is directed opposite to \mathbf{E} , the torque also equals zero, but this equilibrium is unstable.

1.18 Determining the Ponderomotive Forces from the Expression for Energy

1. The motion of bodies in an electric field is attended, generally speaking, both by a change in the energy U of this field and by the work W of the ponderomotive forces of the field. If no transformations of other kinds of energy (thermal, chemical, etc.) occur, the work of the ponderomotive forces W should be done at the expense of the change δU in the energy of the field, so that

$$W = -\delta U \quad (1.120)$$

On the basis of this relationship, in Sec. 1.15 we determined the energy of a field U (1.96); conversely, knowing U , on the basis of Eq. (1.120) we can determine the work W and, consequently, the magnitude of the ponderomotive forces doing this work.

Indeed, Eq. (1.120) shows that the energy of an electric field U plays the part of the *potential* energy in the meaning it has in analytical mechanics whose methods we can take advantage of. Assume that the energy U in the most general case is expressed depending on some "generalized coordinates" j_i characterizing the distribution of charges, conductors, etc.

Hence,

$$\delta U = \sum_i \frac{\partial U}{\partial j_i} \delta j_i \quad \text{and} \quad W = \sum_i \Theta_i \delta j_i \quad (1.121)$$

where Θ_i , according to the terminology of analytical mechanics, are the "generalized forces" acting "in the direction" of the coordinates j_i . Hence, on the basis of Eq. (1.120), provided that the coordinates j_i are independent, it follows that

$$\Theta_i = - \frac{\partial U}{\partial j_i} \quad (1.122)$$

The determination of the ponderomotive forces on the basis of this formula is very often incomparably simpler than their direct determination by the formulas of Sec. 1.17 by integration with respect to the separate elements of the charges.

2. Let us determine the forces of mutual attraction of the plates of a parallel-plate capacitor in this way as a very simple example. The energy of this capacitor equals [see Eq. (1.104) and the results of solving Problem 8 on p. 59]

$$U = \frac{q^2}{2C} = \frac{2\pi q^2 d}{S}$$

When the plates are moved apart, i.e. the distance d between them is increased, the work

$$W = -\delta U = - \frac{2\pi q^2}{S} \delta d$$

is performed.

On the other hand, if f is the force of attraction acting on a unit area of the surface of a capacitor plate, and F is the total force acting on the entire surface S of a plate, then

$$W = -F \delta d = -fS \delta d$$

Equating the expressions obtained, we get

$$F = fS = \frac{2\pi q^2}{S} = 2\pi\sigma^2 S$$

where $\sigma = q/S$ is the density of the charge on the surface of the plates, which agrees quite well with Eq. (1.115).

The same result can be arrived at directly from Eq. (1.122), assuming in this equation that $j = d$ and $\Theta = F$.

3. In the preceding paragraph, we assumed unexplicitly that upon moving of the capacitor plates their charge q remains constant. We would obtain an absolutely different value of the quantity δU if we assumed that not the charge but the potential difference of the capacitor plates $\varphi_2 - \varphi_1$ remains constant when the plates are displaced. The explanation is that the potential difference of the capacitor plates can remain unchanged when the plates are moved (changes in the capacitance) *only* when this potential difference is kept constant by some *extraneous* e.m.f.'s (of a non-electrostatic origin) (see Sec. 3.4) that do a certain additional work when the capacitance C changes. (This occurs, for example, when the plates of a capacitor are connected to the poles of a galvanic cell.) Here Eq. (1.120) can obviously no longer be applied.

For a capacitor of any shape, Eqs. (1.104) and (1.52) yield:
with $q = \text{const}$

$$\delta_q U = \delta \left(\frac{q^2}{2C} \right) = - \frac{q^2}{2C^2} \delta C = - \frac{1}{2} (\varphi_2 - \varphi_1)^2 \delta C$$

with $\varphi_2 - \varphi_1 = \text{const}$

$$\delta_\varphi U = \delta \left\{ \frac{1}{2} C (\varphi_2 - \varphi_1)^2 \right\} = \frac{1}{2} (\varphi_2 - \varphi_1)^2 \delta C$$

i.e.

$$\delta_\varphi U = -\delta_q U = W \quad (1.123)$$

Thus, the changes in the electrical energy of a capacitor with $q = \text{const}$ and with $\varphi_2 - \varphi_1 = \text{const}$ attending the moving of the capacitor plates are equal in value, but opposite in sign. On the other hand, owing to the relationship

$$q = (\varphi_2 - \varphi_1)C$$

the change in the capacitance C with a constant value of $\varphi_2 - \varphi_1$ should be attended by a change in the charge of the capacitor q :

$$\delta q = (\varphi_2 - \varphi_1) \delta C$$

i.e. by the transfer of the charge δq from one of the capacitor plates to the other. When this charge δq passes through the galvanic cell connected to the plates of the capacitor, the chemical energy of this cell, as we shall see in Chap. 3, will diminish, while the e.m.f. of the cell \mathcal{E} performs the work $A = \mathcal{E} \delta q$.

In an open circuit consisting of the cell and the capacitor

$$\mathcal{E} = \varphi_2 - \varphi_1$$

and, consequently,

$$A = (\varphi_2 - \varphi_1)\delta q = (\varphi_2 - \varphi_1)^2\delta C$$

or, in view of Eq. (1.123):

$$A = 2\delta_\varphi U = W + \delta_\varphi U \quad (1.124)$$

Thus, when $\varphi_2 - \varphi_1 = \text{const}$, the work W of the ponderomotive forces of the field is done not at the expense of the energy of the field U , but at the expense of the chemical energy of the galvanic cell (or the energy of another source of an extraneous e.m.f.). In particular, the *positive* work of the forces of the field W is attended by a *growth* in the electrical energy U also occurring at the expense of the energy of the galvanic cell.

Thus, Eq. (1.120) holds only provided that the movement of bodies in an electric field is not attended by the work of extraneous e.m.f.'s of a non-electrostatic origin. In addition, a condition for the applicability of Eq. (1.120) is obviously the *sufficiently slow velocity of movement of the bodies*, namely, such a low velocity that at each given moment of the process the electrical state of the system can be described by equations of electrostatics (i.e. differs by an infinitely small value from the equilibrium state).

Example. Determine the forces acting on a solid dipole on the basis of its energy in the external field. The energy of a dipole in an external field, according to Eq. (1.103), equals

$$U = -\mathbf{p}\mathbf{E} = -pE \cos \theta$$

and is a function (a) of the coordinates of the dipole centre x , y , and z , and (b) of the angle θ between the axis of the dipole and the given direction of the electric field \mathbf{E} (we consider the numerical value of the dipole moment to be constant: the dipole is "solid"). According to the well-known laws of analytical mechanics, the "generalized forces" corresponding to these generalized coordinates are:

(a) the resultant of the forces applied to the dipole:

$$\left. \begin{aligned} F_x &= - \frac{\partial U}{\partial x} \\ F_y &= - \frac{\partial U}{\partial y} \\ F_z &= - \frac{\partial U}{\partial z} \end{aligned} \right\} \quad (1.125)$$

and (b) the moment of the forces applied to the dipole:

$$N = -\frac{\partial U}{\partial \theta} \quad (1.126)$$

that tend to increase the angle θ .

It follows from Eqs. (1.103) and (1.125) that

$$\mathbf{F} = -\nabla U = \nabla(\mathbf{p}\mathbf{E}) \quad (1.127)$$

This expression for an *electrostatic* field, which we are restricting ourselves to here, differs from the previously derived expression (1.118) only in its form:

$$\mathbf{F} = \mathbf{p} \nabla \cdot \mathbf{E}$$

Indeed, for any constant (not depending on the coordinates x, y, z) vector \mathbf{p} , the following relationship holds:

$$\nabla(\mathbf{p}\mathbf{E}) = \mathbf{p} \nabla \cdot \mathbf{E} + [\mathbf{p} \text{ curl } \mathbf{E}] \quad (1.128)$$

To prove that this relationship is true, it is sufficient to consider the component of the vector $\nabla(\mathbf{p}\mathbf{E}) - \mathbf{p} \nabla \cdot \mathbf{E}$ at least along the x -axis:

$$\begin{aligned} & \frac{\partial}{\partial x} (\mathbf{p}\mathbf{E}) - \mathbf{p} \nabla \cdot \mathbf{E} = \\ & = p_x \frac{\partial E_x}{\partial x} + p_y \frac{\partial E_y}{\partial x} + p_z \frac{\partial E_z}{\partial x} - \\ & - \left(p_x \frac{\partial E_x}{\partial x} + p_y \frac{\partial E_x}{\partial y} + p_z \frac{\partial E_x}{\partial z} \right) = \\ & = p_y \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) + p_z \left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \right) = \\ & = p_y \text{curl}_z E - p_z \text{curl}_y E = [\mathbf{p} \text{ curl } \mathbf{E}]_x \end{aligned}$$

For the electrostatic field we are interested in, the vector $\mathbf{E} = -\text{grad } \phi$, and therefore its curl, like the curl of any gradient of a scalar, equals zero: $\text{curl } \mathbf{E} = 0$ [Eq. (1.36)].

Thus, for an electrostatic field, we do have

$$\nabla(\mathbf{p}\mathbf{E}) = \mathbf{p} \nabla \cdot \mathbf{E} \quad (1.129)$$

and Eq. (1.127) is equivalent to Eq. (1.118).

Reverting to the moment of the forces applied to a dipole, we get from Eqs. (1.103) and (1.126):

$$N = -\frac{\partial U}{\partial \theta} = -pE \sin \theta$$

Since $\theta \leq \pi$, the right-hand side of this expression is negative, and, consequently, according to the meaning of Eq. (1.126), the moment \mathbf{N} tends to diminish the angle θ between \mathbf{p} and \mathbf{E} (Fig. 24).

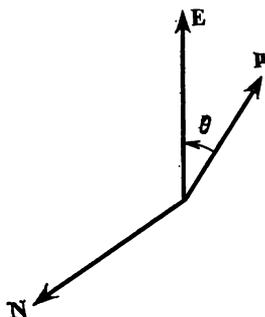


Fig. 24

Therefore, with respect to its magnitude and direction, the vector of the moment of the forces is determined by the formula

$$\mathbf{N} = [\mathbf{p}\mathbf{E}] \quad (1.130)$$

that coincides with Eq. (1.119) derived on an earlier page.

1.19. Instability of Electrical Systems. Constraints

1. The question of the possibility of *stable* configurations of electric charges is exceedingly important for the electron theory of the structure of matter. If matter consists of electric charges—electrons and protons, then the question appears whether a system of such charges can be in stable *static* equilibrium or whether these charges must be in a state of continuous *motion* in the atoms and molecules of all bodies.

For our purposes, it will be sufficient to prove the instability of a static system of *point* charges, although the corresponding laws can also be applied to space charges*.

As indicated in Sec. 1.18, the electrical energy U of a system of charges plays the part of the *potential* energy of this system. On the other hand, on the basis of the general laws of mechanics, a *minimum* of potential energy is the condition for *stable* equilibrium, in our case a minimum of the electrical energy U . The energy of a system

* To explain the stability of the electrons and protons themselves, it was postulated for a long time that there exist forces of a non-electrical origin preventing the flying apart of elements of these charges under the influence of mutual repulsion. The development of the quantum theory resulted in a new formulation of the problem of elementary particles in which the problem of the stability of these particles is treated in an absolutely different way.

of point charges, according to Eq. (1.100), is

$$U = \frac{1}{2} \sum_{i, k} \frac{q_i q_k}{R_{ik}} \quad (i \neq k)$$

and is a function of the coordinates x_i , y_i , and z_i of all the charges q_i of the system because

$$R_{ik} = \sqrt{(x_i - x_k)^2 + (y_i - y_k)^2 + (z_i - z_k)^2}$$

For the function of U to have a minimum at the corresponding values of the coordinates x_i , y_i , and z_i , it is essential, first, that the first derivatives of U with respect to all the coordinates of all the charges become equal to zero, and, second, that the second derivatives of U with respect to the coordinates x_h , y_h , and z_h be positive*:

$$\frac{\partial^2 U}{\partial x_h^2} > 0, \frac{\partial^2 U}{\partial y_h^2} > 0, \frac{\partial^2 U}{\partial z_h^2} > 0 \text{ for any } h$$

But

$$\begin{aligned} \frac{\partial^2 U}{\partial x_h^2} + \frac{\partial^2 U}{\partial y_h^2} + \frac{\partial^2 U}{\partial z_h^2} &= \nabla_h^2 U = \\ &= \frac{1}{2} \sum_{i, k} q_i q_k \nabla_h^2 \left(\frac{1}{R_{ik}} \right) \quad (i \neq k) \end{aligned}$$

where ∇_h stands for a differential operator corresponding to spatial differentiation with respect to the coordinates x_h , y_h , z_h of the charge q_h :

$$\nabla_h = \mathbf{i} \frac{\partial}{\partial x_h} + \mathbf{k} \frac{\partial}{\partial y_h} + \mathbf{j} \frac{\partial}{\partial z_h}$$

It is obvious that when $h \neq i$ and $h \neq k$, we have

$$\nabla_h^2 \left(\frac{1}{R_{ik}} \right) = 0$$

if $h = i$ or if $h = k$, then this relationship nevertheless remains true in view of Eq. (1.74). Consequently,

$$\nabla_h^2 U = \frac{\partial^2 U}{\partial x_h^2} + \frac{\partial^2 U}{\partial y_h^2} + \frac{\partial^2 U}{\partial z_h^2} = 0$$

Thus, the requirement that all the second derivatives of U with respect to the coordinates be positive cannot be observed; the energy U cannot have a minimum and, consequently, *stable static configuration*

* The conditions which derivatives of the kind $\partial^2 U / \partial x_i \partial x_k$ and $\partial^2 U / \partial x_i \partial y_i$ must comply with do not interest us here.

of electric charges is impossible. This statement is called the *Earnshaw theorem**.

The physical meaning of this theorem will become clear if we remember that unlike charges are attracted to each other with a growing force until they coincide, i.e. up to their mutual neutralization or annihilation, whereas like charges repel each other up to infinity.

Let us consider a very simple example as an illustration of the Earnshaw theorem. The system of three charges $q_1 = -4q$, $q_2 = q$, and $q_3 = -4q$, as can easily be seen, will be in static equilibrium if they are on a straight line in the order indicated above and if the distance between q_1 and q_2 equals that between q_2 and q_3 . Upon the slightest displacement, however, for example of the charge q_1 in the direction of the charge q_2 , the attraction which it is subjected to from the side of q_2 becomes greater than the repulsion from the side of q_3 , and thus the forces acting on q_1 (and on the other charges) will no longer be balanced, the charges q_1 and q_2 will be attracted to each other, while q_3 will fly away to infinity.

2. We may consider it to have been established experimentally that the distance between the particles of electricity (electrons and positive nuclei)** in the atoms of material bodies is very great (of the order of magnitude of 10^{-8} cm) in comparison with the dimensions of the particles themselves (not over 10^{-12} cm). Therefore, every atom can be considered as a system of point charges which the above formulation of the Earnshaw theorem can be applied to. Since, on the other hand, the atoms of the chemical elements are undoubtedly stable systems, then, consequently, *matter cannot be constructed from electrical particles within the limits of electrostatics, and an atom should be a dynamic system****. The notion of the stable periodic (or

* The possibility that the function U has a minimum with a configuration of the charges at which all the $\frac{1}{2} n(18n^2 + 9n + 11)$ derivatives of U of the first, second and third orders with respect to all the $3n$ coordinates (where n is the number of charges) become equal to zero, while the derivatives of the fourth order are positive, would correspond to the superposition of at least $\frac{1}{2} n(18n^2 + 9n + 11)$ conditions on the function of $3n$ variables. Since the number of conditions exceeds the number of variables, this possibility cannot be realized. It should also be noted that the above proof of the Earnshaw theorem assumes that all the distances R_{ik} between any pair of charges remain finite.

** In essence, the nuclei of atoms (except for the hydrogen nucleus) consist in turn of several protons and neutrons; owing to the small dimensions of these nuclei, however (not over 10^{-12} cm), each such composite nucleus as a whole can be considered as a single electrical particle.

*** The proof of the Earnshaw theorem is based, in essence, only on the forces of interaction between the points of a system being inversely proportional to the square of the distance between them, so that this theorem can also be applied to material points gravitating according to Newton's law (the Solar system). The stability of the Solar system is also ensured only by *motion* of the planets.

quasiperiodic) motion of electrical particles in the interior of atoms corresponding to this conclusion underlies the modern theory of matter and is confirmed in quite a few physical phenomena. True, this notion within the confines of classical physics leads to internal contradictions because the accelerated motion of electric charges, according to the laws of electrodynamics, is inseparably linked with the emission of electromagnetic energy (waves), i.e. cannot be stable. This contradiction, however, resulted not in the rejecting of the dynamic model of an atom, but in the rejecting of classical mechanics and electrodynamics in favour of quantum mechanics which eliminates contradiction.

3. Let us turn to the macroscopic theory of electrostatic phenomena that is the main subject of the present chapter. According to the Earnshaw theorem, purely electrostatic systems cannot be stable.

To avoid, however, a consideration of the hidden motion of elementary charges, the macroscopic electrostatic theory uses the formal notion of additional forces or *constraints of a non-electrostatic origin* ensuring the required stability of charged systems.

In this respect, there is complete analogy of electrostatics and mechanics, which widely uses notions of constraints accomplished with the aid of supports, fixed axes, inextensible threads, etc. Naturally, similar to how the physical mechanism of the constraints (force of elasticity) is disclosed in the further treatment of mechanics, before our subsequent development of the theory of electricity we are confronted with the task of disclosing the physical meaning of the formally introduced forces of constraint of a non-electrostatic origin.

Within the confines of electrostatics, a sufficiently close first approximation to actual conditions can be achieved by introducing into the consideration two main kinds of constraints. They correspond, first, to *ideal (perfect) conductors*, and, second, to *ideal dielectrics*. As regards conductors, without mentioning the fact explicitly, we actually used the assumption everywhere in our preceding discussion that there are forces of a non-electrostatic origin on the surface of conductors which prevent the charges from escaping beyond the surface of a conductor. Indeed, only these forces ensure stability of the system of charged conductors; otherwise the presence, for example, of an isolated charged conductor would be impossible—the elements of its charge under the influence of mutual repulsion would fly away in different directions to infinity. Using the terminology of mechanics, we can call these retaining forces the reactions of the constraints. They must evidently be equal in magnitude and opposite in direction to the ponderomotive forces acting on the surface of a conductor whose density, according to Eq. (1.115), equals $E^2/8\pi$.

The following chapter will be devoted to the properties of dielectrics. For constraints in dielectrics see, particularly, Sec. 2.8.

2

Dielectrics

2.1 Dielectrics.

Electric Moment and Potential of a Neutral Molecule.

Polarization of a Dielectric

1. Dielectrics are non-conductors of electricity*. Unlike metals and electrolytes, they contain no charges that can move over considerable distances and carry a current.

Dielectrics are constructed either of neutral molecules (all gaseous and liquid dielectrics and part of the solid ones) or of charged ions fixed in definite equilibrium positions (for example at the lattice points (sites) of a crystal). Ionic crystal lattices can be divided into so-called elementary cells each of which contains an equal number of positive and negative charges and as a whole is neutral. In the following, for definiteness, we shall sometimes assume that a dielectric is constructed of neutral molecules. The main postulates of the theory being treated, however, can also be applied to ionic crystalline and amorphous dielectrics. In these cases, we must understand that, for instance, an elementary cell of a crystal is a molecule and only a molecule.

Under the action of an external electric field**, the charges in a dielectric are not torn away by the field from their places, but are only displaced somewhat from the positions of equilibrium to new equilibrium positions.

The resultant of the electric forces acting on a neutral molecule in a *homogeneous* ($E = \text{const}$) electric field obviously equals zero; therefore the centre of gravity of a molecule of a dielectric in a homogeneous field remains stationary. The electrical particles of opposite signs in a molecule of a dielectric, however, should move in opposite directions under the action of the field forces—the molecule deforms. Consequently, to determine the action of a field on a dielectric, it is necessary first of all to find a convenient quantitative characteristic of the distribution of the charges in a neutral molecule.

* In essence, all dielectrics have a certain, though very small, conductivity so that the concept of an ideal non-conductor is only a first approximation to actual conditions.

** If only its intensity is not too great (otherwise, for example, breakdown of the dielectric may occur).

2. The vector of the electric moment \mathbf{p} of a system of charges, neutral as a whole, can serve as such a characteristic. It is determined by the equation

$$\mathbf{p} = \sum_i q_i \mathbf{R}_i \quad (2.1)$$

where summation covers all the elementary charges (electrons and nuclei) in the system, while \mathbf{R}_i is the radius-vector conducted to the charge q_i from an arbitrary initial point O . The system of charges is assumed to be electrically neutral, i.e.

$$\sum_i q_i = 0 \quad (2.2)$$

because only in this condition is the value of the vector \mathbf{p} unambiguously determined by the distribution of the charges, and it does not depend on the choice of the initial point O .

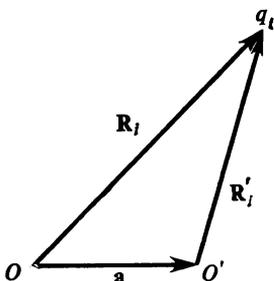


Fig. 25

Indeed, if we shift the reference point from O to O' over an arbitrary distance \mathbf{a} (Fig. 25), then the new radius-vector \mathbf{R}'_i of the charge q_i will be determined by the difference

$$\mathbf{R}'_i = \mathbf{R}_i - \mathbf{a}$$

and, consequently, instead of Eq. (2.1) we get

$$\mathbf{p} = \sum_i q_i \mathbf{R}'_i = \sum_i q_i \mathbf{R}_i - \mathbf{a} \sum_i q_i$$

which when condition (2.2) is observed coincides with Eq. (2.1).

When a system consists of two equal and opposite charges $\pm q$ (Fig. 26) whose radius-vectors are \mathbf{R}_+ and \mathbf{R}_- , the moment of the system obviously equals

$$\mathbf{p} = \sum_i q_i \mathbf{R}_i = q (\mathbf{R}_+ - \mathbf{R}_-) = q \mathbf{l}$$

where $\mathbf{l} = \mathbf{R}_+ - \mathbf{R}_-$ is a vector drawn from the negative charge to the positive one. Thus, the definition of the dipole moment which

we already know from Eq. (1.48) follows in this particular case from Eq. (2.1).

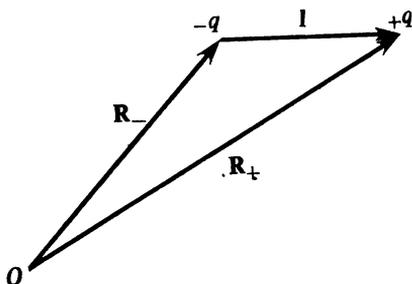


Fig. 26

3. The importance of the concept of the electric moment of a system of charges is due to the fact that the potential φ of the field induced by an arbitrary system of charges, neutral as a whole, with the moment \mathbf{p} at distances that are great in comparison with the dimensions of the system coincides with the potential of the dipole of the same moment \mathbf{p} .

Indeed, the potential of the system of charges q_i at an arbitrary point of the field P is

$$\varphi = \sum_i \frac{q_i}{R'_i}$$

where \mathbf{R}'_i is the distance from the point P to the charge q_i . Let us choose an arbitrary point O in the vicinity of the charges q_i , which

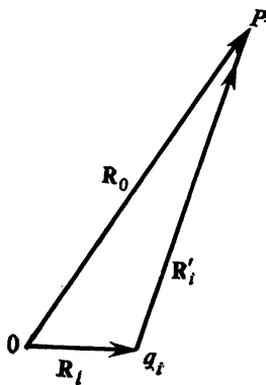


Fig. 27

we shall conditionally call the “centre” of the system, and let \mathbf{R}_0 and \mathbf{R}_i be, respectively, the distances from the centre O to the point P being considered and to the charge q_i , so that (Fig. 27)

$$\mathbf{R}'_i = \mathbf{R}_0 - \mathbf{R}_i; \quad R_i'^2 = R_0^2 - 2\mathbf{R}_0\mathbf{R}_i + R_i^2$$

Further,

$$\begin{aligned} \frac{1}{R_i'} &= R_0^{-1} \left(1 - \frac{2\mathbf{R}_0\mathbf{R}_i}{R_0^2} + \frac{R_i'^2}{R_0^2} \right)^{-1/2} = \\ &= \frac{1}{R_0} \left(1 + \frac{\mathbf{R}_0\mathbf{R}_i}{R_0^2} + \dots \right) \end{aligned}$$

If $R_i \ll R_0$, then, limiting ourselves to the first two terms of the expansion*, we get:

$$\varphi = \frac{1}{R_0} \sum q_i + \frac{\mathbf{R}_0}{R_0^3} \sum q_i \mathbf{R}_i \quad (2.3)$$

or, in view of Eqs. (2.1) and (2.2):

$$\varphi = \frac{\mathbf{R}_0}{R_0^3} \sum q_i \mathbf{R}_i = \frac{\mathbf{p}\mathbf{R}_0}{R_0^3} \quad (2.4)$$

which coincides with the potential of the field of the dipole with the moment \mathbf{p} [Eq. (1.49)].

4. In the first approximation, not only the field induced by a system of charges that is neutral as a whole coincides with the field of an equivalent dipole, but also the forces acting on this system in an electric field and on the dipole equivalent to it are equal to each other.

To convince ourselves that this is true, it is sufficient to show that the potential energy of a neutral system of charges in an *external* electric field coincides with the energy of an equivalent dipole because the ponderomotive forces are unambiguously determined by the expression for the energy (see Sec. 1.18).

The energy of a system of charges in an external field equals

$$U = \sum_i \varphi_i q_i$$

where φ_i is the potential of the external field where the charge q_i is. If the potential of the external field at the "centre" of the system O equals φ_0 , then with an accuracy to values of the second order of magnitude the potential at the point \mathbf{R}_i is

$$\varphi_i = \varphi_0 + \mathbf{R}_i \text{grad } \varphi = \varphi_0 - \mathbf{E}\mathbf{R}_i$$

* The following terms of the expansion diminish inversely proportional to R_0^3 , R_0^4 , etc. Their magnitude is characterized by the so-called quadrupole, octupole, etc. moments of the system of charges, which are similar in a certain respect to its dipole moment \mathbf{p} . Disregarding these terms for purposes of simplification, we shall therefore consider that if $\mathbf{p} = 0$, the field of a system of charges that is neutral as a whole equals zero at great distances from it, although this is naturally not completely true.

and, therefore,

$$U = \varphi_0 \sum q_i - \mathbf{E} \sum q_i \mathbf{R}_i$$

or, according to Eqs. (2.1) and (2.2)

$$U = -\mathbf{E} \sum q_i \mathbf{R}_i = -\mathbf{pE} \quad (2.5)$$

which, as we had to prove, coincides with the expression for the energy of a dipole (1.103).

5. Thus, in the first approximation, *any system of charges, neutral as a whole, whose electric moment equals \mathbf{p} is equivalent to a dipole of the same moment \mathbf{p} both in the active and the passive respect* (i.e. both in respect to the field it induces, and in respect to the forces acting on it).

We shall take advantage of this circumstance in the following and often replace the consideration of a complex of real, generally speaking, very complicated molecules of a dielectric with the consideration of the dipoles equivalent to them.

6. Not only the electrical state of a separate molecule, but also the state of a macroscopic volume of a dielectric consisting of many molecules can be characterized by the electric moment. By the *polarization* of a dielectric \mathbf{P} is meant the electric moment of a unit volume of the dielectric:

$$\mathbf{P} = \sum q_i \mathbf{R}_i \quad (2.6)$$

where summation covers all the charges (electrons and atomic nuclei) in a unit volume of the dielectric.

If a dielectric consists of neutral molecules, summation can be performed in two steps: first summation for the charges in the separate molecules of the dielectric, which gives the moment \mathbf{p} of each molecule*, and then summation over all the molecules in a unit volume. Thus, Eq. (2.6) can also be written as follows:

$$\mathbf{P} = \sum \mathbf{p} \quad (2.7)$$

In other words, the polarization of a dielectric equals the vector sum of the electric moments of the molecules in a unit volume of the dielectric.

Equation (2.7) can also be applied to an ionic crystalline dielectric, assuming that \mathbf{p} in this case stands for the moment of the separate elementary cells of the crystal. Although the division of a crystal into elementary cells is ambiguous, the result \mathbf{P} of summing up the moments of the separate cells does not depend on this arbitrary division and has a quite definite value.

Finally, if the polarization of a dielectric is not uniform, then the value of the vector \mathbf{P} at a given point should evidently be determined

* We must remind our reader that the radius-vectors \mathbf{R}_i of the charges of a neutral molecule in Eq. (2.1) may be conducted from an arbitrary origin.

as the ratio of the electric moment of an element of volume of the dielectric to the magnitude of this element dV (with a sufficiently small dV , see Sec. 2.6). In other words,

$$\mathbf{P} dV = \sum_{dV} q_i \mathbf{R}_i = \sum_{dV} \mathbf{p} \quad (2.8)$$

where summation covers the charges (or correspondingly all the molecules) in the element dV .

2.2 Free and Bound Charges.

Potential of an Electric Field When Dielectrics Are Present. Dependence of Polarization on the Field

1. When considering an electrostatic field that contains dielectrics, we must distinguish two kinds of electric charges, free and bound ones. By *free charges* we shall mean, first, all the electric charges that can move over macroscopic distances under the influence of an electric field (electrons in metals and in a vacuum, ions in gases and electrolytes, etc.), and, second, charges brought in from outside onto the surface of the dielectrics and violating their neutrality (for example, the charges of the intraionic lattice of solid dielectrics formed owing to the lack of ions of a definite sign in this section of the dielectric so that the section as a whole is no longer neutral). We shall call the charges in the neutral molecules of dielectrics and the ions fixed in solid dielectrics near definite equilibrium positions *bound charges**.

The potential φ of an electrostatic field containing dielectrics obviously equals the sum of the potential φ_0 induced by the free charges and the potential φ' induced by the bound electric charges in the dielectrics, i.e.:

$$\varphi = \varphi_0 + \varphi'$$

The potential of the free charges is determined by Eq. (1.86):

$$\varphi_0 = \int \frac{\rho dV}{R} + \int \frac{\sigma dS}{R}$$

where ρ and σ stand for the volume and surface densities of the free charges.

The potential φ' of the field of bound charges is unambiguously determined by the polarization of the dielectrics.

Indeed, let us consider the element of volume of a dielectric dV that is neutral as a whole. The electric moment of this element dV ,

*It must be noted that in introducing this definition we have digressed from the obsolete terminology according to which our free charges are called true ones, while the complex of true and bound charges is called free charges.

according to Eq. (2.8), equals $\mathbf{P} dV$, and therefore the potential of the charges of the dielectric in this element dV , according to Eq. (2.4), equals $(\mathbf{P}\mathbf{R}/R^3) dV$, where \mathbf{R} is the distance to the point of the field being considered from dV . Finally, the potential φ' of all the elements of a polarized dielectric, i.e. of all the bound charges, is evidently determined by the integral

$$\varphi' = \int \frac{\mathbf{P}\mathbf{R}}{R^3} dV$$

which can be extended over the entire infinite space because at the points where there is no dielectric we have $\mathbf{P} = 0$, and the integrand becomes equal to zero. Thus, the resultant potential φ of an electrostatic field when dielectrics are in it is expressed by the equation

$$\varphi = \varphi_0 + \int \frac{\mathbf{P}\mathbf{R}}{R^3} dV \quad (2.9)$$

2. It is expedient to transform this equation somewhat. According to formulas (A.10) and (A.43₂) of vector analysis, we have

$$\frac{\mathbf{P}\mathbf{R}}{R^3} = \mathbf{P} \operatorname{grad}_q \left(\frac{1}{R} \right) = \operatorname{div}_q \left(\frac{\mathbf{P}}{R} \right) - \frac{1}{R} \operatorname{div} \mathbf{P}$$

where the subscript q indicates that upon differentiation of the radius-vector \mathbf{R} it is considered as a position function of its origin, which in the given case coincides with dV . Since the vector \mathbf{P} is a position function of only this "source point", we have discarded the subscript q at $\operatorname{div} \mathbf{P}$ as being superfluous. Thus,

$$\varphi = \varphi_0 - \int \frac{\operatorname{div} \mathbf{P}}{R} dV + \int \operatorname{div}_q \left(\frac{\mathbf{P}}{R} \right) dV$$

or, according to Gauss's theorem (A.17)*

$$\varphi = \varphi_0 - \int \frac{\operatorname{div} \mathbf{P}}{R} dV + \oint_{S+S'_1} \frac{P_n}{R} dS$$

where the last integral should be extended over the outer surface S of the volume V being considered and over the surfaces S'_1 that separate from the volume of integration the surfaces S_1 of discon-

* In deriving Gauss's theorem (A.17), \mathbf{a} is considered to be a position function of the element of integration volume dV (or of the element of the integration surface dS). This formula can be directly applied to our integral because its integrand includes $\nabla_q(1/R)$. When calculating $\nabla_q(1/R)$ the radius-vector \mathbf{R} is considered as a position function of its "source point" coinciding in the given case with dV . If the integral to be transformed included the gradient $\nabla_a(1/R)$, then Gauss's formula could be applied only after substituting $-\nabla_q(1/R)$ for $\nabla_a(1/R)$.

tinuity of the vector \mathbf{P} . If we consider the total field, then the integral with respect to S vanishes, while the integral with respect to the surfaces S_1' when they converge to S_1 becomes, as usual, [cf., for instance, the derivation of Eq. (1.80)] equal to

$$\int_{S_1} \frac{(P_{1n} - P_{2n})}{R} dS$$

Introducing, finally (meanwhile purely formally) the notation

$$-\operatorname{div} \mathbf{P} = \rho_b \quad \text{and} \quad -(P_{2n} - P_{1n}) = \sigma_b \quad (2.10)$$

where the subscript “b” stands for “bound”, and introducing the expression for φ_0 into Eq. (2.9), we finally get

$$\begin{aligned} \varphi &= \varphi_0 + \int \frac{\rho_b}{R} dV + \int \frac{\sigma_b}{R} dS = \\ &= \int \frac{(\rho + \rho_b) dV}{R} + \int \frac{(\sigma + \sigma_b) dS}{R} \end{aligned} \quad (2.11)$$

Thus, the electric field when dielectrics *are present* coincides with the field that would be induced *in the absence* of the dielectrics by the same free charges with the addition to them of the charges ρ_b and σ_b determined by Eqs. (2.10). It is clear that the quantities ρ_b and σ_b are the mean densities of the “bound” charges of the dielectric. In Sec. 2.7, we shall establish the correctness of this statement by direct calculations.

It must be noted that in a uniformly polarized dielectric ($\mathbf{P} = \text{const}$) the density (mean) of the bound charges, according to Eq. (2.10), equals zero. This is quite clear because owing to the identical physical states of adjacent sections of the dielectric in this case no charges of one sign can accumulate anywhere. On the boundary between the polarized dielectric and a vacuum or metal, however, according to Eq. (2.10) a surface bound charge having the density $\sigma = \pm P_n$ will concentrate (because $P = 0$ both in a vacuum and in a metal).

3. Comparing Eq. (2.11) with Eqs. (1.86) and (1.65), we obviously get

$$\nabla^2 \varphi = -4\pi (\rho + \rho_b)$$

Taking into consideration the relationships

$$\mathbf{E} = -\nabla \varphi \quad \text{and} \quad \operatorname{div} \mathbf{E} = -\nabla^2 \varphi \quad (2.12)$$

and also Eq. (2.10), we can rewrite this equation as follows:

$$\operatorname{div} \mathbf{E} = 4\pi \rho - 4\pi \operatorname{div} \mathbf{P}$$

or

$$\operatorname{div} (\mathbf{E} + 4\pi \mathbf{P}) = 4\pi \rho \quad (2.13)$$

4. Differential equations (2.12) and (2.13) are the fundamental equations of an electrostatic field in an arbitrary medium. To obtain a complete system of equations of electrostatics, all that remains to do is to add an equation relating \mathbf{E} and \mathbf{P} to them.

In the absence of external fields, the polarization \mathbf{P} of a dielectric equals zero*: the electric moments of the individual molecules in the absence of an external field, if they do differ from zero, are oriented absolutely chaotically, and their sum equals zero. In the presence of an electric field, the polarization of a dielectric, *as shown by experiments*, is proportional to the field intensity \mathbf{E} :

$$P = \alpha E \quad (2.14)$$

The deviations from proportionality between P and E in the fields we can study are so insignificant that with rare exceptions (see, for example, Sec. 2.10) they can in general be disregarded**. The coefficient α in Eq. (2.14) characterizes the properties of a given dielectric and is called its *polarizability*.

It remains to consider the direction of the vector \mathbf{P} . For isotropic dielectrics, it already follows from considerations of symmetry that the vector \mathbf{P} can be directed only along the single chosen direction of the electric field \mathbf{E} (or in the direction opposite to \mathbf{E}). Since the positive charges are displaced upon polarization of the dielectric in the direction of the field and since the vector of the electric moment is directed from the negative charges to the positive ones (cf. the case of a simple dipole, Fig. 26), then the vector \mathbf{P} is parallel to the field \mathbf{E} , and in the vector form we finally get

$$\mathbf{P} = \alpha \mathbf{E} \quad (2.15)$$

In anisotropic media, however, the direction of the polarization vector may not coincide, and generally speaking does not coincide with the direction of the field. The absolute value of the vector \mathbf{P} in this case too depends not only on the absolute value of the vector \mathbf{E} , but also on the direction of the latter relative to the crystallographic

* We shall not stop here to consider the comparatively rare exclusions from this rule, for example the phenomena of pyro- and piezoelectricity. Particularly, piezoelectricity consists in that crystals, which have definite properties of symmetry, are polarized not only under the action of external electric fields, but also upon mechanical deformation.

** The linear relationship between the polarization \mathbf{P} and the field \mathbf{E} corresponds to linear electrodynamics. In recent years, non-linear electrodynamics has also become widely developed (and, accordingly, non-linear optics), and in this case a non-linear relationship between \mathbf{P} and \mathbf{E} is used. The non-linear nature must be taken into consideration in strong fields which have become available owing to the creation of lasers. In addition, for a number of media with a high polarizability (ferroelectrics, plasma, etc.), non-linearity may occur already in comparatively weak fields—ones that are small in comparison with the characteristic atomic field $E \approx e/a^2 \approx 10^8$ to 10^9 V/cm (here $e = 4.8 \times 10^{-10}$ cgse units is the charge of a proton and $a \approx 1$ to 3×10^{-8} cm is the dimension of an atom or the lattice constant).

axes of the dielectric. The relationship between the components of the vector \mathbf{P} and those of the vector \mathbf{E} remains linear, however, and for anisotropic dielectrics Eqs. (2.14) and (2.15) should be replaced by the following:

$$\left. \begin{aligned} P_x &= \alpha_{11}E_x + \alpha_{12}E_y + \alpha_{13}E_z \\ P_y &= \alpha_{21}E_x + \alpha_{22}E_y + \alpha_{23}E_z \\ P_z &= \alpha_{31}E_x + \alpha_{32}E_y + \alpha_{33}E_z \end{aligned} \right\} \quad (2.16)$$

The values of the coefficients of polarizability α_{ik} depend on the orientation of the axes x , y , and z of the coordinate system relative to the crystallographic axes of the dielectric.

In the following, we shall limit ourselves to a consideration of only isotropic dielectrics which Eq. (2.15) may be applied to.

2.3 Electric Displacement Vector.

Differential Equations of a Field in an Arbitrary Medium. Induction Lines

1. Instead of the polarization \mathbf{P} , it is convenient to introduce the vector \mathbf{D} determined by the formula

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} \quad (2.17)$$

Different authors use different names for this vector: the electric induction, electric polarization [not to be confused with the vector \mathbf{P} , Eq. (2.7)], the *electric displacement*, etc. We shall use the last of these terms. The electric displacement vector is in essence the sum of two absolutely different physical quantities: the field intensity and (multiplied by 4π) the polarization of a unit volume of the medium. Nevertheless, the introduction of this vector enormously simplifies the studying of the field in dielectrics. Particularly, one of the fundamental equations of an electric field—Eq. (2.13)—acquires a very simple form when the notation (2.17) is used in it:

$$\operatorname{div} \mathbf{D} = 4\pi\rho \quad (2.18)$$

For a non-polarizing medium (for instance a vacuum), we have $\mathbf{P} = 0$, the displacement vector \mathbf{D} coincides with the field intensity \mathbf{E} , and Eq. (2.18) coincides with Eq. (1.27):

$$\operatorname{div} \mathbf{E} = 4\pi\rho$$

so that the latter equation is a particular case of Eq. (2.18).

Multiplying Eq. (2.18) by a volume element dV and integrating over the volume confined within the closed surface S , we get, according to Gauss's theorem (A.17)

$$\oint D_n dS = 4\pi \int \rho dV \quad (2.19)$$

This formula is a generalization of Gauss's electrostatic law (1.11) for an arbitrary, particularly a dielectric, medium.

Further, for isotropic dielectrics from Eqs. (2.15) and (2.17), we get

$$\mathbf{D} = (1 + 4\pi\alpha) \mathbf{E} = \varepsilon \mathbf{E} \quad (2.20)$$

where

$$\varepsilon = 1 + 4\pi\alpha \quad (2.21)$$

Thus, the displacement \mathbf{D} is proportional to the field intensity \mathbf{E} . The proportionality factor ε between them is called the *permittivity* (*dielectric constant*) of a dielectric*. It follows from Eqs. (2.15) and (2.21) that

$$\mathbf{P} = \frac{\varepsilon - 1}{4\pi} \mathbf{E} \quad (2.22)$$

2. In the limiting case of surface charges, Eq. (2.18), in accordance with Eqs. (1.29) and (1.30), becomes

$$\text{Div } \mathbf{D} = D_{2n} - D_{1n} = 4\pi\sigma. \quad (2.23)$$

where σ is the surface density of the *free* charges.

Using Eq. (2.20), we can write Eq. (2.23) as follows:

$$D_{2n} - D_{1n} = \varepsilon_2 E_{2n} - \varepsilon_1 E_{1n} = 4\pi\sigma \quad (2.24)$$

where ε_1 and ε_2 are the values of the permittivity at both sides of the surface of discontinuity. In particular, if $\sigma = 0$, i.e. if there are no *free* charges on this surface, then

$$\varepsilon_2 E_{2n} = \varepsilon_1 E_{1n} \quad \text{and} \quad D_{2n} = D_{1n}$$

Consequently, on an *uncharged* interface between two different media ($\varepsilon_1 \neq \varepsilon_2$), the normal component of the electric displacement remains continuous, whereas the normal component of the field intensity undergoes a jump.

As regards the tangential component of the vector \mathbf{E} , the reasoning which in Sec. 1.7 resulted in Eq. (1.37) remains in force because in an arbitrary dielectric too the work of the electric forces does not depend on the shape of the path. Consequently,

$$E_{2t} = E_{1t} \quad \text{or} \quad \frac{1}{\varepsilon_2} D_{2t} = \frac{1}{\varepsilon_1} D_{1t} \quad (2.25)$$

In other words, the tangential component of the field intensity always remains continuous, while the tangential component of the displace-

* Equation (2.17) also holds for anisotropic dielectrics, but upon excluding \mathbf{P} from it with the aid of Eqs. (2.16), we get three equations of the same kind for the components of the vector \mathbf{D} that include nine permittivities ε_{ik} .

ment at the interface between two different media ($\epsilon_2 \neq \epsilon_1$) undergoes a jump.

Since upon electrostatic equilibrium, \mathbf{E} (and therefore \mathbf{D}) inside conductors equals zero, then it follows from Eq. (2.25) that at the outer surface of conductors the vector \mathbf{E} (and therefore \mathbf{D}) is always perpendicular to this surface. Therefore, Eq. (2.24) acquires the following form *for the surface* of conductors:

$$\epsilon \mathbf{E} = \mathbf{D} = 4\pi\sigma \mathbf{n} \quad (2.26)$$

where \mathbf{n} = outward normal to the surface

ϵ = permittivity of the medium in contact with it.

Equation (1.38) is obviously a particular case of this formula.

3. Equations (2.12), (2.18), (2.20), and (2.23)

$$\left. \begin{aligned} \mathbf{E} &= -\text{grad } \varphi, \quad \mathbf{D} = \epsilon \mathbf{E} \\ \text{div } \mathbf{D} &= 4\pi\rho, \quad D_{2n} - D_{1n} = 4\pi\sigma \end{aligned} \right\} \quad (\text{A})$$

supplemented with the requirement of the continuity of the potential φ^* are a *complete system* of equations of an electrostatic field in an arbitrary medium. This means that if we are given the values of the densities of the free charges ρ and σ and the value of the permittivity ϵ at each point of space, and if the conditions (1.85) (i.e. ER^2 remains finite at $R \rightarrow \infty$) are complied with, then the system (A) unambiguously determines the electric field, i.e. the values of φ , \mathbf{E} and \mathbf{D} at each point of space; conversely, if we are given the values (a) of the permittivity ϵ and (b) the field intensity \mathbf{E} (or the potential φ , or the displacement \mathbf{D}) at each point of space, then the system (A) unambiguously determines the distribution of the free charges ρ and σ .

The truth of the second statement is obvious. To prove the first one, let us assume that there are two solutions $\mathbf{E}, \mathbf{D}, \varphi$ and $\mathbf{E}', \mathbf{D}', \varphi'$ of the system (A) with given values of ϵ, ρ and σ . Introducing both solutions into (A) and then subtracting the relevant equations from one another, we get

$$\left. \begin{aligned} \mathbf{E}'' &= -\text{grad } \varphi'', \quad \mathbf{D}'' = \epsilon \mathbf{E}'', \\ \text{div } \mathbf{D}'' &= 0, \quad D_{2n}'' = D_{1n}'' \end{aligned} \right\} \quad (\text{A}')$$

where

$$\mathbf{E}'' = \mathbf{E} - \mathbf{E}', \quad \mathbf{D}'' = \mathbf{D} - \mathbf{D}', \quad \varphi'' = \varphi - \varphi',$$

* Because we are not considering electrical double layers, i.e. surfaces of discontinuity of the potential. It is not difficult, however, to include them in the theory being treated. It should be noted that in accordance with Sec. 1.7, the equation $\mathbf{E} = -\text{grad } \varphi$ ensures the continuity of the tangential components of the vector \mathbf{E} .

Further, on the basis of the system (A') and Eq. (A.43₂), we can write the following chain of equations:

$$\begin{aligned}\varepsilon E''^2 &= \mathbf{D}'' \mathbf{E}'' = -\mathbf{D}'' \text{grad } \varphi'' = \\ &= -\text{div} (\mathbf{D}'' \varphi'') + \varphi'' \text{div} \mathbf{D}'' = -\text{div} (\mathbf{D}'' \varphi'')\end{aligned}$$

Hence, the integral of $\varepsilon E''^2$ over the arbitrary volume confined by the surface S , on the basis of Eq. (A.17), will equal

$$\begin{aligned}\int \varepsilon E''^2 dV &= -\int \text{div} (\mathbf{D}'' \varphi'') dV = \\ &= -\oint_S D_n'' \varphi'' dS\end{aligned}$$

Here the surface integral should be taken only over the boundary surface S because in the entire field both φ'' and, according to (A'), D_n'' remain continuous. If we now extend integration over the volume of the *total field*, then the integral over the surface S vanishes. Hence,

$$\int \varepsilon E''^2 dV = 0$$

which can occur only if at all the points of the field $\mathbf{E}'' = \mathbf{E} - \mathbf{E}'$ becomes equal to zero. This proves the single-valued nature of the solution of the system (A).

Thus, the vector \mathbf{E}'' , which satisfies the system (A'), identically equals zero. Since in the absence of free charges ($\sigma = \rho = 0$) the system (A) takes on the form of the system (A'), then, consequently, *in the absence of free charges the electric field identically equals zero*. Thus, the presence of dielectrics can only alter the field of the free charges; in the absence of the latter the polarization of a dielectric diminishes, becomes equal to zero, and the electric field vanishes. (See the first footnote on page 116.)

4. We shall note in conclusion that it is not convenient to use force lines of a field for the graphical depicting of an electric field in dielectrics, i.e. lines of the vector \mathbf{E} (see Sec. 1.10). The reason is that the divergence (volume and surface) of this vector when dielectrics are present may differ from zero not only at the points of the field where *free* (space and surface) charges are, but also at the points where the bound charges of the dielectric are whose density, in turn, depends on the intensity of the field, the non-homogeneity of the medium, etc. Therefore, the so-called *displacement lines*, i.e. lines of the electric displacement vector \mathbf{D} , are used for the graphical depicting of a field in a dielectric.

Since according to Eq. (2.20) the vector \mathbf{D} at each point of space (except for anisotropic media) is parallel to the vector \mathbf{E} , then each displacement line is also a force line, and vice versa. Therefore, in

particular, from the fact that closed force lines cannot exist it follows that *closed displacement lines also cannot exist*. If, however, as is customary practice, we draw the lines of force and the lines of displacement so that the number of these lines intersecting an element of surface area dS is as far as possible proportional to the flux of the relevant vector (\mathbf{D} or \mathbf{E}) through this element, then the density of the lines of displacement and those of force will, generally speaking, change differently from one section of space to another. In particular, with such a way of drawing, some of the force lines will have to be terminated at the bound negative charges of the dielectric, whereas the corresponding displacement lines will pass through and beyond these charges until they meet free charges.

Indeed, since the dependence of the volume and surface divergence of the vector \mathbf{D} on the distribution of the free charges in an arbitrary medium coincides with the dependence of $\text{div } \mathbf{E}$ and $\text{Div } \mathbf{E}$ on ρ and σ in the absence of dielectrics, then according to the results of Sec. 1.10, the displacement lines can begin and terminate only at the points of the field where there are *free* electric charges, or extend to infinity. (See, however, the footnote on p. 63.) In a vacuum, the vector \mathbf{D} is identical to the vector \mathbf{E} , so that the lines of displacement coincide with those of force.

Problem 16. Show that at the interface of two dielectrics the force lines (i.e. the lines of the direction of the vector \mathbf{E}) are diffracted, and

$$\tan \beta_1 : \tan \beta_2 = \varepsilon_1 : \varepsilon_2$$

where β_1 = angle formed by the direction of the force line in the first dielectric with a normal to the interface

ε_1 = permittivity of the first medium

β_2 and ε_2 = relevant quantities for the second medium.

Problem 17. Show that the intensity of the field \mathbf{E}' in the middle part of a long and narrow slot made in a solid dielectric equals the intensity of the field \mathbf{E} in the dielectric if the slot is parallel to the vector \mathbf{E} and that \mathbf{E}' equals the displacement \mathbf{D} in the dielectric if the slot is perpendicular to \mathbf{E} .

Problem 18. Show that for the unambiguous determination of the electrostatic field in an arbitrary medium it is sufficient to set, first, the arrangement and shape of the conductors, the value of the permittivity at each point of the medium, and the distribution of the *free* space and surface charges in the dielectric, and, second, either the potential φ_i of each conductor [problem (A)], or the total charge q_i of each conductor [problem (B)] (cf. Sec. 1.13).

Problem 19. The space between the plates of a parallel-plate capacitor is filled with two layers of a dielectric. The permittivities

of these layers are ε_1 and ε_2 , and their thicknesses are d_1 and d_2 whose sum $d_1 + d_2 = d$, the distance between the capacitor plates. Show that the capacitance of the capacitor C is determined by the relationship:

$$\frac{1}{C} = \frac{4\pi}{S} \left(\frac{d_1}{\varepsilon_1} + \frac{d_2}{\varepsilon_2} \right)$$

where S is the surface area of its plates.

2.4 Electric Field in a Homogeneous Dielectric

1. Let us consider the very simple case when the entire field, i.e. all the regions of space in which the vector \mathbf{E} does not equal zero, is filled with a *homogeneous* dielectric. An example is a system of conductors immersed in an infinite homogeneous dielectric (because upon electrical equilibrium within the conductors $\mathbf{E} = 0$) or a dielectric confined by a closed metal shell (electrostatic shield). In this case in all the differential equations for a field, the constants ε and α can be put before the derivative sign, and, for instance, from Eqs. (2.18) and (2.20) it follows that

$$\begin{aligned} \operatorname{div} \mathbf{D} &= \operatorname{div} \varepsilon \mathbf{E} = \varepsilon \operatorname{div} \mathbf{E} = 4\pi\rho \\ \operatorname{div} \mathbf{E} &= -\nabla^2\varphi = \frac{1}{\varepsilon} \cdot 4\pi\rho \end{aligned} \quad (2.27)$$

This means that with a given distribution of the free charges, the potential and the intensity of the field in a homogeneous dielectric are $1/\varepsilon$ -th of the potential and intensity of the field in a vacuum. This postulate often serves as the foundation of the whole formal theory of dielectrics.

It directly follows from it that the potential and intensity of the field of a point charge in a *homogeneous* dielectric equal

$$\varphi = \frac{q}{\varepsilon R} \quad \text{and} \quad E = \frac{q}{\varepsilon R^2} \quad (2.28)$$

(the so-called *generalized Coulomb law*). Further, the potential difference between the plates of a capacitor when a homogeneous dielectric fills the space between them should diminish ε times if the charges of the plates remain unchanged. This means that the capacitance of the capacitor C grows ε times:

$$C = \varepsilon C_0 \quad (2.29)$$

Finally, we must remind our reader that the value of the electric displacement vector \mathbf{D} in a homogeneous medium does not depend on the permittivity of the medium and is completely determined

by the distribution of the free charges because it follows from Eqs. (2.20) and (2.28) that

$$D = \varepsilon E = \frac{q}{R^2} \quad (2.30)$$

2. It must be remembered without fail, however, that Eqs. (2.27)-(2.30) are *not applicable at all* to a *non-homogeneous* dielectric. For instance,



Fig. 28

if we introduce a piece of dielectric **D** into the field of a charge q (Fig. 28), then owing to the polarization of this dielectric the field intensity at the points P_1 and P_2 will not diminish, as would correspond to Eq. (2.27), but will grow. Indeed, the negative charges of the dipoles will shift in the dielectric to the left, and the positive charges to the right, so that the direction of the resultant field of these charges at the points P_1 and P_2 will coincide with the direction of the field of the charge q . At the point P_3 , however, the polarization of the dielectric will result in weakening of the original field of the charge q .

In general for a *non-homogeneous* medium, we cannot establish a simple relationship between the field and the arrangement of only the *free* charges, i.e. a relationship similar to Coulomb's law (2.28). Only by reverting to differential equations of a field, i.e. to equations relating the values of the quantities characterizing the field at *adjacent* points of space, can we arrive at comparatively simple relationships between these quantities [system of equations (A), p. 119] because only differential relationships are completely determined by the properties of a given element of a medium regardless of the properties of its remote regions.

3. Let us consider another example when the infinite half-space above the plane $z = 0$ is filled with a homogeneous dielectric having the permittivity ε_1 , while the half-space under this plane is filled with a different homogeneous dielectric having the permittivity ε_2 (Fig. 29). We shall determine the field of the charge q at an arbitrary point P . We select the coordinate axes so that the z -axis passes through the point P . The coordinates of this point will be $x = y = 0$ and $z = z_0$. Let us assume for definiteness that $z_0 > 0$, i.e. the charge is in the top half-space where $\varepsilon = \varepsilon_1$.

Let the potentials in the top and bottom half-spaces be φ_1 and φ_2 , respectively. The condition of continuity of the potential at the interface of the dielectrics is

$$(\varphi_1)_{z=0} = (\varphi_2)_{z=0} \quad (2.31)$$

Further, since in our case $\sigma = 0$, then by expressing \mathbf{E} in Eq. (2.34) through $\text{grad } \varphi$, we get

$$\epsilon_1 \left(\frac{\partial \varphi_1}{\partial z} \right)_{z=0} = \epsilon_2 \left(\frac{\partial \varphi_2}{\partial z} \right)_{z=0} \quad (2.32)$$

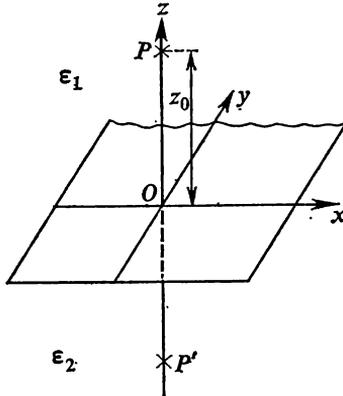


Fig. 29

Finally, since ϵ is constant in each half-space, then according to Eq. (2.27) we get

$$\nabla^2 \varphi_1 = -\frac{4\pi\rho}{\epsilon_1}, \quad \nabla^2 \varphi_2 = -\frac{4\pi\rho}{\epsilon_2} \quad (2.33)$$

In our case $\rho = 0$ everywhere except for the point P ; hence,

$$\nabla^2 \varphi_2 = 0 \quad (2.34)$$

while the first of the equations (2.33) will obviously be satisfied if we assume that

$$\varphi_1 = \frac{q}{\epsilon_1 R} + \varphi'_1, \quad \nabla^2 \varphi'_1 = 0 \quad (2.35)$$

where R is the distance from the point of observation to the point P : $R = \sqrt{x^2 + y^2 + (z - z_0)^2}$.

The potential at each point of space will depend, first, on the distance from it to the point P , i.e. on R , and, second, on the distance from it to the interface plane, i.e. on z . From considerations of symmetry, instead of the variable z it is convenient to introduce the distance $R' = \sqrt{x^2 + y^2 + (z + z_0)^2}$ from an arbitrary point of space to the point P' symmetrical to P relative to the interface. The coordinates of the point P' are $0, 0, -z_0$. The equation of the interface plane in the variables R and R' will acquire the form

$$R = R' \quad (2.36)$$

while the expressions

$$\varphi_1' = \frac{\lambda q}{\varepsilon_1 R'} \quad \text{and} \quad \varphi_2 = \frac{\mu q}{\varepsilon_2 R} \quad (2.37)$$

where λ and μ are constants, will evidently be the solutions of Eqs. (2.34) and (2.35) having the required symmetry. Indeed, the potential φ_2 in the lower half-space cannot contain a term proportional to $1/R'$ because it would not satisfy Eq. (2.34). Similarly, φ_1' cannot contain a term proportional to $1/R$. Using Eqs. (2.35) and (2.37) in Eq. (2.31) and taking into account Eq. (2.36), we get $(1 + \lambda)/\varepsilon_1 = \mu/\varepsilon_2$. Similarly, from Eq. (2.32) after elementary calculations, we get $1 - \lambda = \mu$. Thus, all our equations will be satisfied if we assume that $\lambda = (\varepsilon_1 - \varepsilon_2)/(\varepsilon_1 + \varepsilon_2)$ and $\mu = 2\varepsilon_2/(\varepsilon_1 + \varepsilon_2)$, i.e. if we assume that

$$\left. \begin{aligned} \varphi = \varphi_1 &= \frac{q}{\varepsilon_1 R} + \frac{(\varepsilon_1 - \varepsilon_2)}{\varepsilon_1(\varepsilon_1 + \varepsilon_2)} \frac{q}{R'} \quad \text{when } z > 0 \\ \varphi = \varphi_2 &= \frac{2}{\varepsilon_1 + \varepsilon_2} \frac{q}{R} \quad \text{when } z < 0 \end{aligned} \right\} \quad (2.38)$$

On the basis of the uniqueness theorem (Sec. 2.3), the expressions obtained are the only solutions of our problem (up to the additive constant in the potential that does not affect the field intensity).

If the entire space is filled with a homogeneous dielectric ($\varepsilon_1 = \varepsilon_2$), then Eq. (2.38) transforms into an elementary expression of the generalized Coulomb law (2.38). When $\varepsilon_1 \neq \varepsilon_2$, the influence of non-homogeneity of the dielectric on the potential φ_1 in the upper half-space is equivalent to the influence of an additional fictitious charge of the magnitude $(\varepsilon_1 - \varepsilon_2)(\varepsilon_1 + \varepsilon_2)^{-1}q$ placed at the point P' symmetrical to P . (Cf. the solution of a similar problem in Sec. 1.13 by the method of images.)

2.5 Direct Calculation of a Field When a Dielectric Is Present (in Very Simple Cases)

1. To determine the influence of polarization of a dielectric on the field intensity, we shall calculate the field directly in two very simple cases. For simplicity, we shall assume that all the charges in the dielectric are secured in place equally firmly and have the same absolute value q . We shall denote the number of charges of each sign in a unit volume of the dielectric by N .

2. Let us consider an elementary derivation of the generalized Coulomb law for a *homogeneous* dielectric. Assume that the entire field is filled with a homogeneous dielectric and the "free" positive charge q_0 is in the homogeneous dielectric at the point O ; it is necessary

to determine the field \mathbf{E} of this charge at the point P at a distance R from it. Under the action of the field \mathbf{E} , the negative charges of the dielectric will shift toward the centre O while the positive charges will move away from it. Owing to the spherical symmetry of the field, the entire space can be divided into concentric spherical layers with their centre at O , and the density of arrangement of the charges in each of them is constant. The field of all the charged spherical layers *external* with respect to the point P equals zero at this point, while the field of all the *internal* layers at the point P is such as if the entire charge of these layers were concentrated at the centre O [see Eq. (1.17)]. What is the total magnitude of the charge of these internal layers, i.e. of the charge inside of the sphere S passing through the point P ?

Since, according to our assumption, all the charges in the dielectric are identical in magnitude and are secured in place with the same strength, then when polarization occurs all the positive charges at the distance R from the centre O would have to move outward over a certain distance δR , and all the negative charges would have to move inward over the same distance. Naturally, the displacement δR diminishes with an increasing distance from O . When polarization occurs, all the *positive* charges in the layer with the thickness δR adjoining the sphere S with the radius R from inside should pass *outward* through the surface of the sphere S . According to our condition, the number of these charges is $4\pi R^2 N \delta R$.

The same number of *negative* charges will evidently pass from outside *into* the sphere S . Thus, the total negative charge δq appearing in the sphere S owing to polarization of the dielectric will be

$$\delta q = -8\pi R^2 q N \delta R$$

The intensity of the field of these charges at the point P , as indicated above, will be such as if all the charges were concentrated at the centre of symmetry O . Hence, the total field intensity at the point P will be

$$E = \frac{q_0 + \delta q}{R^2}$$

On the other hand, the electric moment of a unit volume of dielectric $\mathbf{P} = \sum q_i \mathbf{R}_i$ equalled zero prior to polarization, and after polarization it will become equal to

$$\mathbf{P} = \sum q_i (\mathbf{R}_i \pm \delta \mathbf{R}) = 2Nq\delta \mathbf{R}$$

Thus, the numerical value of the vector \mathbf{P} is

$$P = 2Nq\delta R = -\frac{\delta q}{4\pi R^2}$$

and, therefore,

$$E = \frac{q_0 + \delta q}{R^2} = \frac{q_0}{R^2} - 4\pi P$$

Thus,

$$D = E + 4\pi P = \frac{q_0}{R^2} \quad \text{and} \quad E = \frac{1}{\epsilon} D = \frac{q_0}{\epsilon R^2}$$

Q.E.D.

3. Let us determine the field of a uniformly polarized sphere. Assume that the polarization \mathbf{P} is constant in magnitude and direction at all the points of a sphere having the radius a . When $\mathbf{P} = 0$, the positive

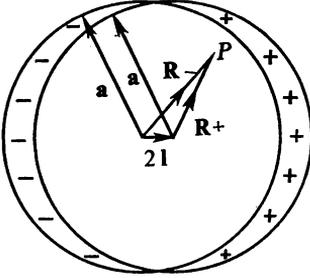


Fig. 30

and negative charges of the dielectric are identically distributed over the volume of the sphere and their fields compensate one another. When polarization appears, the positive charges shift through a certain distance l , and the negative ones through the distance $-l$. Thus, after displacement, the negative charges of the dielectric will fill a sphere having the radius a whose centre is displaced over the distance $2l$ relative to the centre of a sphere of the same radius filled with positive charges (Fig. 30). Consequently, the field of a uniformly polarized sphere should be identical to the field of two spheres having the radius a , displaced over the distance $2l$ relative to each other, and uniformly charged with electricity of opposite signs. Since according to our assumption there are N charges of each sign per unit volume of the dielectric, then the total charge of each sphere in absolute value will be $q' = qNV$, where $V = \frac{4}{3} \pi a^3$ is the volume of the sphere.

The external field of a uniformly charged sphere is such as if the entire charge of the sphere q' were concentrated at its centre [Eq. (1.18)]. Hence, the external field of a polarized sphere is such as if two point charges $\pm q'$ were at a distance of $2l$ from each other, i.e. it is identical with the field of a dipole having the moment $\mathbf{p} = 2q'l = 2VqNl$. Thus, the potential outside of a uniformly polarized sphere having the volume V , according to Eq. (1.49), will be ($R \geq a$)

$$\varphi_e = 2qNV \frac{lR}{R^3}$$

where \mathbf{R} is a radius-vector from the centre of the sphere to the point of the field being investigated.

On the other hand, the electric moment of a unit volume of the sphere $\mathbf{P} = \sum q_i \mathbf{R}_i$ prior to polarization equalled zero; after polarization it will become equal to $\mathbf{P} = \sum q_i (\mathbf{R}_i \pm \mathbf{l}) = 2qN\mathbf{l}$. Hence, we finally have ($R \geq a$)

$$\varphi_e = V \frac{\mathbf{P}\mathbf{R}}{R^3} \quad (2.39)$$

A similar formula is used to determine the potential of the inner points of a sphere ($R \leq a$) if we assume V to be the volume not of the entire sphere, but only of its part that is closer to the centre than the point of the field being considered, i.e. if in Eq. (2.39) we assume that $V = \frac{4}{3} \pi R^3$:

$$\varphi_i = \frac{4\pi}{3} R^3 \frac{\mathbf{P}\mathbf{R}}{R^3} = \frac{4\pi}{3} \mathbf{P}\mathbf{R} \quad (2.40)$$

Indeed, the potential of the field of the charged sphere formed by the positive charges of the dielectric inside this sphere is [Eq. (1.51)]

$$\varphi_+ = 2\pi\rho \left(a^2 - \frac{R_+^2}{3} \right)$$

where $\rho = qN$ is the density of the positive charges in the dielectric. The potential of the sphere formed by the negative charges of the dielectric is

$$\varphi_- = -2\pi\rho \left(a^2 - \frac{R_-^2}{3} \right)$$

Here \mathbf{R}_+ and \mathbf{R}_- are the distances from a point of the field to the centre of the corresponding spheres: $\mathbf{R}_+ = \mathbf{R} - \mathbf{l}$, and $\mathbf{R}_- = \mathbf{R} + \mathbf{l}$. Consequently, the resultant potential of all the charges of the dielectric when $R \leq a$ is

$$\varphi_i = \varphi_+ + \varphi_- = \frac{-2\pi\rho}{3} (R_+^2 - R_-^2)$$

Since $R_+^2 - R_-^2 = -4\mathbf{R}\mathbf{l}$, the potential of the field inside the polarized sphere is

$$\varphi_i = \frac{8\pi\rho}{3} \mathbf{R}\mathbf{l} = \frac{8\pi Nq}{3} \mathbf{R}\mathbf{l} = \frac{4\pi}{3} \mathbf{P}\mathbf{R}$$

Q.E.D.

It is obvious that when $R = a$ the expressions for φ and φ_i take on identical values, i.e. that the potential φ of a polarized sphere is a continuous position function.

Finally, the intensity of the field of a polarized sphere inside it will be ($R \leq a$)

$$\mathbf{E}_i = -\nabla\varphi_i = -\frac{4\pi}{3} \nabla(\mathbf{P}\mathbf{R})$$

Since the vector \mathbf{P} is constant in magnitude and direction, then $\nabla(\mathbf{P}\mathbf{R}) = \mathbf{P}$, and therefore we finally have ($R \leq a$)

$$\mathbf{E}_i = -\frac{4\pi}{3} \mathbf{P} \quad (2.41)$$

Thus, the intensity of the field of a uniformly polarized sphere is constant in magnitude and direction at all its internal points.

Naturally, the problem we have considered could have been solved proceeding directly from the general equations for the field in dielectric media. We invite our reader to acquaint himself with this method of solution, and also to consider the question of how to maintain a uniform polarization in a dielectric sphere in the following example.

Example. A sphere having the radius a made of a homogeneous dielectric is placed in a homogeneous external field E_0 directed along the z -axis. On the basis of the differential equations of the field, it is proved that the sphere will be polarized uniformly, and its polarization will equal

$$P = \frac{3(\varepsilon - 1)}{4\pi(\varepsilon + 2)} E_0 \quad (2.42)$$

while the potential of the field will equal the sum of the potential of the external field $\varphi_0 = -E_0z$ and of the potential of a polarized sphere determined by Eqs. (2.39) and (2.40):

$$\left. \begin{aligned} \varphi_e &= -E_0z + \frac{V\mathbf{P}\mathbf{R}}{R^3} = -E_0z \left(1 - \frac{\varepsilon - 1a^3}{\varepsilon + 2R^3} \right), & R \geq a \\ \varphi_i &= -E_0z + \frac{4\pi}{3} \mathbf{P}\mathbf{R} = -\frac{3E_0z}{\varepsilon + 2}, & R \leq a \end{aligned} \right\} \quad (2.43)$$

To prove the correctness of Eqs. (2.43), it is sufficient to show, first, that φ_e and φ_i satisfy the Poisson equation: $\nabla^2\varphi_e = \nabla^2\varphi_i = 0$, and, second, that on the surface of the dielectric, i.e. when $R = a$, the potential is continuous: $\varphi_e = \varphi_i$, and the normal component of the electric displacement is also continuous:

$$D_{en} = -\frac{\partial\varphi_e}{\partial R} = -\varepsilon \frac{\partial\varphi_i}{\partial R} = D_{in}$$

We shall let our reader do this (when differentiating with respect to R it is convenient to express z through $R \cos \theta$).

It follows from Eq. (2.43) that inside a sphere

$$E_i = \frac{3E_0}{\epsilon + 2}$$

Introducing this value of E_i into Eq. (2.22), we get Eq. (2.42).

2.6 Micro- and Macroscopic Values of Physical Quantities

1. We shall devote this and the following sections up to Sec. 2.10 to a stricter derivation of the equations for a *macroscopic* field in dielectrics from those for a *microscopic* field, and also to determining how the permittivity of a medium depends on its atomistic structure, temperature, etc.

Up to now, we did not give sufficient attention to the circumstance that the field of each molecule of a dielectric in direct proximity to it should exceedingly rapidly change from point to point (for example when passing from the positive to the negative charges of the molecule). True, these changes in the field occur on a microscopic scale and are inaccessible for our macroscopic observation. When measuring, for instance, the field in a liquid dielectric by immersing a test charge such as a sufficiently small charged metal sphere into it, we obviously measure the *average* of the values which the field intensity \mathbf{E} has on the surface of the sphere.

2. To define more precisely the concept of the average or mean value, we shall introduce the following terminology proposed by H. Lorentz. *Unlike mathematically infinitely small quantities (infinitesimals)*, we shall call such elements of volumes, surfaces, and lines *physical infinitesimals* that simultaneously meet the following two requirements:

(a) Physically infinitely small* elements should be exceedingly *great* in comparison with the distances between the molecules of a medium and, consequently, *in comparison with the microscopic heterogeneities of the medium and the field*.

(b) At the same time infinitely small elements should be exceedingly *small in comparison with the macroscopic heterogeneities* of the field and the medium; in other words, the *mean* values of physical quantities (for example φ , \mathbf{E} , ϵ , etc.) in any of these elements should differ by an infinitely small value from the mean values of these quantities in their neighbouring elements**.

* Here and below the concept "infinitely small" is used in its physical meaning.

** Except for elements separated from one another by surfaces of discontinuity, if only in general we shall want to take such surfaces (in essence fictitious ones) into consideration.

Even in gases, not mentioning liquids and solids, the distances between their molecules are so small in comparison with the macroscopic heterogeneities of the fields which are usually studied that it is almost always possible to simultaneously comply with both these conditions. Cases are naturally possible when the above conditions mutually exclude each other; for example, the wavelength of hard X-rays that can be a measure of the heterogeneity of the field of this radiation is less than the distance between the molecules of material bodies.

3. Leaving aside such exclusive cases, in the following by *macroscopic quantities* we shall understand the *mean values of physical quantities in an infinitely small volume*. In other words, by the macroscopic value of an arbitrary physical (scalar or vector) quantity Ψ (for instance φ , \mathbf{E} , ρ , etc.) at a given point of space P we shall understand the mean of the true or microscopic values of this quantity in the infinitely small volume V surrounding the point P :

$$\Psi_{\text{macro}} = \bar{\Psi}_{\text{micro}} = \frac{1}{V} \int \Psi_{\text{micro}} dV \quad (2.44)$$

It should be noted that only by introducing this definition do we impart a definite meaning to all our preceding reasoning. Particularly, the atomistic structure of electricity also manifests itself inside conductors in the exceedingly rapid fluctuations of the microscopic values of physical quantities at adjacent points of space; for instance, the microscopic electric charge density ρ differs from zero only inside electrons and atomic nuclei. Thus, when speaking of the charge density in the surface layer of conductors, of the constancy of the potential inside them, etc., we are in essence speaking of the mean values of these quantities determined by equations similar to Eq. (2.44).

4. In the following, we shall repeatedly have to find equations for macroscopic quantities on the basis of differential equations for microscopic ones. We shall have to use the following rule: the mean value of an arbitrary quantity Ψ with respect to a coordinate (and also with respect to time) equals the derivative of the mean value of this quantity:

$$\frac{\partial \bar{\Psi}}{\partial x} = \bar{\frac{\partial \Psi}{\partial x}} \quad (2.45)$$

where the bar on top denotes the mean value.

We shall prove this theorem assuming that Ψ is a scalar quantity. After next applying the theorem to the separate components of an arbitrary vector, we shall be able to see that it also remains in force for vector quantities. Finally, the corresponding theorem also holds for derivatives with respect to time. That this is true can be seen by simple differentiation of Eq. (2.44).

According to Eq. (2.44), the mean value of Ψ at the point P equals

$$\bar{\Psi}_P = \frac{1}{V} \int_V \Psi dV$$

where for determinacy we shall consider that the integral is taken over the volume V of an infinitely small sphere having the surface area S with its centre at P . Similarly,

$$\bar{\Psi}_{P'} = \frac{1}{V'} \int_{V'} \Psi dV$$

where V' is the volume of an equal sphere having the surface S' with its centre at P' (Fig. 31 shows a section of these spheres by a central plane). Hence,

$$\begin{aligned} \frac{\partial}{\partial x} (\bar{\Psi}_P) &= \lim_{PP' \rightarrow 0} \left(\frac{\bar{\Psi}_{P'} - \bar{\Psi}_P}{PP'} \right) = \\ &= \frac{1}{V} \lim_{dx \rightarrow 0} \left(\frac{\int_{V'} \Psi dV - \int_V \Psi dV}{dx} \right) \end{aligned}$$

But

$$\int_{V'} \Psi dV - \int_V \Psi dV = \int_{V_+} \Psi dV - \int_{V_-} \Psi dV,$$

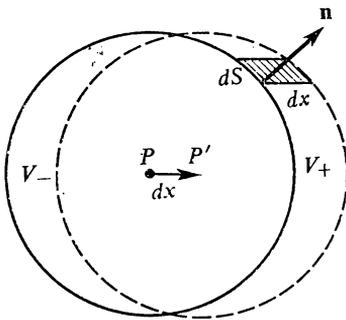


Fig. 31

where V_+ and V_- are, respectively, the volumes of the front and the rear layers confined between the surfaces S and S' . Further, the element of volume dV_+ adjoining the element dS of the sphere S equals, when $\cos(\mathbf{n}, \mathbf{x}) \geq 0$,

$$dV_+ = dS dx \cos(\mathbf{n}, \mathbf{x})$$

where \mathbf{n} is the outward normal to the sphere S , and correspondingly, when $\cos(\mathbf{n}, x) \leq 0$,

$$dV_- = -dS \cos(\mathbf{n}, x)$$

Consequently,

$$\begin{aligned} \int_{V_+} \Psi dV - \int_{V_-} \Psi dV &= \int_{\cos(\mathbf{n}, x) > 0} \Psi \cos(\mathbf{n}, x) dS dx + \\ &+ \int_{\cos(\mathbf{n}, x) < 0} \Psi \cos(\mathbf{n}, x) dS dx = \\ &= dx \oint \Psi \cos(\mathbf{n}, x) dS \end{aligned}$$

where the last integral relates to the entire surface of the sphere S . Thus, we finally get

$$\frac{\partial}{\partial x} (\bar{\Psi}_P) = \frac{1}{V} \oint \Psi \cos(\mathbf{n}, x) dS$$

On the other hand,

$$\frac{\partial \bar{\Psi}}{\partial x} = \frac{1}{V} \int_V \frac{\partial \Psi}{\partial x} dV = \frac{1}{V} \int_V \mathbf{i} \operatorname{grad} \Psi \cdot dV$$

where \mathbf{i} is a unit vector along the x -axis. Owing to the constancy of this vector,

$$\mathbf{i} \operatorname{grad} \Psi = \operatorname{div}(\mathbf{i}\Psi)$$

whence on the basis of Gauss's theorem (A.17), we obtain

$$\begin{aligned} \frac{\partial \bar{\Psi}}{\partial x} &= \frac{1}{V} \int_V \operatorname{div}(\mathbf{i}\Psi) dV = \\ &= \frac{1}{V} \oint_S (\mathbf{n}\mathbf{i}) \Psi dS = \\ &= \frac{1}{V} \oint_S \Psi \cos(\mathbf{n}, x) dS \end{aligned}$$

which coincides with the expression for $\partial(\bar{\Psi})/\partial x$. Thus, we have proved Eq. (2.45).

2.7 Derivation of Equations for the Field in Dielectrics by Averaging the Microscopic Field

1. Developing the ideas set out in the preceding section, we shall now take on the task of deriving equations for the mean *macroscopic* values of the quantities \mathbf{E} , φ , ρ , etc. characterizing a field by the direct averaging of the equations for a true *microscopic* field. In other words, we shall derive the equations which we obtained in a somewhat different way in Sec. 2.3. We shall proceed from the assumption that the fundamental equations for an electrostatic field

$$\mathbf{E} = -\text{grad } \varphi \quad (1.59)$$

$$\nabla^2 \varphi = -\text{div } \mathbf{E} = -4\pi\rho \quad (1.63)$$

are strictly *correct* for a true *microscopic field* if by ρ we understand the true density of electric charges differing from zero only inside separate atoms (more correctly, inside electrons and atomic nuclei)*.

2. It directly follows from Eq. (1.45) that the mean value of the electric vector \mathbf{E} equals the gradient of the mean value of the potential φ , i.e. that Eq. (1.59) also remains true for macroscopic quantities. Equation (1.63) becomes

$$\nabla^2 \bar{\varphi} = -\text{div } \bar{\mathbf{E}} = -4\pi\rho \quad (2.46)$$

where ρ stands for the mean value of the true charge density, i.e. the sum of the densities of the free and bound electric charges:

$$\rho = \rho_f + \rho_b \quad (2.47)$$

The density of the electric charges bound in a dielectric, however, is determined by this polarization, i.e. depends in turn on the field intensity $\bar{\mathbf{E}}$. It will therefore be convenient to use Eq. (2.46) only if we delete the density of the bound electricity from it, for which purpose we must determine how ρ_b depends on the field intensity $\bar{\mathbf{E}}$.

3. Let us separate in a dielectric a certain volume V whose dimensions are great in comparison with the distances between the molecules, using the surface S for this purpose. Let us further assume, for simplicity's sake, that the dielectric consists of neutral molecules and that the volume contains no free charges not bound to the molecules of the dielectric. Generally speaking, the surface S will intersect a certain number of the molecules so that some of the charges of these molecules will be outside of the volume V and the others inside of it. Therefore, notwithstanding the fact that each molecule of the dielectric as a whole is neutral, the total charge of the volume V and, conse-

* These electrons and nuclei are assumed to be at rest (electrostatics!). In other words, from the standpoint of the classical electron theory, we must meanwhile forget about the motion of these elementary particles or, more correctly, determine the *time-average* of the quantities we are interested in. From the standpoint of quantum mechanics, atoms in the normal state are indeed a stationary electrical system.

quently, the mean charge density in it may differ from zero. To determine the magnitude of this charge, let us make matters simpler by replacing the molecules of the dielectrics with equivalent dipoles (Fig. 32) and consider an infinitely small element dS of the surface S .

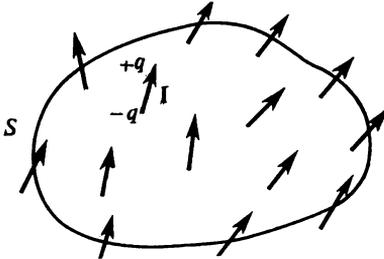


Fig. 32

The mean vector length \mathbf{l} of the dipoles in the infinitely small layer of the dielectric adjoining dS will be related to the polarization \mathbf{P} of this layer by Eq. (2.7):

$$\mathbf{P} = \sum \mathbf{p} = N\mathbf{p} = Nq\mathbf{l}$$

whence

$$\mathbf{l} = \frac{\mathbf{P}}{Nq}$$

The element dS will intersect all the dipoles whose centres are in the layer with the thickness $l|\cos(\mathbf{l}, \mathbf{n})|$ (Fig. 33). The vertical lines in the latter expression denote that the absolute value of the cosine should be taken. The volume of this layer is $dS l|\cos(\mathbf{l}, \mathbf{n})|$.

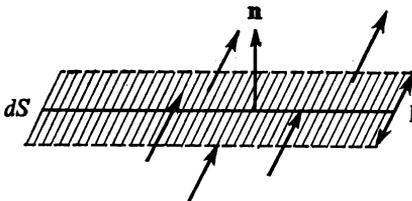


Fig. 33

Consequently, the number of dipoles* cut in half by the element dS equals $Nl dS|\cos(\mathbf{l}, \mathbf{n})|$, while the absolute value of the uncom-

*The number of molecules in a microscopically thin layer can be taken equal to the product of their average number N in a unit volume and the volume of the layer only if the arrangement of the molecules obeys the laws of chance as in gases and liquids. In crystalline substances where the arrangement of the molecules is strictly ordered, the number of molecules in such a layer may appreciably depend on the position of the boundaries of this layer with respect to the points of the crystal lattice. Consequently, strictly speaking, the derivation of Eq. (2.48) requires a more accurate approach for solid dielectrics.

compensated charge in the volume V corresponding to these dipoles is

$$|dq| = NqldS |\cos(\mathbf{l}, \mathbf{n})| = P|\cos(\mathbf{P}, \mathbf{n})| dS = |P_n| dS$$

If the angle between \mathbf{P} and \mathbf{n} (\mathbf{n} is an *outward* normal to dS) is acute, i.e. $\cos(\mathbf{P}, \mathbf{n}) > 0$, then the negative charges of the cut dipoles are inside of the surface S ($dq < 0$); otherwise, when $\cos(\mathbf{P}, \mathbf{n}) < 0$, the charges are positive ($dq > 0$). Hence, the algebraic value of the uncompensated charge dq is

$$dq = -P_n dS$$

4. To determine the total charge q in the volume V , it is obviously sufficient to integrate the expression obtained over the entire boundary surface of this volume S ; next, using Eq. (A.17), we get

$$q = -\oint_S P_n dS = -\int_V \operatorname{div} \mathbf{P} \cdot dV$$

On the other hand, it is evident that

$$q = \int_V \bar{\rho}_b dV$$

so that we finally get Eq. (2.10) which we have already become acquainted with:

$$\bar{\rho}_b = -\operatorname{div} \mathbf{P} \quad (2.48)$$

Thus, the density of the bound charges in a dielectric is determined by the divergence of its polarization. Particularly, if \mathbf{P} is constant, then upon considering, for instance, the surface of a cube, it is easy to see directly that the front and rear (in the direction of \mathbf{P}) faces of the cube are intersected by an identical number of dipoles. Consequently, the algebraic sum of the charges and, therefore, the mean density of the electric charges inside the cube, will equal zero.

It should be noted that according to Eq. (2.15)

$$\operatorname{div} \mathbf{P} = \operatorname{div} \alpha \mathbf{E} = \alpha \operatorname{div} \mathbf{E} + \mathbf{E} \operatorname{grad} \alpha$$

and, consequently, $\operatorname{div} \mathbf{P}$ and ρ_b can differ from zero only if the dielectric is not homogeneous ($\operatorname{grad} \alpha \neq 0$), or if $\operatorname{div} \mathbf{E} \neq 0$. The latter inequality, as follows from Eq. (1.27), means (with $\alpha = \text{const}$) that the dielectric contains free charges not bound to its molecules.

5. Equation (2.48) is based on the assumption, naturally, that the vector \mathbf{P} is continuous; on surfaces of discontinuity of this vector, Eq. (2.48), in accordance with Eqs. (1.29) and (1.30), becomes

$$\bar{\sigma}_b = -\operatorname{Div} \mathbf{P} = -(P_{2n} - P_{1n}) \quad (2.49)$$

where $\bar{\sigma}_b$ = mean surface density of the bound charges on the surface of discontinuity

Div \mathbf{P} = surface divergence of the vector \mathbf{P} .

Since $\mathbf{P} = \alpha\mathbf{E}$, then the surfaces of discontinuity of this vector, i.e. the surface charges of the bound electricity, must coincide either with the interfaces of the two media (a jump in the polarizability α), or with the surfaces of discontinuity of the vector \mathbf{E} . The latter surfaces, as follows from Eq. (2.24), coincide in turn (when $\alpha = \text{const}$) with the surface charges of *free* electricity.

We can also verify the correctness of Eq. (2.49) by direct consideration of the arrangement of the dipoles induced in the molecules of the dielectric. The appearance of surface bound charges is evidently explained by the fact that the charges of the dipoles adjoining opposite sides of the surface of discontinuity of the polarization vector cannot compensate each other.

6. Let us return to Eq. (2.46). Using Eqs. (2.47) and (2.48), we can write it in the following form:

$$\text{div } \bar{\mathbf{E}} = 4\pi(\bar{\rho}_f + \bar{\rho}_b) = 4\pi\bar{\rho}_f - 4\pi \text{div } \mathbf{P}$$

or

$$\text{div } (\bar{\mathbf{E}} + 4\pi\mathbf{P}) = 4\pi\bar{\rho}_f$$

This equation coincides with Eq. (2.18) because according to its meaning by $\bar{\mathbf{E}}$ here we should understand the *mean* intensity of a microscopic field \mathbf{E} , and by $\bar{\rho}_f$ the *mean* density of the *free* electric charges ρ_f .

A system of differential equations for a macroscopic field in a dielectric [system (A), Sec. 2.3] directly follows from Eq. (2.13) and from Eq. (1.59), Q.E.D.

2.8 Two Classes of Dielectrics. Quasi-Elastic Dipoles

1. We shall devote this and the following two sections to the molecular theory of dielectrics that makes it possible to relate the properties of a dielectric to its molecular structure, and also to determine how the permittivity depends on the temperature of the dielectric, its density, etc.

Two kinds or two classes of dielectrics (with neutral molecules) should be distinguished: (1) dielectrics whose molecules are so symmetrically constructed from electric charges that their electric dipole moment equals zero in the absence of an external electric field, and (2) dielectrics whose molecules have a definite moment p_0 in the absence of an external field. The first class includes, for example, the gases N_2 , H_2 , CO_2 , CH_4 , and CCl_4 in the gaseous and liquid

states; the second class includes, for example, the gases SO_2 , H_2S , NH_3 , and the liquids water, nitrobenzene, ethers, esters, and organic acids, which in the majority of cases have a high permittivity.

2. In the presence of an external field \mathbf{E} , molecules of class one dielectrics are polarized, i.e. the symmetry in the arrangement of their charges is violated, and each molecule acquires an electric moment \mathbf{p} whose magnitude (with not too great fields) is proportional to the field intensity:

$$\mathbf{p} = \beta \mathbf{E} \quad (2.50)$$

The factor β is called the *polarizability of molecules*.

As regards class two molecules, the magnitude of their electric moment p_0 will also change under the action of an external field. The rotation of the molecule axis in the direction of the field under the action of the moment of the forces applied to it

$$\mathbf{N} = [\mathbf{p}_0 \mathbf{E}]$$

[cf. Eq. (1.119)], however, will be of much greater significance. In comparison with this rotation of the axis, i.e. with the change in the direction of the vector \mathbf{p}_0 , the change in the numerical value of the moment under the action of the force plays such an insignificant part in virtually all cases that it may be disregarded in general [except for rapidly varying fields (see Sec. 7.11)].

Accordingly, we shall consider only two limiting cases below: (1) molecules whose moment is proportional to the field intensity \mathbf{E} (we shall call them *quasi-elastic* for brevity's sake), and (2) molecules having a non-zero constant moment \mathbf{p}_0 (we shall call them *rigid* molecules). Dielectrics of the second class will be considered in greater detail in Sec. 2.10.

For quasi-elastic molecules, the direction of the moments of all the molecules is parallel to the field, and, consequently, it follows from Eq. (2.50) that

$$\mathbf{P} = \sum \mathbf{p} = N \mathbf{p} = N \beta \mathbf{E} \quad (2.51)$$

where N is the number of molecules per unit volume. Comparing Eq. (2.51) with Eq. (2.15), we arrive at the following relationship between the polarizability of a dielectric α and the polarizability of its molecules β :

$$\alpha = N \beta \quad (2.52)$$

3. As already mentioned above, in the following we shall sometimes, for purposes of simplification, replace a consideration of a complex of the real molecules of a dielectric with a consideration of the complex of dipoles having the same moment \mathbf{p} equivalent to them.

When we have to do with quasi-elastic molecules, the moment of an equivalent dipole will be determined on the basis of Eq. (2.50) by the formula

$$\mathbf{p} = q \mathbf{l} = \beta \mathbf{E} \quad (2.53)$$

Particularly, the length of a dipole should vanish in the absence of an external field, and its charges should neutralize each other, moving apart only in the presence of an external field E .

A dipole whose moment is proportional to the field is called a *quasi-elastic dipole* because Eq. (2.53) is equivalent to the assumption that a quasi-elastic force of mutual attraction

$$F = \kappa l \quad (2.54)$$

proportional to the distance l between the charges of a dipole acts between them.

Indeed, apart from F , the force of the external field $\pm qE$ acts on each charge, and it must balance the force F . Consequently, the length of a quasi-elastic dipole should be determined by the relationship

$$\kappa l = qE$$

where κ is a measure of the elastic forces or the "coefficient of elasticity" of a dipole. Thus, the value of the moment of a dipole should equal

$$p = ql = \frac{q^2}{\kappa} E \quad (2.55)$$

which coincides with Eq. (2.53) if we assume that

$$\beta = \frac{q^2}{\kappa} \quad \text{and} \quad \kappa = \frac{q^2}{\beta} \quad (2.56)$$

4. Each molecule is actually a complicated dynamic system of charges obeying quantum laws. In the relationships we are interested in, however, these molecules are equivalent to a very simple electrostatic model—an elementary dipole (see Sec. 2.1). For the dependence of the moment of an equivalent dipole on the external field to be the same as that of the moment of a real quasi-elastic molecule on this field, we must assume that certain forces of a non-electrostatic origin act between the charges of the dipole which *together* with the Coulomb attraction of these charges are equivalent to the quasi-elastic force. According to the Earnshaw theorem (Sec. 1.19), only the introduction of such non-electrostatic forces corresponding to the concept of constraints in analytical mechanics can ensure the stability of the electrostatic model of a molecule.

2.9 Difference of the Field Acting on a Dipole from the Mean One*

1. In the preceding section in deriving Eq. (2.52), we tacitly permitted an inaccuracy. Namely, in Eq. (2.51)

$$\mathbf{P} = N\mathbf{p} = \alpha\mathbf{E} \quad (2.57)$$

* This section may be omitted when reading the book the first time.

we must evidently understand \mathbf{p} to be the *mean* value of the moments of the dipoles in an infinitely small volume of a dielectric, and \mathbf{E} to be the intensity of the *mean macroscopic* field. To obtain Eq. (2.52), i.e.

$$\alpha = N\beta$$

we introduced Eq. (2.50)

$$\mathbf{p} = \beta\mathbf{E}$$

holding for a separate dipole into Eq. (2.57). Hence, we tacitly assumed that Eq. (2.50) also remains true for the relationship between the *mean* moment of a dipole and the *mean* macroscopic intensity of a field.

By definition, however, β is the proportionality factor between the moment of a dipole equivalent to a molecule and the electric force acting on its charges. This force is determined by the intensity of the *external* (relative to a given dipole) electric field at the place where the dipole is, or, briefly speaking, at the *centre* of the dipole. Consequently, the mean moment of the dipoles should equal

$$\mathbf{p} = \beta\mathbf{E}' \quad (2.58)$$

where \mathbf{E}' is the *mean* intensity of the field at the points where the *centres of the dipoles* are, it being the *external* field with respect to each individual dipole. By \mathbf{E} , however, is meant the mean field intensity in a dielectric, when calculating which we take into consideration the field of *all* the dipoles at *all* the points of the dielectrics, and not only at the centres of the dipoles. Generally speaking, \mathbf{E}' will obviously differ from \mathbf{E} , and Eq. (2.51) should be replaced by the formula

$$\mathbf{P} = N\mathbf{p} = N\beta\mathbf{E}'$$

Since \mathbf{E}' is evidently proportional to \mathbf{E} , then the fundamental equation (2.15) naturally remains in force, but the polarizability of a unit volume of a dielectric α *does not equal* $N\beta$. Letting α_0 stand for the polarizability calculated without account taken of the difference between \mathbf{E}' and \mathbf{E} :

$$\alpha_0 = N\beta \quad \text{and} \quad \mathbf{P} = \alpha_0\mathbf{E}' \quad (2.59)$$

we shall pose the problem of finding the relationship between α and α_0 , for which purpose, in turn, it is necessary to find the relationship between \mathbf{E}' and \mathbf{E} .

2. To determine the field intensity \mathbf{E}' at the centre O of a dipole, let us circumscribe from this centre a sphere S having an infinitely small radius*. The field \mathbf{E}' at the point O will consist, first, of the field

* The result of our calculations obviously cannot depend on the shape of the surface S . We have chosen a spherical shape only for the convenience of the calculations.

\mathbf{E}_1 of all the charges outside of the sphere S , and, second, of the field \mathbf{E}_2 of the charges inside of S except for the charges of the dipole O itself.

If we were to cut out and remove the sphere S from the dielectric, the field in the spherical cavity formed would obviously be equal to the field \mathbf{E}_1 . Since the charges inducing this field are outside of the sphere S , then in determining \mathbf{E}_1 we may disregard the atomistic structure of the dielectric and replace the complex of its molecules with an electric moment of the density \mathbf{P} continuously distributed through its volume. Further, since the sphere has infinitely small dimensions, the polarization \mathbf{P} of the dielectric surrounding the sphere will have a value that is constant in magnitude and direction. Thus, \mathbf{E}_1 equals the intensity of the field in a spherical cavity cut out inside a uniformly polarized dielectric. Prior to removal of the sphere, the field in the uniformly polarized dielectric is homogeneous and equals the mean intensity of the macroscopic field \mathbf{E} . Upon the removal of the sphere S from this field, it will obviously be necessary to subtract the field of the uniformly polarized sphere S whose intensity, according to Eq. (2.41), equals $-\frac{4\pi}{3}\mathbf{P}$. Hence,

$$\mathbf{E}_1 = \mathbf{E} - \left(-\frac{4\pi}{3}\mathbf{P}\right) = \mathbf{E} + \frac{4\pi}{3}\mathbf{P} \quad (2.60)$$

Thus, the field \mathbf{E}_1 is constant over the entire volume of the sphere S and does not depend on its diameter.

As regards the quantity \mathbf{E}' , which we for brevity shall call the intensity of the electric field "acting on a dipole" or of the "effective" electric field, according to what has been said above it equals

$$\mathbf{E}' = \mathbf{E}_1 + \mathbf{E}_2 = \mathbf{E} + \frac{4\pi}{3}\mathbf{P} + \mathbf{E}_2 \quad (2.61)$$

Thus, the problem of determining the effective field \mathbf{E}' consists in determining the mean field \mathbf{E}_2 induced at the centre O of a dipole by the other charges inside of the sphere S . The intensity of this field may appreciably depend on the structure of the dielectric, particularly on the mutual arrangement of its dipoles, and, therefore, strictly speaking, there is no universal dependence of \mathbf{E}_2 on β , N , and \mathbf{E} . With certain very simple assumptions on the structure of a dielectric, however, as we shall now show, the field \mathbf{E}_2 will be found to equal zero.

3. Let us introduce Cartesian coordinates with the centre at O . According to Eq. (1.61), the component E_x of the field induced at the point O by a separate dipole having the coordinates x , y , and z is

$$E_x = \frac{3(\mathbf{pR})_x - p_x R^2}{R^5} = \frac{p_x(2x^2 - y^2 - z^2) + 3x(p_y y + p_z z)}{R^5}$$

Hence, the component of the field \mathbf{E}_2 along the x -axis will be

$$\begin{aligned} \mathbf{E}_{2x} = & \sum p_x \frac{x^2 - y^2}{R^5} + \sum p_x \frac{x^2 - z^2}{R^5} + \\ & + 3 \sum p_y \frac{xy}{R^5} + 3 \sum p_z \frac{xz}{R^5} \end{aligned} \quad (2.62)$$

where summation should extend over all the dipoles within an infinitely small sphere S (except for the dipole at its centre)*.

Let us assume that the dipoles of a dielectric are arranged at the points of a *cubic* space lattice. In this case, the moments \mathbf{p} of all the dipoles inside an infinitely small sphere S will be identical in magnitude and direction, and in Eq. (2.62) we can put p_x , p_y , and p_z outside the summation sign. Let us make the coordinate axes x , y , and z coincide with the principal axes of the crystal. If the coordinates of one of the dipoles inside of S are $x = a$, $y = b$, and $z = c$, then inside of S a dipole will also be found having the coordinates $x = b$, $y = a$, and $z = c$. Hence it follows that $\sum(x^2 - y^2)/R^5 = 0$. In addition, inside of S there will also be a dipole having the coordinates $x = -a$, $y = b$, and $z = c$, whence it follows that $\sum xy/R^5 = 0$, etc. Thus, in this case, indeed, $\mathbf{E}_2 = 0$.

The same result ($\mathbf{E}_2 = 0$) is also obtained with an absolutely chaotic arrangement of the dielectric molecules (a gaseous dielectric). Indeed, if all the positions of any earmarked molecule inside the sphere S are equally probable and do not depend on the position of the other molecules, then the mean value of the field of any molecule at the centre O of the sphere equals zero. It must be borne in mind, however, that the assumption on the absence of any correlation between the positions of different molecules is equivalent to disregarding (1) the finite dimensions of the molecules, and (2) the dipole interaction of the molecules with one another.

4. When $\mathbf{E}_2 = 0$, Eq. (2.61) acquires the form

$$\mathbf{E}' = \mathbf{E} + \frac{4\pi}{3} \mathbf{P} \quad (2.63)$$

Using this in Eq. (2.59), we get

$$\mathbf{P} = \alpha_0 \mathbf{E}' = \alpha_0 \mathbf{E} + \frac{4\pi}{3} \alpha_0 \mathbf{P}$$

* If in determining the mean value of \mathbf{E}_2 we take into account all the dipoles inside the sphere S at *all* points of this sphere and not only at the centre O of the dipole, then this mean value of \mathbf{E}_2 will obviously equal $-\frac{4\pi}{3} \mathbf{P}$, so that $\mathbf{E}_1 + \mathbf{E}_2$ will equal \mathbf{E} .

whence

$$\mathbf{P} = \frac{\alpha_0}{1 - \frac{4\pi}{3} \alpha_0} \mathbf{E} = \alpha \mathbf{E}$$

and, consequently,

$$\alpha = \frac{\alpha_0}{1 - \frac{4\pi}{3} \alpha_0} \quad (2.64)$$

Thus, the fact that the “effective” field \mathbf{E}' differs from the mean field \mathbf{E} affects only the numerical value of the dielectric polarizability. In particular, the relationship between the permittivity ε and the polarizability α_0 is obtained from Eq. (2.21):

$$\varepsilon = 1 + 4\pi\alpha = 1 + \frac{4\pi\alpha_0}{1 - \frac{4\pi}{3} \alpha_0}$$

whence

$$\frac{\varepsilon - 1}{\varepsilon + 2} = \frac{4\pi\alpha_0}{3} \quad (2.65)$$

This equation and Eq. (2.63) are called the *Lorenz-Lorentz equations*. Substituting for α_0 in Eq. (2.65) its value from Eq. (2.59), we get

$$\frac{\varepsilon - 1}{\varepsilon + 2} = \frac{4\pi}{3} N\beta \quad (2.66)$$

When $\alpha_0 \ll 1$ (i.e. with ε close to unity), we may assume that α equals α_0 and \mathbf{E}' equals \mathbf{E} , and we return to the previous equations which are thus found to be quite correct for only slightly polarizing dielectrics. This is quite understandable because in slightly polarizing dielectrics the influence of the dipoles on one another does not play an appreciable part. At the same time, it is exactly the interaction between the dipoles of a dielectric taken into consideration by the second term of Eq. (2.63) that underlies, in essence, the difference between \mathbf{E}' and \mathbf{E} .

5. The corollaries following from the Lorenz-Lorentz equation (2.65) will be treated in Sec. 2.10. Meanwhile we shall note once more that its derivation is based on a number of assumptions. Particularly, when replacing the field of a separate molecule or atom of a dielectric with the field of an equivalent dipole [Eq. (2.62)], we disregard the fact that this replacement is valid only for distances noticeably exceeding the dimensions of an atom (see Sec. 2.1), whereas in solid and liquid dielectrics the distances between adjacent atoms are compatible with the dimensions of these atoms.

Thus, it follows from the preceding material only that it is necessary to take into account the difference between the mean macroscopic field \mathbf{E} and the mean effective field \mathbf{E}' . As regards the Lorenz-Lorentz equation (2.65), although its correctness has been strictly proved only with quite special assumptions, it is nevertheless, as we shall see in Sec. 2.10, well justified in practice when applied to liquid dielectrics with *quasi-elastic dipoles* [for gaseous media ϵ is so close to unity that Eq. (2.65) is virtually identical to Eq. (2.21) with $\alpha = \alpha_0$].

For the impossibility of applying the Lorenz-Lorentz equation to dielectrics with rigid dipoles see the following section.

2.10 Polarization of Dielectrics Whose Molecules Have a Constant Electric Moment. Temperature Dependence of Permittivity

1. As we have mentioned in Sec. 2.8, if the molecules of a dielectric have an electric moment p_0 even in the absence of an external field, then the change in the numerical value of this moment p_0 under the action of the field may be ignored. The polarization of these dielectrics is due to the circumstance that the forces of the electric field tend to turn the axes of the molecules in the direction of the vector \mathbf{E} , i.e. tend to ensure an ordered orientation of the electric moments of the molecules. The polarization of a dielectric, however, does not immediately reach a maximum corresponding to arrangement of all the molecules in the direction of the field, but grows in proportion to the field \mathbf{E} because the chaotic thermal motion (in particular rotation) of the molecules and their collisions with one another tend to violate the ordered orientation of their axes, i.e. tend to depolarize the dielectric. Thus, the actual magnitude of polarization is determined by the relationship between the ordering action of the field and the opposite action of thermal motion. The polarization of dielectrics of this class will evidently sharply diminish with increasing temperature.

2. Let a unit volume of a dielectric contain N molecules having an electric moment p_0 that is constant in magnitude. Let us circumscribe a sphere of a unit radius around an arbitrary point of the dielectric and introduce polar coordinates on it: θ — the polar angle and α — the longitude, having selected the polar axis parallel to the vector \mathbf{E} . The direction of the moment p_0 of an arbitrary molecule will be characterized by the coordinates θ and α of the “trace” of the axis of this molecule on our unit sphere, i.e. by the coordinates of the point of intersection of the surface of this sphere with the continuation of the axis of the molecule (Fig. 34). If the field \mathbf{E} were absent, the axes of the molecules would be uniformly distributed in all directions. This means that the number of molecules, the trace of whose axis is on a given element of the unit sphere, would be

proportional to the size of this element. Particularly, the number dN of molecules having the polar angle of their axes within the limits from θ to $\theta + d\theta$, i.e. whose traces are on the spherical belt of the surface $2\pi \sin \theta d\theta$ contained between the two parallel circles θ and $\theta + d\theta$ ought to equal

$$dN = c \sin \theta d\theta \quad (2.67)$$

where c is a certain constant factor.

To determine the distribution of the axes of the molecules in the presence of an external field \mathbf{E} orienting them, we must turn to a

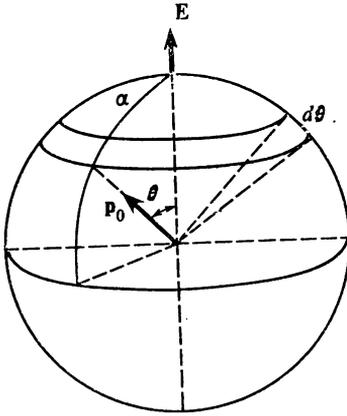


Fig. 34

well-known theorem of statistical mechanics, the so-called *Boltzmann theorem*. According to the latter, in conditions of thermodynamic equilibrium, the law of distribution of molecules in the presence of a conservative force field (in our case an electrostatic field) differs from the law of their distribution in the absence of this field by the factor $\exp(-V/kT)$, where V is the potential energy of a molecule in the force field being considered, T is the absolute temperature, and k is the Boltzmann constant equal to 1.38×10^{-23} J/K or 1.38×10^{-16} erg/K.

In our case, the potential energy of a dipole molecule in an electric field of forces equals, according to Eq. (1.103)

$$V = -p_0 E \cos \theta$$

Hence, the distribution of the molecules will be determined, in view of Eq. (2.67), by the equation

$$\begin{aligned} dN &= c \exp\left(\frac{p_0 E \cos \theta}{kT}\right) \sin \theta d\theta = \\ &= c \exp(a \cos \theta) \sin \theta d\theta \end{aligned} \quad (2.68)$$

where dN is the number of molecules the angles of whose axes with the direction of the field are within the limits from θ to $d\theta$, and where

$$a = \frac{p_0 E}{kT} \quad (2.69)$$

Thus, the deviation of the distribution of molecules from a uniform one determined by the value of the factor $\exp\left(\frac{p_0 E}{kT} \cos \theta\right)$ is the greater, the higher is the field intensity and the lower is the temperature. This is quite understandable because an increase in the temperature is attended by a growth in the energy of thermal motion that disrupts the ordering of distribution.

3. The difference between the field E' acting on a dipole (Sec. 2.9) and the mean macroscopic field E may be disregarded absolutely for gases and dilute solutions of dipole liquids. At normal temperatures, the quantity a in all practically accessible external fields (for molecular fields see the end of the present section) is considerably smaller than unity. Consequently, we may with sufficient accuracy replace the factor $\exp(a \cos \theta)$ with the first two terms of its expansion into a series by the powers of a . Equation (2.68) thereupon becomes (when $a \ll 1$)

$$dN = c(1 + a \cos \theta) \sin \theta d\theta$$

The proportionality constant c can be found from the condition that the total number of all the molecules in a unit volume should equal N :

$$\int_0^\pi dN = \int_0^\pi c(1 + a \cos \theta) \sin \theta d\theta = 2c = N$$

Consequently, Eq. (2.68) can finally be written as follows (when $a \ll 1$):

$$dN = \frac{N}{2} (1 + a \cos \theta) \sin \theta d\theta \quad (2.70)$$

Thus, knowing the directional distribution of the axes of the molecules, we can easily also determine their resultant electric moment, i.e. the polarization of the dielectric \mathbf{P} . The vector \mathbf{P} is parallel to the field intensity \mathbf{E} , therefore its numerical value should equal the sum of the projections of the moments of all N molecules onto the direction \mathbf{E} . The total moment of dN molecules whose axes are between θ and $\theta + d\theta$ equals $p_0 dN$, while the projection of this moment onto the direction \mathbf{E} equals

$$p_0 dN \cos \theta$$

Consequently, the polarization of the dielectric equals

$$P = \int p_0 \cos \theta dN = \frac{p_0 N}{2} \int_0^\pi \cos \theta (1 + a \cos \theta) \times \\ \times \sin \theta d\theta = \frac{N p_0 a}{3}$$

or, after introducing the value of a from Eq. (2.69), we have

$$P = \frac{N p_0^2}{3kT} E = \alpha E \quad (2.71)$$

where

$$\alpha = \frac{N p_0^2}{3kT} \quad (2.72)$$

It must be noted that the above calculations are true with an accuracy up to the second order of magnitude with respect to the parameter a inclusively, and not to the first order as appears at first sight. Indeed, if we retain the quadratic terms in the expansion of the function $c \exp(a \cos \theta)$, and then calculate the normalization constant c in the conventional way, then instead of Eq. (2.70) we shall obtain

$$dN = \frac{N}{2 \left(1 + \frac{a^2}{6}\right)} \left(1 + a \cos \theta + \frac{a^2 \cos^2 \theta}{2}\right) \sin \theta d\theta$$

or with the same accuracy (with $a \ll 1$)

$$dN = \frac{N}{2} \left(1 + a \cos \theta + \frac{a^2 \cos^2 \theta}{2} - \frac{a^2}{6}\right) \quad (2.73)$$

After this, the integral

$$\frac{p_0 N}{2} \int_0^\pi \cos \theta \left(\frac{a^2 \cos^2 \theta}{2} - \frac{a^2}{6}\right) \sin \theta d\theta$$

will be added in the expression for P and the integral will evidently vanish. We shall take advantage of this remark in Sec. 2.12 when dealing with the question of the energy of gaseous dielectrics with rigid dipoles. Thus, taking into account the quadratic terms does not change the final expression for P .

We shall note once more that the possibility of applying Eqs. (2.71) and (2.72) is limited by the condition

$$a = \frac{p_0 E}{kT} \ll 1 \quad (2.74)$$

When this condition is violated in very strong fields or at very low temperatures, P stops growing proportionally to E and approaches its maximum possible value corresponding to arrangement of all the dipoles in the direction of the field. The magnitude of this "saturation polarization" obviously equals

$$P_{\text{sat}} = Np_0^*$$

4. We shall now take into consideration the circumstance that in actual conditions the molecules of all bodies regardless of whether or not they have a certain constant electric moment p_0 also have a certain quasi-elastic polarizability β . Therefore, the polarization of a dielectric with rigid dipoles consists of the quasi-elastic polarization [Eq. (2.52)] and of the polarization corresponding to ordering of the orientation of the rigid dipoles [Eq. (2.72)], i.e. is determined by the expression

$$\varepsilon - 1 = 4\pi\alpha = 4\pi N \left(\beta + \frac{1}{3} \frac{p_0^2}{kT} \right) \quad (2.75)$$

To determine how the permittivity ε depends on the temperature T and the density τ of a dielectric, let us express the number N of molecules in a unit volume through its density τ , its molecular weight M , and the Avogadro constant $N_A = 6.06 \times 10^{23}$, equal to the number of molecules in a gramme-molecule. Since τ and M equal, respectively, the mass of a unit volume and the mass of a gramme-molecule, then

$$N = \frac{\tau N_A}{M} \quad (2.76)$$

Introducing this expression into Eq. (2.75), we get

$$\frac{\varepsilon - 1}{\tau} = \frac{4\pi N_A}{M} \left(\beta + \frac{1}{3} \frac{p_0^2}{kT} \right) \quad (2.77)$$

We can clearly see from inspection of this equation that the polarization of rigid dipoles is not an intramolecular process as with quasi-elastic dipoles, but is determined by the relationship between the orienting action of an electric field and the disorienting action of thermal motion. Namely, the permittivity of a dielectric with quasi-elastic dipoles ($p_0 = 0$) having a constant density τ does not at all depend on the temperature**, whereas in dielectrics with rigid

* The polarization of substances with rigid dipoles, in strong fields will be considered in greater detail in Sec. 5.13 in connection with the magnetizing of para- and ferromagnetics.

** When considering dielectrics with elastic dipoles, we took no account of the thermal motion of the molecules, which, naturally, is permissible in a first approximation. It can be shown, however, that even upon more exact consideration of the question, the mean moment of each molecule $p = \beta E$ does not depend on the temperature. See, for instance, P. Debye, *Polar Molecules*. New York, Dover (1945).

dipoles the fraction of polarizability $(\epsilon - 1)/4\pi$ due to these dipoles at constant τ is inversely proportional to the absolute temperature.*

The dependence of ϵ on T and τ expressed by Eq. (2.77) is well confirmed in experiments both for dielectrics with quasi-elastic dipoles (N_2 , H_2 , CO_2 , CH_4 , CCl_4 , etc.) and for dielectrics with rigid dipoles (H_2O , SO_2 , HCl , NH_3 , etc.) if the dielectrics being studied are in the gaseous state (for liquids see below). Particularly, using Eq. (2.77), we can calculate the value p_0 of the electric moment of a dielectric molecule according to the measured dependence of ϵ on T ; for most molecules with rigid dipoles it is of the order of magnitude of 10^{-18} absolute (electrostatic) units. The same results are obtained upon the direct determination of the moment of separate molecules based on measuring the deviation of a beam of molecules flying into a non-uniform electric field in a vacuum (Stern's method).

5. It must be noted that Eqs. (2.72) and (2.77) do not take into consideration the difference between the mean value of the electric field intensity E and the mean value E' of the field acting on the molecules of a dielectric (Sec. 2.9). These equations can therefore be applied only for weakly polarizing media ($a \leq 1$). For dielectrics with quasi-elastic dipoles, when we take into account the difference between the effective field E' and the mean field E , we arrive at the Lorenz-Lorentz equation (2.65):

$$\frac{\epsilon - 1}{\epsilon + 2} = \frac{4\pi}{3} N\beta = \frac{4\pi N_A \tau}{3M} \beta \quad (2.78)$$

which is well confirmed for dielectric liquids of this class (whose polarizability α is not small in comparison with unity).

At one time, it was assumed that the difference between the effective field E' and the mean field E is expressed by Eq. (2.63) not only for dielectrics with quasi-elastic dipoles, but also for those with rigid dipoles. It followed from this assumption that when α differs considerably from unity, Eq. (2.77) should be replaced by the equation

$$\frac{\epsilon - 1}{\epsilon + 2} = \frac{4\pi N_A \tau}{3M} \left(\beta + \frac{p_0^2}{3kT} \right) \quad (2.79)$$

which is a generalization of the Lorenz-Lorentz equation (2.66) and can be applied to dielectrics of any class.

The results of experimental investigation of dipole liquids (water, alcohols, ethers, esters, etc.), however, do not at all agree with Eq.

* In a later section (Sec. 7.11), we shall see that dielectrics of these two kinds also appreciably differ in their behaviour in fast-varying electric fields.

(2.79)*. This is quite understandable because the initial Lorentz relationship (2.63) between the effective and the mean fields E' and E used for this equation cannot be applied to dielectrics with rigid dipoles. Indeed, each molecule (dipole) in dielectrics of this class is subjected to a very strong orienting action on the part of the adjacent molecules (dipoles). Although in the absence of an external electric field, the axes of the molecules are distributed uniformly in all directions, there is a definite correlation, however, between the directions of the axes of *neighbouring* molecules. This correlation also changes only slightly in an external field, which is not taken into account by Eq. (2.72)**.

6. We shall note in conclusion that the electrical interaction of *neighbouring* ions plays an appreciable part in the peculiar properties of *ferroelectrics*. They include, first, crystals of Seignette (Rochelle) salt and substances related to it, which are complex organic compounds, and, second, a number of crystals of the type KH_2PO_4 . Within a definite temperature interval, limited for crystals of the first kind by two Curie points from the sides of both high and low temperatures (for crystals of the second kind, the lower Curie point is absent), ferroelectrics have anomalous electrical properties similar to the magnetic properties of ferromagnetics. For instance, within this temperature interval, the permittivity of ferroelectrics in weak electric fields reaches values having the order of magnitude of several thousands; with a growth in the field, ϵ rapidly drops, while the polarization of the ferroelectrics tends to saturation, etc.

2.11 Energy of the Electric Field in Dielectrics

1. In Sec. 1.15, we derived expression (1.101) for the energy of an electric field in the absence of dielectrics:

$$U = \frac{1}{2} \int \rho \varphi dV + \frac{1}{2} \int \sigma \varphi dS \quad (2.80)$$

This formula also holds for an electric field in an arbitrary medium if by ρ and σ we understand the density of *free* charges. The influence of the dielectric tells in that with the same distribution of the free

* An equation similar to Eq. (2.79) may be approximately applied for greatly diluted solutions of substances having rigid dipoles in non-dipole liquids, namely

$$\frac{\epsilon - 1}{\epsilon + 2} = \frac{4\pi}{3} \left(N\beta \frac{N'p_0^2}{3kT} \right)$$

where N and N' signify the number of solvent and solute molecules in a unit volume of the solution, respectively, β is the polarizability of the solvent molecules, and p_0 is the electric moment of the solute molecules. The condition for applying this equation, however, is the small value of the dipole term (the second term in the parentheses).

** See C. Kittel, *Introduction to Solid State Physics*. New York, Wiley (1971), Chap. 7 and the references to it.

charges the value of the potential φ in a dielectric differs from its value in a vacuum. Particularly, with the same distribution of the free charges, the potential φ and at the same time, according to Eq. (2.80), the energy U in a *homogeneous* dielectric is $1/\epsilon$ -th of that in a vacuum.

2. To prove the correctness of Eq. (2.80) when dielectrics are present, it would be necessary to calculate the work W of the forces of a field upon movements of both the free charges and the dielectrics themselves and show that

$$W = -dU \quad (2.81)$$

The proof of the correctness of this relationship with arbitrary movements of the free charges, but with the dielectrics remaining stationary, is similar to the reasoning that led us in Sec. 1.15 to Eq. (1.101). Considering the motion of the charge q_1 in the field of the charge q_2 , we get Eq. (1.97):

$$U = q_1\varphi_1$$

where φ_1 is the potential of the field of the charge q_2 at the point where the charge q_1 is. Considering the motion of the charge q_2 , we get

$$U = q_2\varphi_2$$

Both expressions for U should obviously be equal*:

$$q_1\varphi_1 = q_2\varphi_2 \quad (2.82)$$

Thus, we again get Eqs. (1.98), (1.99), and (1.101), Q.E.D.

* We proved the correctness of this "reciprocity relationship" in the absence of dielectrics in Sec. 1.15 by substituting for φ its expression q/R . The correctness of Eq. (2.82) of an arbitrary medium can be proved as follows.

Let $\varphi^{(k)}$ and $\mathbf{D}^{(k)}$ stand for the potential and the displacement of the field induced by the charge q_k . Let us separate infinitely small volumes V_1 and V_2 around each of the charges q_1 and q_2 and denote the surfaces confining these volumes by S_1 and S_2 . In the entire space V' outside of V_1 and V_2 we have

$$\operatorname{div} \mathbf{D}^{(1)} = \operatorname{div} \mathbf{D}^{(2)} = 0$$

and, consequently,

$$\varphi^{(2)} \operatorname{div} \mathbf{D}^{(1)} = \varphi^{(1)} \operatorname{div} \mathbf{D}^{(2)}$$

or according to Eq. (A.43)

$$\operatorname{div} (\varphi^{(2)} \mathbf{D}^{(1)}) - \mathbf{D}^{(1)} \nabla \varphi^{(2)} = \operatorname{div} (\varphi^{(1)} \mathbf{D}^{(2)}) - \mathbf{D}^{(2)} \nabla \varphi^{(1)} \quad (a)$$

Since

$$\begin{aligned} -\mathbf{D}^{(1)} \nabla \varphi^{(2)} &= \mathbf{D}^{(1)} \mathbf{E}^{(2)} = \epsilon \mathbf{E}^{(1)} \mathbf{E}^{(2)} = \\ &= \mathbf{E}^{(1)} \mathbf{D}^{(2)} = -\mathbf{D}^{(2)} \nabla \varphi^{(1)} \end{aligned}$$

Eq. (a) is equivalent to

$$\operatorname{div} (\varphi^{(2)} \mathbf{D}^{(1)}) = \operatorname{div} (\varphi^{(1)} \mathbf{D}^{(2)}) \quad (b)$$

Integrating both sides of Eq. (b) over the entire space V' outside of volumes V_1 and V_2 , we get on the basis of Gauss's theorem (A.17)

$$\oint_{S_1+S_2} \varphi^{(2)} D_n^{(1)} dS = \oint_{S_1+S_2} \varphi^{(1)} D_n^{(2)} dS \quad (c)$$

Naturally, Eq. (1.100) is no longer true in the presence of dielectrics because $\varphi_1 \neq q_2/R_{12}$.

3. For an exhaustive substantiation of Eq. (2.80), it would be necessary to calculate also the work W of the forces of a field upon arbitrary movements of the dielectrics and convince ourselves that it also equals $-dU$. A strict performance of these calculations, however, would not only be very intricate, but also could not be executed without definite assumptions on the structure and properties of dielectrics. We shall therefore satisfy ourselves with the above substantiation of Eq. (2.80) and reverse the problem, i.e. we shall consider this expression for the energy [or, more exactly, expression (2.83)—see below] in the following as one of the *postulates of the macroscopic theory* of a field whose corollaries are justified experimentally. Particularly, we shall use Eq. (2.81) to determine the work of the forces of a field upon the movements of dielectrics *on the basis* of Eq. (2.80), for energy assuming it to be given.

Apart from this, we shall show in this and the following sections that in the very simple cases when it is not difficult to directly calculate the work of the forces of a field upon the movements of dielectrics, the results of these calculations coincide with the corollaries ensuing from Eq. (2.80).

On the infinitely small surface S_1 surrounding the charge q_1 , the potential $\varphi^{(2)}$ of the charge q_2 may be considered constant. Denoting it by φ_1 , we get on the basis of Eq. (2.19)

$$\oint_{S_1} \varphi^{(2)} D_n^{(1)} dS = \varphi_1 \oint_{S_1} D_n^{(1)} dS = -4\pi q_1 \varphi_1$$

(the minus sign appears because the outward normal to S_1 relative to the volume of integration V' is directed toward the charge q_1 instead of away from it). Further, the surface S_2 can be chosen so that the potential $\varphi^{(2)}$ on it would have a constant value. Hence,

$$\oint_{S_2} \varphi^{(2)} D_n^{(1)} dS = \varphi^{(2)} \oint_{S_2} D_n^{(1)} dS = 0$$

because there are no charges inducing the field $\mathbf{D}^{(1)}$ inside of S_2 . Thus,

$$\oint_{S_1+S_2} \varphi^{(2)} D_n^{(1)} dS = -4\pi q_1 \varphi_1$$

In an absolutely similar way, we get

$$\oint_{S_1+S_2} \varphi^{(1)} D_n^{(2)} dS = -4\pi q_2 \varphi_2$$

Introducing this into Eq. (c), we get the reciprocity relationship (2.82).

4. We have already mentioned in Sec. 1.16 that the expression for the electrical energy (2.80) corresponds in its form to the notion of the interaction of charges at a distance. As for the case when dielectrics are absent, however, this expression can be transformed so that in accordance with the notions of the theory of short-range interaction, the field may be considered as distributed with a definite volume density u over the entire space in which the field differs from zero. Indeed, the integrand of the first of the integrals in Eq. (2.80), on the basis of Eqs. (2.18), (A.43₂), and (1.59) can be written in the following form:

$$\begin{aligned}\rho\varphi &= \frac{1}{4\pi} \varphi \operatorname{div} \mathbf{D} = \frac{1}{4\pi} \{ \operatorname{div} (\varphi \mathbf{D}) - \mathbf{D} \operatorname{grad} \varphi \} = \\ &= \frac{1}{4\pi} \{ \operatorname{div} (\varphi \mathbf{D}) + \mathbf{DE} \}\end{aligned}$$

whence on the basis of Gauss's theorem (A.17), we have

$$\begin{aligned}\frac{1}{2} \int \rho\varphi dV &= \frac{1}{8\pi} \int \mathbf{DE} dV + \frac{1}{8\pi} \int \operatorname{div} (\varphi \mathbf{D}) dV = \\ &= \frac{1}{8\pi} \int \mathbf{DE} dV + \frac{1}{8\pi} \oint D_n \varphi dS\end{aligned}$$

The last integral must be extended, first, over the surface S limiting the volume of integration V , and, second, over the surfaces S'_1 separating from this volume the surfaces of discontinuity of the integrand, i.e. the surfaces of discontinuity of the normal component of the vector \mathbf{D} (because the potential φ should be continuous since we are not considering electrical double layers). If we are considering the *total field*, then the integral over the surface S confining it vanishes (p. 92). The surfaces of discontinuity of the normal component of the vector \mathbf{D} , however, are surfaces, that are charged with free electricity, and the jump of this component D_n is determined by Eq. (2.24). Contracting the surfaces S'_1 in the usual way up to their complete contact with the surfaces of discontinuity S_1 (see p. 73), we get the equation

$$\begin{aligned}\lim_{S'_1 \rightarrow S_1} \frac{1}{8\pi} \int_{S'_1} D_n \varphi dS &= \frac{1}{8\pi} \int_{S_1} \varphi (D_{1n} - D_{2n}) dS = \\ &= -\frac{1}{2} \int_{S_1} \varphi \sigma dS\end{aligned}$$

Thus

$$\frac{1}{2} \int \rho\varphi dV = \frac{1}{8\pi} \int \mathbf{DE} dV - \frac{1}{2} \int \sigma\varphi dS,$$

and, consequently, the energy of the *total field*, according to Eq. (2.80), equals

$$U = \frac{1}{8\pi} \int \mathbf{DE} \, dV \quad (2.83)$$

This expression can be interpreted in the sense that the energy of an electric field is distributed over the entire space it occupies with a volume density equal to

$$u = \frac{1}{8\pi} \mathbf{DE} = \frac{\epsilon}{8\pi} E^2 \quad (2.84)$$

This equation is one of the fundamental formulas of the theory of electricity. Equation (1.109) is a particular case of it when $\epsilon = 1$.

The expressions (2.80) and (2.83) are equivalent *only* for constant electric fields and, as we shall see in the following, are violated for variable fields. A variable electric field in general cannot be characterized by a single-valued scalar potential φ , and for this reason Eq. (2.80), which includes φ , loses its meaning for a variable field. On the other hand, Eq. (2.83) for the energy of an electric field remains true for variable fields, and this is why it should be considered as the fundamental definition of the energy of an electric field.

2.12 Energy Transformations Connected with the Polarization of Dielectrics. Free Energy of an Electric Field

1. According to Eq. (2.84), at a given intensity \mathbf{E} , the energy of the field in a dielectric is ϵ times greater than that of the field in a vacuum. If the carrier of electric energy, however, is an electric field, as is assumed by the theory of short-range action, then this energy should seem to depend only on the field intensity, and not on the properties of the medium in the field. Particularly, it should seem that with an identical field intensity its energy should be the same both in a perfect vacuum and when the vacuum contains separate molecules of a dielectric. There are, however, two reasons for which the density of the field energy in a dielectric depends on the intensity of this field \mathbf{E} in a different way than in a vacuum.

First, according to the macroscopic theory, by \mathbf{E} we understand the *mean* intensity of an electric field (see Sec. 2.6): $\mathbf{E} = \overline{\mathbf{E}}_{\text{micro}}$. Since the mean density of electric energy is

$$u_{e1} = \frac{1}{8\pi} \overline{E}_{\text{micro}}^2 \quad (2.85)$$

and since the mean square of the field intensity, generally speaking, does not equal the square of the mean intensity

$$\overline{E}_{\text{micro}}^2 \neq (\overline{E}_{\text{micro}})^2 = E^2$$

then the true density u_{e1} of the electric energy in a dielectric is not expressed by Eq. (1.109):

$$u_{e1} \neq \frac{1}{8\pi} E^2$$

Second, by the energy of a field is meant the entire energy that must be spent to induce the field (or, which is the same, the entire energy that is liberated when the field vanishes). In the presence of dielectrics, not all of this energy is electric energy in the direct meaning of this word.

2. Indeed, let us first consider a dielectric with *quasi-elastic dipoles*. The appearance of a field in such a dielectric is inseparably linked with the polarization of its molecules, i.e. with spreading apart of the charges of the dipoles equivalent to molecules. According to the basic assumption, forces of both an electrostatic and a *non-electrostatic* origin act between the charges of each dipole. Their resultant is the quasi-elastic force of mutual attraction of these charges. Hence, upon spreading apart the charges of a dipole over the distance l , the forces of an electric field do a definite work, while the dipole acquires the elastic energy $\kappa l^2/2$ equal to this work (here κ is a measure of the elastic forces or the "coefficient of elasticity" of a dipole). According to Eqs. (2.56) and (2.53), we have

$$\frac{\kappa l^2}{2} = \frac{q^2}{2\beta} \left(\frac{\beta E'}{q} \right)^2 = \frac{1}{2} \beta E'^2 = \frac{1}{2} p E' \quad (2.86)$$

where E' is the intensity of the field acting on a dipole. The internal energy of a polarized molecule will have the same value. The total quasi-elastic energy u_q of all the molecules in a unit volume of the dielectric will be

$$u_q = \frac{1}{2} N p \bar{E}' \quad (2.87)$$

Since by the energy of an electric field we understand the entire energy that must be spent to induce the field, then the total energy u of the field in a dielectric should evidently be taken equal to the sum of the electric energy proper u_{e1} and the elastic energy u_q of the polarized dipoles inseparably linked with it:

$$u = u_{e1} + u_q = \frac{1}{8\pi} \bar{E}_{\text{micro}}^2 + \frac{1}{2} N p \bar{E}'$$

If we disregard the difference between \bar{E}_{micro} , E' , and the mean intensity of a macroscopic field E and take into account that $P = Np$, we get

$$u_q = \frac{1}{2} N p E = \frac{1}{2} P E = \frac{1}{8\pi} (\epsilon - 1) E^2$$

and

$$u = u_{e1} + u_q = \frac{1}{8\pi} E^2 + \frac{1}{8\pi} (\epsilon - 1)E^2 = \frac{\epsilon E^2}{8\pi}$$

which coincides with Eq. (2.84).

We have given this derivation of formula (2.84) owing to its great clarity. But it is absolutely unlawful in essence to disregard the difference between E_{micro} , E' , and E . We shall therefore give a stricter proof of the fact that the energy of the field (2.84) in a dielectric with quasi-elastic molecules equals the sum of the electric energy proper of the field and the elastic energy stored in the polarized molecules of the dielectric.

3. Let us separate an infinitely small sphere S inside a dielectric and resolve the field $\mathbf{E}_{\text{micro}}$ into two components—the field \mathbf{E}_1 of the charges outside of the sphere S and the field \mathbf{E}_{in} of the dipoles of the dielectric inside of S . The mean density of the energy inside of S will be

$$\begin{aligned} u &= \frac{1}{8\pi} \overline{E_{\text{micro}}^2} = \frac{1}{8\pi} \overline{(\mathbf{E}_1 + \mathbf{E}_{\text{in}})^2} = \\ &= \frac{1}{8\pi} (\overline{\mathbf{E}_1^2} + 2\overline{\mathbf{E}_1 \mathbf{E}_{\text{in}}} + \overline{\mathbf{E}_{\text{in}}^2}) \end{aligned}$$

The field \mathbf{E}_1 of the external charges, according to Eq. (2.60), is constant over the entire sphere S , and

$$\overline{\mathbf{E}_1} = \mathbf{E}_1 = \mathbf{E} + \frac{4\pi}{3} \mathbf{P}$$

Further,

$$\overline{\mathbf{E}_{\text{in}}} = \mathbf{E} - \mathbf{E}_1 = -\frac{4\pi}{3} \mathbf{P}$$

Consequently, the density of the energy of the field \mathbf{E}_1 is

$$u_{11} = \frac{1}{8\pi} \overline{\mathbf{E}_1^2} = \frac{1}{8\pi} \left(\mathbf{E} + \frac{4\pi}{3} \mathbf{P} \right)^2$$

and the density of the mutual energy of the fields \mathbf{E}_1 and \mathbf{E}_{in} is

$$u_{12} = \frac{1}{4\pi} \overline{\mathbf{E}_1 \cdot \mathbf{E}_{\text{in}}} = \frac{1}{4\pi} \mathbf{E}_1 \cdot \overline{\mathbf{E}_{\text{in}}} = -\frac{1}{3} \mathbf{P} \left(\mathbf{E} + \frac{4\pi}{3} \mathbf{P} \right)$$

As regards the energy of the field \mathbf{E}_{in} of the dipoles in S , the total energy of this field equals the sum of (1) the mutual energy of the dipoles; (2) the proper energy of the positive and negative charges in the dipoles, and, finally, (3) the energy of the interaction by pairs of the charges in the same dipole. According to the initial fundamentals of the theory, the complex of the electrostatic and non-electrostatic

interaction of the charges of each dipole yields a quasi-elastic force; hence, the energy of this interaction equals the elastic energy of a dipole. The intrinsic energy of the dipole charges does not depend on the field intensity and the polarization of a dielectric. In the macroscopic theory, this additive constant is not taken into consideration. Finally, the energy of interaction of a system of dipoles equals ($k \neq i$)

$$-\frac{1}{2} \sum_{i,k} \mathbf{p}_i \mathbf{E}_{ki}$$

[cf. Eq. (1.103)], where \mathbf{E}_{ki} is the intensity of the field of the k -th dipole of the system at the place where the i -th dipole \mathbf{p}_i is.

The known fraction of this energy of dipole interaction is localized outside of the sphere S occupied by the dipoles. This fraction equals the integral $\frac{1}{8\pi} \int \mathbf{E}_{in}^2 dV$ taken over the entire space outside of the sphere. When determining the field intensity \mathbf{E}_{in} in the outer space, we can consider the electric moment of the dipoles to be uniformly distributed throughout the volume of the sphere S , i.e. consider this sphere to be uniformly polarized. Consequently, according to Eq. (2.39), the field \mathbf{E}_{in} in the external space equals the field of a dipole having the moment $V\mathbf{P}$, where V is the volume of the sphere S . Using Eq. (1.61), we get

$$\frac{1}{8\pi} \int \mathbf{E}_{in}^2 dV = \frac{V^2}{8\pi} \int \frac{3(\mathbf{P}\mathbf{R})^2 + P^2 R^2}{R^8} dV = \frac{4\pi}{9} P^2 V$$

where both integrals are taken over the entire space outside of the sphere S . Thus, the fraction of the mutual energy of the dipoles localized inside of the sphere S equals

$$-\frac{1}{2} \sum_{i,k} \mathbf{p}_i \mathbf{E}_{ki} = \frac{4\pi}{9} P^2 V$$

and its density is

$$u'_{22} = -\frac{1}{2V} \sum_{i,k} \mathbf{p}_i \mathbf{E}_{ki} = \frac{4\pi}{9} P^2$$

The density of the elastic energy of the dipoles [cf. Eq. (2.86)] is

$$u_q = \frac{1}{2V} \sum_i \mathbf{P}_i \mathbf{E}'_i$$

where \mathbf{E}'_i is the intensity of the field acting on the i -th dipole. Summation of the expressions obtained yields

$$u = u_{11} + u_{12} + u'_{22} + u_q = \frac{1}{8\pi} \left(E^2 - \frac{(4\pi)^2}{3} P^2 \right) + \\ + \frac{1}{2V} \sum_i \mathbf{p}_i \left(\mathbf{E}'_i - \sum_k \mathbf{E}_{ki} \right)$$

It is obvious that

$$\mathbf{E}'_i - \sum_k \mathbf{E}_{ki} = \mathbf{E}_1 = \mathbf{E} + \frac{4\pi}{3} \mathbf{P}$$

the field intensity \mathbf{E}_1 being constant over the entire volume of the sphere. Hence,

$$\begin{aligned} \frac{1}{2V} \sum_i \mathbf{p}_i (\mathbf{E}'_i - \sum_k \mathbf{E}_{ki}) &= \frac{1}{2} \left(\mathbf{E} + \frac{4\pi}{3} \mathbf{P} \right) \frac{\sum \mathbf{p}_i}{V} = \\ &= \frac{1}{2} \left(\mathbf{E} + \frac{4\pi}{3} \mathbf{P} \right) \mathbf{P} \end{aligned}$$

Introducing this into the preceding equation, we get

$$U = \frac{1}{8\pi} (E^2 + 4\pi \mathbf{E}\mathbf{P})$$

which, in view of Eq. (2.22), coincides with Eq. (2.84).

4. Let us consider in conclusion the energy of the electric field in a dielectric with *rigid dipoles*. This case appreciably differs from the one we have just studied because rigid dipoles, unlike quasi-elastic ones, do not have a reserve of internal energy that changes depending on the field intensity. On the other hand, however, the following must be taken into account. With quasi-elastic dipoles, we have the right to assume that there is no interaction between the polarization of a dielectric and the thermal motion of its molecules*. For molecules with rigid dipoles, however, the thermal motion, as we have seen, has an appreciable influence on polarization, preventing its saturation. Consequently, in the opposite case, the appearance of polarization, i.e. the alignment of the dipole axes in the direction of an external field, should affect the thermal motion of the dielectric molecules and be related with a change in the energy of this motion, i.e. with the absorption or liberation of heat.

For example, in gaseous dielectrics under the action of the electric field \mathbf{E} , the axis of each dipole (if only it is not parallel to \mathbf{E}) should perform pendular oscillations about the direction of this field. When the dielectric molecules collide, partial exchange occurs between the energy of these oscillations and that of the translational thermal motion of the molecules until equilibrium sets in corresponding to a definite degree of polarization of the dielectric (depending on both \mathbf{E} and the temperature).

Thus, *the work of the external forces* W_e^{**} done to change the intensity of the field in a dielectric with rigid dipoles and to polarize

* See the second footnote on page 148.

** We shall retain the symbol W without a subscript for the work of the ponderomotive forces of an electric field which for reversible processes is equal in magnitude but opposite in sign to the work of the external forces W_e .

it should be spent not only for a growth in the proper electric energy of the field, but also for heating (or cooling) the dielectric. If the latter is kept at a constant temperature, the part of the work W_e exceeding the change in the electric energy dU_{e1} will be transferred to the surrounding bodies in the form of a certain amount of heat Q :

$$W_e = dU_{e1} + Q$$

According to the second law of thermodynamics, the quantity of heat Q transferred by a system to the surrounding bodies in an arbitrary *reversible* process is proportional to the growth in the entropy S of the system during the process:

$$Q = -T dS \quad (2.88)$$

where T is the absolute temperature of the system.

Consequently*,

$$W_e = dU_{e1} - T dS$$

or, since according to our assumption $T = \text{const}$, we have

$$W_e = d(U_{e1} - TS) \quad (2.89)$$

It is known from thermodynamics that the function of state of a system whose increment in a reversible isothermal process equals the work of the external forces done in this process is called the *free (Helmholtz) energy* of the system A . Consequently, the part of the free energy of a *unit volume* of a dielectric that depends on the intensity of the electric field and the polarization of the dielectric equals

$$A_u = u_{e1} - Ts \quad (2.90)$$

where s is the portion of the entropy of a unit volume of the dielectric which depends on its polarization.

We can now define more precisely the meaning of Eq. (2.84) for the density of electric energy. One of the main *postulates of the macroscopic theory of electricity* is the statement that the expression

$$u = \frac{\epsilon E^2}{8\pi} = A_u \quad (2.91)$$

determines the density of the *free energy* of an electric field in dielectrics**.

* Strictly speaking, this expression for W_e is true only if the volume of a dielectric is kept constant during its polarization. If polarization is conducted at a constant pressure p , it is generally attended by a certain change dV in the volume of the dielectric V (*electrostriction*) for which the additional work $-p dV$ is done.

** It is known from thermodynamics that the free (Helmholtz) energy A_u coincides with the energy of a system only when A_u does not depend on the temperature T (with a constant volume V of the system); in the given case this condition denotes that ϵ does not depend on T with constant V [quasi-elastic dipoles, see Eq. (2.66), and also footnote** on p. 148].

The correctness of this postulate, as of any postulate in general, can be proved only by comparing the complex of corollaries ensuing from it with experimental facts; the theoretical considerations in its favour consist in the following.

First, at the beginning of Sec. 2.11, we proved that the work of the forces of an electric field upon the displacement of *free* electric charges equals the decrease of expression (2.80) or of expression (2.83) equal to it. It was assumed in this proof that the dielectric is stationary and that the value of ϵ at each point of the field *remains unchanged* upon the displacements of the free charges. Since ϵ , generally speaking, depends on the temperature, the constancy of ϵ means that the dielectrics are kept at a constant temperature, i.e. that the heat dQ liberated in them upon changes in the field due to displacements of the free charges is removed from them. Thus, the conclusions of Sec. 2.11 relate to *isothermal processes*, i.e. Eqs. (2.83) and (2.84) determine the *free* energy of an electric field.

Second, we have shown in this section that for dielectrics consisting of quasi-elastic dipoles whose polarization is not connected with heat effects, Eq. (2.84) for the density of the free energy, as should be expected, coincides with that for the density of the internal energy. Finally, we shall now show that when certain simplifying assumptions are made, the direct calculation of the *free* energy of the field in a dielectric with ideal rigid dipoles leads to Eq. (2.84) or (2.91).

Let us consider a gaseous dielectric having such a low polarizability that the difference between $(\bar{E}_{\text{micro}})^2 = E^2$ and $\overline{E_{\text{micro}}^2}$, and also between E and E' (Sec. 2.9) in it may be disregarded. We consider the dielectric molecules to be rigid dipoles. The density of the proper electric energy in such a dielectric is

$$u_{e1} = \frac{1}{8\pi} E^2$$

The entropy of an arbitrary body is related to the thermodynamic probability Π of its state by the Boltzmann equation:

$$S = k \ln \Pi$$

where k is the Boltzmann constant. Assume that the distribution of the axes of the molecules by directions is determined by the expression

$$dN = f(\theta) \sin \theta d\theta$$

where dN is the number of molecules whose axes form an angle ranging from θ to $\theta + d\theta$ with a certain definite direction in space. Therefore, the logarithm of the thermodynamic probability of this distribution will be expressed as follows*:

$$\ln \Pi = - \int_0^\pi f \ln f \sin \theta d\theta \quad (2.92)$$

* See, for example, A. Sommerfeld. *Thermodynamik und Statistik*, Weisbaden (1952).

In our case when $a \ll 1$, according to Eq. (2.73), we have

$$f = \frac{N}{2} \left(1 + a \cos \theta + \frac{a^2 \cos^2 \theta}{2} - \frac{a^2}{6} \right)$$

Let us insert this value into Eq. (2.92) and limit ourselves to the first two terms of the expansion of $\ln f$ by the powers of a ; we get

$$\ln f = a \cos \theta - \frac{a^2}{6} + \ln N - \ln 2$$

Upon integration, we may omit the insignificant additive constant since it does not depend on the electric field intensity, whereas our entire calculations are aimed at computing only the part of the entropy that appears owing to the action of the electric field. Integration yields

$$\ln \Pi = - \frac{Na^2}{2 \times 3}$$

Assuming here that N equals the number of molecules in a unit volume, we get the logarithm of the thermodynamic probability for a unit volume.

On the other hand, according to Eqs. (2.69) and (2.71), we have

$$a = \frac{p_0 E}{kT}$$

$$p = \frac{Np_0^2}{3kT} E$$

Consequently,

$$\ln \Pi = - \frac{Na^2}{2 \times 3} = - \frac{Np_0^2 E^2}{2 \times 3(kT)^2} = - \frac{PE}{2kT}$$

Hence,

$$TS = kT \ln \Pi = - \frac{1}{2} PE$$

or in view of Eq. (2.22)

$$TS = - \frac{\varepsilon - 1}{8\pi} E^2$$

Thus, the free energy of a unit volume of a dielectric is

$$A_u = \frac{E^2}{8\pi} - TS = \frac{E^2}{8\pi} + \frac{\varepsilon - 1}{8\pi} E^2 = \frac{\varepsilon}{8\pi} E^2$$

Generally speaking, Eq. (2.84) determines not the “internal” energy of an electric field, but its free energy, which is a measure of

the work connected with isothermal and only isothermal changes of the field. If the heat liberated upon the polarization of a dielectric is not completely absorbed by the surrounding bodies, then the work of the electric forces does not equal the reduction in the quantity u determined by Eq. (2.83). It must be noted, however, that such an insignificant amount of heat is liberated upon the polarization of a dielectric that this circumstance does not play a practicable part in the majority of cases.

2.13 Ponderomotive Forces in Dielectrics

1. A force equal to the sum of the forces applied to the separate molecules of a dielectric should obviously act on each element of its volume in an electric field. Let us replace, as previously, these molecules with equivalent dipoles and use Eq. (1.118) determining the resultant of the forces applied to a dipole:

$$\mathbf{F} = \mathbf{p} \nabla \cdot \mathbf{E}'$$

where \mathbf{E}' is the intensity of the field acting on a dipole.

The *density of the forces* applied to a dielectric, i.e. the force related to a unit volume of the dielectric, will therefore equal

$$\mathbf{f} = \sum \mathbf{F} = \sum \overline{\mathbf{p} \nabla \cdot \mathbf{E}'} = N \overline{\mathbf{p} \nabla \cdot \mathbf{E}'} \quad (2.93)$$

where summation must be performed over all the dipoles (molecules) in a unit volume; N is the number of molecules in a unit volume, while the bar indicates the mean value. For weakly polarized dielectrics ($\alpha \ll 1$), we can in the first approximation, on one hand, replace the *mean value of the product* $\overline{\mathbf{p} \nabla \cdot \mathbf{E}'}$ with the *product of the mean values* $\overline{\mathbf{p}}$ and $\overline{\nabla \cdot \mathbf{E}'}$, and, on the other, disregard the difference between $\overline{\nabla \cdot \mathbf{E}'}$ and $\nabla \cdot \mathbf{E}$, where \mathbf{E} is the intensity of the mean macroscopic field (see Sec. 2.9). In this approximation

$$\mathbf{f} = N \overline{\mathbf{p}} \nabla \cdot \mathbf{E} = \mathbf{P} \nabla \cdot \mathbf{E}$$

or, on the basis of Eq. (2.22)

$$\mathbf{f} = \frac{\epsilon - 1}{4\pi} \mathbf{E} \nabla \cdot \mathbf{E}$$

Further, according to Eq. (A. 47)

$$\mathbf{E} \nabla \cdot \mathbf{E} = \frac{1}{2} \nabla E^2 - [\mathbf{E} \operatorname{curl} \mathbf{E}]$$

In an electrostatic field, the last term equals zero because $\mathbf{E} = -\nabla\phi$, while the curl of the gradient equals zero [Eq. (A.42₁)]. Hence, in an electrostatic field

$$\mathbf{E}\nabla\cdot\mathbf{E} = \frac{1}{2}\cdot\nabla E^2 \quad (2.94)$$

$$\mathbf{f} = \frac{\epsilon - 1}{8\pi}\nabla E^2 \quad (2.95)$$

Thus, in the above approximation, the density of the ponderomotive forces in a dielectric is proportional to the *gradient of the square* of the field intensity. This is quite understandable because, first, in a homogeneous field the sum of the forces applied to each dipole equals zero, and, second, an increase in the field is attended by a growth not only of the forces of the field, but also of the polarization, i.e. the vector sum of the dielectric dipole moments. The force \mathbf{f} is directed toward the growth of the *absolute* value of the vector \mathbf{E} *regardless of the direction of this vector*. The reason is that a change in the direction of the vector \mathbf{E} is attended by a change in the direction of polarization \mathbf{P} . Thus, in an electric field, a dielectric *is carried along into the region of the maximum field intensity*. These ponderomotive forces, for example, are the reason why charged conductors attract pieces of paper, pith balls, and the like.

2. The above derivation of formula (2.95) is based on a number of assumptions and simplifications. A general expression for the ponderomotive forces can be obtained from the expression for the energy of a field U [Eq. (2.80) or (2.83)] by considering the change in the energy δU connected with an infinitely small (virtual) displacement \mathbf{s} of the bodies (conductors and dielectrics) in the field. Naturally, different points of these bodies may be displaced differently so that \mathbf{s} is an arbitrary but continuous function of a point. When observing the conditions considered in Sec. 1.18, this change in energy should equal the work of the ponderomotive forces of the field W taken with the reverse sign [Eq. (1.120)]:

$$\delta U = -W$$

which, in turn, obviously equals

$$W = \int \mathbf{sf} dV$$

where the integral should be taken over the entire volume of the *total field*, $\mathbf{f} dV$ being the force applied to an element of volume dV , and \mathbf{s} the displacement of this element. Hence,

$$\delta U = -\int \mathbf{sf} dV \quad (2.96)$$

where \mathbf{f} is the density of the ponderomotive forces of the field. Having determined δU from the expression for the energy, we can find \mathbf{f} with the aid of Eq. (2.96).

This is the most general way of calculating the ponderomotive forces which we shall repeatedly use in the following.

3. Applying this general method to the calculation of the ponderomotive forces of an electric field, we shall assume in this section that both the permittivity ε and the field intensity are continuous everywhere, i.e. that the value of ε smoothly changes in the interfaces between different media and that surface free charges σ are absent ($\sigma = 0$). The question of the forces applied to surfaces of discontinuity will be treated in Sec. 2.15.

On the basis of the assumption that surfaces of discontinuity are absent in all the surface integrals which we shall encounter in the present section, integration will be extended only over the external boundary surface of the field. In addition, we shall agree to consider the *total field* so that all these surface integrals will vanish. In addition, all the integrals of the type

$$\int \operatorname{div} \mathbf{a} dV$$

will vanish because according to Gauss's theorem (A.17), these integrals can be transformed into surface ones:

$$\int \operatorname{div} \mathbf{a} dV = \oint \mathbf{a}_n dS = 0 \quad (2.97)$$

4. According to Eqs. (2.80) and (2.83), the energy of an electrostatic field, owing to the absence of surface charges ($\sigma = 0$) which we have assumed in this section, can be expressed by one of the following equations:

$$U_1 = \frac{1}{2} \int \rho \varphi dV$$

and

$$U_2 = \frac{1}{8\pi} \int \varepsilon E^2 dV = \frac{1}{8\pi} \int \varepsilon (\operatorname{grad} \varphi)^2 dV$$

here, naturally, $U_1 = U_2$. Consequently, the change in the energy upon an arbitrary infinitely small displacement \mathbf{s} of the bodies in the field is

$$\delta U_1 = \frac{1}{2} \int \varphi \delta \rho dV + \frac{1}{2} \int \rho \delta \varphi dV$$

or

$$\delta U_2 = \delta U_1 = \frac{1}{8\pi} \int (\operatorname{grad} \varphi)^2 \delta \varepsilon dV + \frac{1}{4\pi} \int \operatorname{grad} \varphi \delta (\operatorname{grad} \varphi) dV$$

where $\delta\rho, \delta\varphi, \delta\varepsilon, \dots$ are the changes in the quantities $\rho, \varphi, \varepsilon, \dots$ caused by the displacement \mathbf{s} .

The sequence of performing operations of differentiation and variation may be changed without changing their result, so that

$$\begin{aligned}\delta(\text{grad } \varphi) &= \delta \left(\mathbf{i} \frac{\partial \varphi}{\partial x} + \mathbf{j} \frac{\partial \varphi}{\partial y} + \mathbf{k} \frac{\partial \varphi}{\partial z} \right) = \\ &= \mathbf{i} \frac{\partial(\delta\varphi)}{\partial x} + \mathbf{j} \frac{\partial(\delta\varphi)}{\partial y} + \mathbf{k} \frac{\partial(\delta\varphi)}{\partial z} = \text{grad}(\delta\varphi)\end{aligned}$$

On the other hand,

$$\varepsilon \text{ grad } \varphi = -\varepsilon \mathbf{E} = -\mathbf{D}$$

Consequently, on the basis of Eq. (A.43₂), the integrand of the last integral can be transformed as follows:

$$\varepsilon \text{ grad } \varphi \cdot \delta(\text{grad } \varphi) = -\mathbf{D} \text{ grad}(\delta\varphi) = -\text{div}(\mathbf{D} \delta\varphi) + \delta\varphi \text{ div } \mathbf{D}$$

The integral of the first addend of the right-hand side, according to Eq. (2.97), equals zero, so that, using Eq. (2.18), we finally get

$$\begin{aligned}\delta U_2 &= \frac{1}{8\pi} \int (\text{grad } \varphi)^2 \delta\varepsilon dV + \frac{1}{4\pi} \text{div } \mathbf{D} \delta\varphi dV = \\ &= \frac{1}{8\pi} \int E^2 \delta\varepsilon dV + \int \rho \delta\varphi dV\end{aligned}$$

Since δU_2 equals δU_1 , the energy increment δU can evidently also be written in the following form:

$$\delta U = 2\delta U_1 - \delta U_2 = \int \varphi \delta\rho dV - \frac{1}{8\pi} \int E^2 \delta\varepsilon dV \quad (2.98)$$

Thus, we have reduced the calculation of δU to determination of the change in the density of the free electricity ρ and the permittivity ε upon a virtual displacement \mathbf{s} of the bodies in the field.

5. The quantities $\delta\rho$ and $\delta\varepsilon$ in Eq. (2.98) are called the *local changes* in the quantities ρ and ε in contrast to the so-called *material changes* which we shall denote by $\delta'\rho$ and $\delta'\varepsilon$. The difference between these concepts consists in the following. The *local* change which a quantity experiences upon displacements of the medium is the change in the value of this quantity at a certain definite point of space not participating in this displacement. A *material* change, on the other hand, is the change in the value of the given quantity ρ in a certain definite material element of the moving medium. In other words, if the element dV of a material medium that before displacement was at the point P moves to the point P' , then $\delta'\rho$ equals the difference between the value which ρ had at the point P before the displacement and the value which it has at the point P' after the displacement \mathbf{s} .

On the other hand, $\delta\rho$ equals the difference between the values which ρ had at the same point P before and after the displacement \mathbf{s} . Accordingly, the local change $\delta\rho$ of a quantity ρ at the point P can be represented as the sum of two addends. First, after the displacement \mathbf{s} , the element of volume of the medium will be at the point P that was previously at the point P'' at the distance $-\mathbf{s}$ from P at which the quantity ρ obviously had the value

$$\rho_{P''} = \rho_P + (-\mathbf{s}) \text{ grad } \rho = \rho_P - \mathbf{s} \text{ grad } \rho$$

In addition, the value of the quantity ρ in this volume undergoes the material change $\delta'\rho$. Consequently*,

$$\delta\rho = \delta'\rho - \mathbf{s} \text{ grad } \rho \quad (2.99)$$

Thus, the determination of the local change now consists in determining the material change.

If the displacement of a medium is not attended by its deformation, then the material change $\delta'\rho$ obviously equals zero. If, however, an element V of the medium having the charge dq undergoes compression or expansion in displacement so that its volume becomes equal to $V + \delta'V$, then it is obvious that

$$dq = \rho V = (\rho + \delta'\rho)(V + \delta'V)$$

After transformations, and ignoring the product $\delta'\rho \cdot \delta'V$, we get

$$\delta'\rho = -\rho \frac{\delta'V}{V} \quad (2.100)$$

To determine the relative change $\delta'V/V$ in the volume of an element of the medium, we must note that upon the displacement \mathbf{s} of the element dS of the surface S confining the volume V , the value of this volume grows by $s_n dS$, i.e. by an amount equal to the volume of the cylinder circumscribed by the element dS upon this displacement. The quantity $s_n dS$ may naturally have either a positive or a negative value depending on the magnitude of the angle between \mathbf{s} and the outward normal \mathbf{n} to the element dS . The total change in the volume V upon the displacement \mathbf{s} will evidently be

$$\delta'V = \oint s_n dS$$

and, consequently [see the definition of the divergence of a vector, Eq. (A.18)], in the limit when $V \rightarrow 0$, we have

$$\frac{\delta'V}{V} = \frac{1}{V} \oint s_n dS = \text{div } \mathbf{s} \quad (2.101)$$

* Owing to the assumed infinitely small value of the displacement \mathbf{s} , we have the right to take no account of the difference between $(\delta'\rho)_{P''}$ and $(\delta'\rho)_P$, which are values of the second order of smallness.

Using this value in Eq. (2.100), we obtain

$$\delta' \rho = -\rho \operatorname{div} \mathbf{s} \quad (2.102)$$

It is quite obvious that an equation of this kind can be used to determine not only the local change in the density of electricity ρ , but also, for example, the change in the density τ of the medium itself (i.e. the change in the mass of a unit volume):

$$\delta' \tau = -\tau \operatorname{div} \mathbf{s}$$

6. Introducing Eq. (2.102) into (2.99), we get with the aid of Eq. (A.43₂)

$$\delta \rho = -\rho \operatorname{div} \mathbf{s} - \mathbf{s} \operatorname{grad} \delta = -\operatorname{div} (\rho \mathbf{s}) \quad (2.103)$$

On the other hand, similar to Eq. (2.99), we have

$$\delta \varepsilon = \delta' \varepsilon - \mathbf{s} \operatorname{grad} \varepsilon$$

Since the value of ε depends on the density of the dielectric*, the material change in ε will be determined by the material change $\delta' \tau$ in the density τ :

$$\delta' \varepsilon = \frac{\partial \varepsilon}{\partial \tau} \delta' \tau = -\frac{\partial \varepsilon}{\partial \tau} \tau \operatorname{div} \mathbf{s}$$

and, consequently,

$$\delta \varepsilon = -\frac{\partial \varepsilon}{\partial \tau} \tau \operatorname{div} \mathbf{s} - \mathbf{s} \operatorname{grad} \varepsilon \quad (2.104)$$

Substituting for $\delta \varepsilon$ and $\delta \rho$ in Eq. (2.98) their values from Eqs. (2.103) and (2.104), we get

$$\begin{aligned} \delta U &= \frac{1}{8\pi} \int \mathbf{s} E^2 \operatorname{grad} \varepsilon \, dV + \frac{1}{8\pi} \int E^2 \frac{\partial \varepsilon}{\partial \tau} \tau \operatorname{div} \mathbf{s} \, dV - \\ &- \int \varphi \operatorname{div} (\rho \mathbf{s}) \, dV \end{aligned}$$

The integrands of the last two integrals can be transformed with the aid of Eq. (A.43₂):

$$E^2 \frac{\partial \varepsilon}{\partial \tau} \tau \operatorname{div} \mathbf{s} = \operatorname{div} \left(E^2 \frac{\partial \varepsilon}{\partial \tau} \tau \mathbf{s} \right) - \mathbf{s} \operatorname{grad} \left(E^2 \frac{\partial \varepsilon}{\partial \tau} \tau \right)$$

* In solid dielectrics, the permittivity ε may depend (at a given temperature) not only on the density of the dielectric, but also on its deformations not connected with changes in the density. The investigation of this circumstance requires that account be taken of the anisotropy of the deformed dielectric and has been conducted, for example, in a book by J. A. Stratton, *Electromagnetic Theory*. New York, McGraw-Hill (1941).

and

$$\varphi \operatorname{div}(\rho \mathbf{s}) = \operatorname{div}(\varphi \rho \mathbf{s}) - s \rho \operatorname{grad} \varphi$$

According to Eq. (2.97), the integrals of the first addends of these expressions become equal to zero. Taking into account that $\operatorname{grad} \varphi = -\mathbf{E}$, we get

$$\delta U = \int \mathbf{s} dV \left\{ \frac{1}{8\pi} E^2 \operatorname{grad} \varepsilon - \frac{1}{8\pi} \operatorname{grad} \left(E^2 \frac{\partial \varepsilon}{\partial \tau} \tau \right) - \rho \mathbf{E} \right\} \quad (2.105)$$

7. Equation (2.105) should coincide with Eq. (2.96) when the dependence of the displacement \mathbf{s} on the coordinates of a point is arbitrary. Hence, it follows that the expressions which \mathbf{s} is multiplied by in both integrals are equal

$$\mathbf{f} = \rho \mathbf{E} - \frac{1}{8\pi} E^2 \operatorname{grad} \varepsilon + \frac{1}{8\pi} \operatorname{grad} \left(E^2 \frac{\partial \varepsilon}{\partial \tau} \tau \right) \quad (2.106)$$

It is exactly this formula that is the required expression for the density of the ponderomotive forces \mathbf{f} . It consists of two parts $\mathbf{f}^{(1)}$ and $\mathbf{f}^{(2)}$, namely,

$$\mathbf{f}^{(1)} = \rho \mathbf{E} \quad (2.107)$$

acting on the free electric charges, and

$$\mathbf{f}^{(2)} = \frac{1}{8\pi} \operatorname{grad} \left(E^2 \frac{\partial \varepsilon}{\partial \tau} \tau \right) - \frac{1}{8\pi} E^2 \operatorname{grad} \varepsilon \quad (2.108)$$

depending on $\partial \varepsilon / \partial \tau$ and $\operatorname{grad} \varepsilon$ and differing from zero only in dielectrics.

It follows from Eq. (2.107) that the density $\mathbf{f}^{(1)}$ of the ponderomotive forces acting on the *free* charges is determined in a dielectric, as in a vacuum, by the intensity of the electric field. We proceeded from this assumption in Sec. 2.11 when calculating the work done by a field in the displacement of free charges. As regards the density $\mathbf{f}^{(2)}$ of the ponderomotive forces acting on a dielectric, for a weakly polarizing dielectric ($\alpha \ll 1$) Eq. (2.108) coincides with the expression (2.95) that we obtained by direct calculation of the ponderomotive forces in these dielectrics.

Indeed, according to Eq. (2.79),

$$\frac{\varepsilon - 1}{\varepsilon + 2} = \frac{C\tau}{3}$$

where the coefficient C does not depend on the density of the dielectric τ . When $\alpha \ll 1$, i.e. when $\varepsilon \approx 1$, we can assume with sufficient accuracy that

$$\varepsilon - 1 = C\tau$$

Hence

$$\tau \frac{\partial \varepsilon}{\partial \tau} = \tau C = \varepsilon - 1$$

and Eq. (2.108) becomes

$$\mathbf{f}^{(2)} = \frac{1}{8\pi} \text{grad} \{E^2(\varepsilon - 1)\} - \frac{1}{8\pi} E^2 \text{grad} \varepsilon$$

But according to Eq. (A.43₁),

$$\text{grad} \{E^2(\varepsilon - 1)\} = (\varepsilon - 1) \text{grad} E^2 + E^2 \text{grad} \varepsilon$$

Consequently,

$$\mathbf{f}^{(2)} = \frac{\varepsilon - 1}{8\pi} \text{grad} E^2$$

which indeed coincides with Eq. (2.95).

Thus, the condition for the applicability of Eq. (2.95) is the linear dependence of the permittivity on the density of the dielectric, which, strictly speaking, occurs only in gases.

We must note in conclusion that J. Maxwell and a number of other authors, for example M. Abraham, did not take into consideration the dependence of the permittivity on the density of the medium, owing to which the expression for the ponderomotive forces in dielectrics which they used

$$\mathbf{f} = -\frac{1}{8\pi} E^2 \text{grad} \varepsilon \quad (2.109)$$

differed from Eq. (2.108) in the absence of the first term

$$\mathbf{f}'' = \frac{1}{8\pi} \text{grad} \left(E^2 \frac{\partial \varepsilon}{\partial \tau} \tau \right) \quad (2.110)$$

This term \mathbf{f}'' for solid and liquid dielectrics is compatible in value with Eq. (2.109), so that, generally speaking, it is impossible to disregard it.

It must be borne in mind, however, that the difference of Eq. (2.108) from Maxwell's formula (2.109) tells only on the distribution of the forces over the volume of a dielectric; the resultant of the forces \mathbf{f}'' not taken into consideration by Maxwell and applied to a body either equals zero (if the body is in a vacuum) or is balanced by the hydrostatic pressure appearing in the surrounding medium under the influence of the electric field. This statement will be proved in Sec. 2.15.

2.14 Reduction of Body Forces to Tensions *

1. As we have already mentioned in Sec. 1.16, the mechanistic theory of the electromagnetic field developed in the last century looked for the causes of electrical phenomena in the elastic deformations of a hypothetical medium—ether. A feature of elastic forces, as, in general, of short-range forces is the possibility of reducing them to *tensile* forces or stresses appearing in the deformed media, i.e. the possibility of reducing the forces acting on an arbitrary section of a medium to tensile forces resisted by the *surface* of the section (in particular, pressure is negative tension).

Accordingly, the mechanistic theory of a field was confronted with the task of reducing the ponderomotive forces of a field to elastic tensions of the medium. We shall show that it is indeed possible to reduce these forces to tensile ones. True, this circumstance does not in any way save the mechanistic theory of a field, which as a whole was found to be untenable. In considering many questions, however, the replacement of ponderomotive forces with the tensile forces equivalent to them is quite expedient.

In this section, we shall treat the question of reducing body forces to tensile forces in the general form so as to apply the results obtained to the case of an electric field we are interested in in the next section.

2. Let us consider a certain volume of a medium V confined within the surface S . If \mathbf{f} is the volume density of the forces, then the resultant of all the forces applied to bodies within the volume V will be

$$\mathbf{F} = \int_V \mathbf{f} dV \quad (2.111)$$

On the other hand, if the body forces in general can be reduced to tensile forces, then the complex of tensile stresses acting *from outside* on the closed surface S should equal the same quantity.

The tensile force acting on an element of surface area of an arbitrary section of a medium is proportional to the magnitude of this element of area dS and depends not only on the position of the element, but also on its direction. In other words, it is not only a position function, but also a function of the direction of a normal \mathbf{n} to the element dS . Particularly, when the element of area dS turns through 180 degrees, i.e. when the direction of the normal \mathbf{n} is reversed, the tensile force changes its sign. This is an expression of the fact that mutually balanced equal and opposite tensile forces always act on the opposite sides of an arbitrary element of the surface dS **.

* This and the next sections may be omitted when reading the book the first time.

** Except for elements of surfaces of discontinuity which we are not treating in the present section.

Let us denote by \mathbf{T}_n the force acting from outside on a unit area of the surface (stress), an outward normal to which is directed along \mathbf{n} . We shall denote the components of this stress vector along the axes of coordinates by T_{xn}, T_{yn}, T_{zn} *. Consequently, the resultant of all the tensile forces applied from outside to the surface will evidently be expressed by

$$\mathbf{F} = \oint_S \mathbf{T}_n dS \quad (2.112)$$

where \mathbf{n} is an outward normal to the element dS . Equating expressions (2.111) and (2.112):

$$\mathbf{F} = \int_V \mathbf{f} dV = \oint_S \mathbf{T}_n dS \quad (2.113)$$

we can find the relationship between the density of the body forces \mathbf{f} and the stress \mathbf{T}_n .

3. Let us choose an arbitrary system of Cartesian coordinates and denote by \mathbf{T}_x , \mathbf{T}_y , and \mathbf{T}_z the stress vector acting from outside on a surface, the outward normal to which is directed along the x -axis, y -axis, and z -axis, respectively. Let T_{xx}, T_{yx}, T_{zx} , etc. stand for the components of these stresses along the axes of coordinates so that

$$\mathbf{T}_x = \mathbf{i}T_{xx} + \mathbf{j}T_{yx} + \mathbf{k}T_{zx}$$

$$\mathbf{T}_y = \mathbf{i}T_{xy} + \mathbf{j}T_{yy} + \mathbf{k}T_{zy}$$

$$\mathbf{T}_z = \mathbf{i}T_{xz} + \mathbf{j}T_{yz} + \mathbf{k}T_{zz}$$

Thus, for instance, T_{xz} is the component along the x -axis of the stress \mathbf{T}_z acting on the surface, the outward normal to which is directed along the z -axis.

It is simple to show (see any textbook on the theory of elasticity) that the stress \mathbf{T}_n acting from outside on an arbitrarily oriented surface, an outward normal to which has the direction \mathbf{n} , is related to the stresses \mathbf{T}_x , \mathbf{T}_y , and \mathbf{T}_z by the following expression:

$$\mathbf{T}_n = \mathbf{T}_x \cos(\mathbf{n}, x) + \mathbf{T}_y \cos(\mathbf{n}, y) + \mathbf{T}_z \cos(\mathbf{n}, z) \quad (2.114)$$

so that the component of this stress, for example, along the x -axis is

$$T_{xn} = T_{xx} \cos(\mathbf{n}, x) + T_{xy} \cos(\mathbf{n}, y) + T_{xz} \cos(\mathbf{n}, z) \quad (2.115)$$

Thus, the nine quantities T_{xx}, T_{xy} , etc. completely characterize the system of stresses at a given point of space. The values of these quantities completely determine the stresses acting on an arbitrarily oriented surface. The quantities T_{xx}, T_{xy} , etc. are called the *compo-*

* These components are often denoted by T_{nx}, T_{ny} , and T_{nz} . Our choice of the sequence of the subscripts corresponds to what is adopted in the general theory set out in Sec. 7.15.

nents of the stress tensor, and the totality of them is called the *stress tensor*, which we shall denote by the letter \mathbf{T} (without any subscripts). The components of the stress tensor can be written in the following symmetrical form:

$$\mathbf{T} = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix} \quad (2.116)$$

4. To go over to differential relationships from the integral one (2.113) between the stresses and the body forces, we must evidently first transform the surface integral at the right into a volume one (or vice versa). Using Eqs. (2.112) and (2.115), we can express the component of the resultant \mathbf{F} along the x -axis through the components of the stress tensor as follows:

$$F_x = \oint_S T_{xn} dS = \oint_S \{T_{xx} \cos(\mathbf{n}, x) + T_{xy} \cos(\mathbf{n}, y) + T_{xz} \cos(\mathbf{n}, z)\} dS$$

Let us now use Gauss's theorem (A.17) in the expanded form:

$$\begin{aligned} & \oint_S \{a_x \cos(\mathbf{n}, x) + a_y \cos(\mathbf{n}, y) + a_z \cos(\mathbf{n}, z)\} dS = \\ & = \int_V \left(\frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} \right) dV \end{aligned}$$

Since this theorem holds for any continuous functions of a point a_x , a_y , and a_z , then assuming in it that $a_x = T_{xx}$, $a_y = T_{xy}$, and $a_z = T_{xz}$, we get (provided that the components of the stress tensor inside the volume V are continuous)

$$F_x = \oint_S T_{xn} dS = \int_V \left(\frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} + \frac{\partial T_{xz}}{\partial z} \right) dV$$

Introducing this expression into Eq. (2.113) and equating the integrands owing to the arbitrary value of the volume V , we finally get

$$\left. \begin{aligned} f_x &= \frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} + \frac{\partial T_{xz}}{\partial z} \\ \text{and similarly} \\ f_y &= \frac{\partial T_{yx}}{\partial x} + \frac{\partial T_{yy}}{\partial y} + \frac{\partial T_{yz}}{\partial z} \\ f_z &= \frac{\partial T_{zx}}{\partial x} + \frac{\partial T_{zy}}{\partial y} + \frac{\partial T_{zz}}{\partial z} \end{aligned} \right\} \quad (2.117)$$

It is exactly these equations that establish the required differential relationships between the density of the body forces \mathbf{f} and the components of the stress tensor \mathbf{T} .

5. It follows from Eq. (2.117) that the density of the body forces is determined not by the absolute value of the tensile stresses, but by the nature of the change in the stresses in space (upon displacements of the point of observation). In particular, \mathbf{f} equals zero if the components of the stress tensor \mathbf{T} have constant values in the given section of the medium. This is quite clear, because if we mentally separate an arbitrary parallelepiped in a medium, then upon constancy of the tensor \mathbf{T} , stresses equal in magnitude and opposite in direction will act on its opposite faces. Hence, the resultant of the forces applied to the parallelepiped will equal zero.

6. For equivalence of the body forces and tensile stresses, it is essential that not only the *resultant* of the forces applied to an arbitrary volume remain constant, but also the *moment* of these forces when body forces are replaced with equivalent stresses. This circumstance imposes an additional restriction on the components of the stress tensor.

The moment \mathbf{N} of the body forces applied to an arbitrary volume V is

$$\mathbf{N} = \int [\mathbf{R}\mathbf{f}] dV \quad (2.118)$$

where \mathbf{R} is the distance from the element dV to the point O relative to which the moment of the forces is being determined. If the body forces \mathbf{f} are equivalent to the tensile stresses \mathbf{T} , then Eq. (2.117) should be observed. Hence, the component of \mathbf{N} , for example, along the x -axis should be

$$\begin{aligned} N_x &= \int_V (yf_z - zf_y) dV = \\ &= \int_V \left\{ y \left(\frac{\partial T_{zx}}{\partial x} + \frac{\partial T_{zy}}{\partial y} + \frac{\partial T_{zz}}{\partial z} \right) - \right. \\ &\quad \left. - z \left(\frac{\partial T_{yx}}{\partial x} + \frac{\partial T_{yy}}{\partial y} + \frac{\partial T_{yz}}{\partial z} \right) \right\} dV \end{aligned}$$

The integrand at the right can be written as follows:

$$\begin{aligned} &\frac{\partial}{\partial x} (yT_{zx} - zT_{yx}) + \frac{\partial}{\partial y} (yT_{zy} - zT_{yy}) + \\ &+ \frac{\partial}{\partial z} (yT_{zz} - zT_{yz}) - T_{zy} + T_{yz} \end{aligned}$$

Since the first three terms of the expression coincide in form with the expression for the divergence of a vector having the components

$$a_x = yT_{zx} - zT_{yx}, \quad a_y = yT_{zy} - zT_{yy}, \quad a_z = yT_{zz} - zT_{yz}$$

then the volume integral can be transformed with the aid of Gauss's theorem (A.17):

$$N_x = \oint_S a_n dS + \int_V (T_{yz} - T_{zy}) dV$$

and on the basis of Eq. (2.115)

$$\begin{aligned} a_n &= a_x \cos(x, \mathbf{n}) + a_y \cos(y, \mathbf{n}) + a_z \cos(z, \mathbf{n}) = \\ &= y \{ T_{zx} \cos(x, \mathbf{n}) + T_{zy} \cos(y, \mathbf{n}) + T_{zz} \cos(z, \mathbf{n}) \} - \\ &- z \{ T_{yx} \cos(x, \mathbf{n}) + T_{yy} \cos(y, \mathbf{n}) + T_{yz} \cos(z, \mathbf{n}) \} = \\ &= yT_{zn} - zT_{yn} \end{aligned}$$

We finally get

$$\begin{aligned} N_x &= \int_V (yf_z - zf_y) dV = \oint_S (yT_{zn} - zT_{yn}) dS + \\ &+ \int_V (T_{yz} - T_{zy}) dV \end{aligned} \quad (2.119)$$

The surface integral at the right equals the moment of the tensile forces \mathbf{T}_n applied to the surface S of the volume V . The moment of these tensile forces will equal that of the body forces only if the last integral at the right equals zero. Owing to the arbitrary nature of the volume V , this will occur solely when the integrand equals zero at all points of space:

$$T_{yz} = T_{zy}$$

Repeating the same reasoning for the components of \mathbf{N} along the axes y and z , we get the following relationships:

$$T_{yz} = T_{zy}, \quad T_{zx} = T_{xz}, \quad \text{and} \quad T_{xy} = T_{yx} \quad (2.120)$$

Tensors whose components comply with the relationships (2.120) are called *symmetrical*.

Thus, the necessary and sufficient conditions for a system of body forces and a system of tensile stresses to be equivalent to each other with respect to both the resultant of the forces applied to an arbitrary volume and the moment of these forces consist, first, in the relationships (2.117) and, second, in the symmetry of the stress tensor. If the stress tensor is not symmetrical, the system of stresses cannot be replaced with an equivalent distribution of body forces.

This, by the way, already follows from the fact that if the components of the tensor \mathbf{T} are constant in magnitude, then the body forces according to Eq. (2.117) become equal to zero, whereas the moment

of the tensile forces applied to an arbitrary volume will differ from zero when $T_{ik} \neq T_{ki}$ even when T_{ik} is constant*.

7. In our preceding discussion, we used an arbitrarily selected coordinate system and did not mention the question of the law of transformation of the tensor components upon transformation of the coordinates. This law can be found from the requirement (following from the very definition of the concept of tensile stress) that the components $T_{xn} dS$, $T_{yn} dS$, and $T_{zn} dS$ of the force $\mathbf{T}_n dS$ acting on an arbitrarily arranged and arbitrarily oriented element dS be transformed according to vector transformation rules**.

We shall not stop here to consider the derivation of this law of transformation; we shall only note that it can help us to become convinced that both Eq. (2.117) and condition (2.120) for the symmetry of a tensor retain their form upon any transformation of the Cartesian coordinates.

2.15 Stress Tensor of an Electric Field

1. Let us now turn to the task posed at the beginning of the preceding section of reducing the ponderomotive forces of an electric field to tensile forces. For this purpose, it is convenient to expand the general expression (2.120) for the volume density of these forces into two components:

$$\left. \begin{aligned} \mathbf{f} &= \mathbf{f}' + \mathbf{f}'' \\ \mathbf{f}' &= \rho \mathbf{E} - \frac{1}{8\pi} E^2 \text{grad } \varepsilon, \\ \mathbf{f}'' &= \frac{1}{8\pi} \text{grad} \left(E^2 \frac{\partial \varepsilon}{\partial \tau} \tau \right) \end{aligned} \right\} \quad (2.121)$$

Our problem will evidently be solved if we find such a tensor \mathbf{T} that upon inserting its components into the right-hand side of equations (2.117), their left-hand sides will coincide with the components of the density of the body forces \mathbf{f} determined by Eqs. (2.121).

2. Let us express ρ in Eq. (2.121) for \mathbf{f}' through $\text{div } \mathbf{D}$ with the aid of Eq. (2.18) and consider the component of the force density \mathbf{f}' in any direction, for instance along the x -axis:

$$f'_x = \frac{1}{4\pi} E_x \text{div } \mathbf{D} - \frac{1}{8\pi} E^2 \frac{\partial \varepsilon}{\partial x}$$

* It must be noted that in anisotropic media the stress tensor of an electric field, generally speaking, is not symmetrical.

** However, for example the quantity T_x with the components T_{xx} , T_{yx} , and T_{zx} that determines the stress acting on a certain area is not a vector by itself because the direction of the area itself depends on the arbitrarily selected direction of the coordinate x -axis.

It is easy to see [cf. Eq. (A.43₂)], where φ in our case corresponds to E_x , that

$$\begin{aligned} E_x \operatorname{div} \mathbf{D} &= \frac{\partial}{\partial x} (E_x D_x) + \frac{\partial}{\partial y} (E_x D_y) + \\ &+ \frac{\partial}{\partial z} (E_x D_z) - \mathbf{D} \nabla \cdot E_x \end{aligned}$$

Further, using Eq. (2.94) with a view to Eq. (1.36), we get

$$\mathbf{D} \nabla \cdot E_x = \varepsilon \mathbf{E} \nabla \cdot E_x = \frac{\varepsilon}{2} \frac{\partial}{\partial x} E^2$$

and, therefore,

$$\begin{aligned} & - \frac{1}{4\pi} \mathbf{D} \nabla \cdot E_x - \frac{1}{8\pi} E^2 \frac{\partial \varepsilon}{\partial x} = \\ & = - \frac{1}{8\pi} \left\{ \varepsilon \frac{\partial E^2}{\partial x} + E^2 \frac{\partial \varepsilon}{\partial x} \right\} = - \frac{1}{8\pi} \frac{\partial (\varepsilon E^2)}{\partial x} \end{aligned}$$

Thus,

$$\begin{aligned} f'_x &= \frac{\partial}{\partial x} \left(E_x D_x - \frac{\varepsilon E^2}{2} \right) + \frac{\partial}{\partial y} (E_x D_y) + \\ & + \frac{\partial}{\partial z} (E_x D_z) \end{aligned}$$

This equation coincides in form with Eq. (2.117) if we assume that

$$T'_{xx} = \frac{\varepsilon}{4\pi} \left(E_x^2 - \frac{E^2}{2} \right), \quad T'_{xy} = \frac{\varepsilon}{4\pi} E_x E_y,$$

$$T'_{xz} = \frac{\varepsilon}{4\pi} E_x E_z$$

The remaining components of the tensor T' can be determined in a similar way, and their aggregate can be written in the form of an array of type (2.116):

$$4\pi T' = \left\{ \begin{array}{ccc} \varepsilon \left(E_x^2 - \frac{E^2}{2} \right) & \varepsilon E_x E_y & \varepsilon E_x E_z \\ \varepsilon E_y E_x & \varepsilon \left(E_y^2 - \frac{E^2}{2} \right) & \varepsilon E_y E_z \\ \varepsilon E_z E_x & \varepsilon E_z E_y & \varepsilon \left(E_z^2 - \frac{E^2}{2} \right) \end{array} \right\} \quad (2.122)$$

As regards the second component \mathbf{f}'' of the body forces, it can be seen to coincide in form with Eq. (2.117); only those components of the stress tensor \mathbf{T}'' equivalent to it differing from zero that are on the principal diagonal of matrix (2.116):

$$T''_{xx} = T''_{yy} = T''_{zz} = \frac{1}{8\pi} E^2 \frac{\partial \varepsilon}{\partial \tau} \tau, \quad (2.123)$$

$$T_{ik} = 0 \text{ when } i \neq k$$

Thus, both \mathbf{T}' and \mathbf{T}'' (and, therefore, also their sum $\mathbf{T} = \mathbf{T}' + \mathbf{T}''$) are symmetrical tensors, i.e. comply with condition (2.120).

We have thus proved the equivalence of body forces (2.121) to a system of stresses (2.122) and (2.123), and we can state that the total force \mathbf{F} acting on an arbitrary section of a medium V is determined by the state of the field *on the boundaries of this section*, i.e. can be reduced to a system of forces or stresses applied to its surface S .
3. Ponderomotive forces were reduced for the first time to tensile stresses by J. Maxwell who, however, did not take into account the dependence of the permittivity ε on the density of the dielectric (see the end of Sec. 2.13). Therefore, *Maxwell's stress tensor* corresponds only to a part of our stress tensor \mathbf{T} , namely to \mathbf{T}' .

To distinguish the forces \mathbf{f}'' and the tensor \mathbf{T}'' corresponding to them from Maxwellian forces \mathbf{f}' and Maxwellian stresses \mathbf{T}' , we shall call them *striction* forces and the *striction* tensor. It should be noted that the striction stresses \mathbf{T}'' are equivalent to the uniform pressure*,

$$p'' = - \frac{1}{8\pi} E^2 \frac{\partial \varepsilon}{\partial \tau} \tau \quad (2.124)$$

because the pressure is negative stress directed along a normal to an arbitrary area inside a body and not depending on the orientation of this area.

In liquids and solids, the striction forces and stresses are of the same order of magnitude as the Maxwellian forces and stresses (because $\frac{\partial \varepsilon}{\partial \tau} \tau$ is of the same order of magnitude as ε). This was noted by H. Helmholtz, J. Jeans, and others, but it remained unclear why the authors who disregarded the striction forces and stresses did not obtain results that were erroneous to an appreciable extent. We shall show what the matter is here.

Let us assume that we are interested only in the resultant \mathbf{F} of all the forces of an electric field applied to an arbitrary body A and in the resultant moment \mathbf{N} of these forces. According to Eqs. (2.113)

* For solid dielectrics, our expressions for the striction forces and stresses taking into account only volume striction must be supplemented with terms whose magnitude is determined by the dependence of ε on the shear strains not attended by changes in the density of the medium.

and (2.119), the resultant \mathbf{F} and the moment \mathbf{N} of these forces are unambiguously determined by the values of the tensor \mathbf{T} at the outer side of the surface S of the body. The striction stresses \mathbf{T}'' in a vacuum are evidently equal to zero (because in a vacuum $\frac{\partial \varepsilon}{\partial \tau} \tau$ should be considered equal to zero). Therefore, if the body is surrounded by a vacuum, the striction forces \mathbf{f}'' and stresses \mathbf{T}'' affect only the distribution of the forces over the volume of the body*, but do not affect the magnitude of the resultant of all the forces \mathbf{F} or their moment \mathbf{N} .

If body A is surrounded by a dielectric B instead of by a vacuum, then the resultant \mathbf{F}'' of the striction forces \mathbf{f}'' applied to A , generally speaking, does not equal zero, the resultant being determined by the tensor \mathbf{T}'' . If the system of bodies A and B is in mechanical equilibrium, however, then this force \mathbf{F}'' left out of consideration by Maxwell is exactly compensated by the additional *mechanical* forces, also left out of consideration by Maxwell, acting on the body A from the side of the body B owing to the fact that the body B itself is subjected to electrostriction.

We shall explain this by a very simple example. Let the body A be a charged ball submerged in an infinite homogeneous liquid dielectric B ; the hydrostatic pressure in the liquid B far away from the ball A is given and equals p_2^0 . We ignore the weight of the liquid B . Hence, "according to Maxwell", the surface S of the ball A will be acted upon, first, by the electric stresses \mathbf{T}' and, second, the hydrostatic pressure of the liquid p_2^0 . Actually, however, we must also take into account, first, the electrostriction stresses \mathbf{T}'' acting on the surface of the body A and equivalent, according to Eq. (2.124), to the pressure p'' ; second, since the liquid is subjected to the pressure p'' in the electric field, then the *hydrostatic* pressure p_2 of the liquid near the ball A will not equal the hydrostatic pressure p_2^0 at infinity: the condition for equilibrium of the liquid will be the constancy of the total pressure in it:

$$p'' + p_2 = \text{const} = p_2^0$$

Thus, the sum of the electrostriction stresses \mathbf{T}'' equivalent to the pressure p'' and the actual hydrostatic pressure of the liquid p_2 acting on the surface of a body equals the hydrostatic pressure of the liquid p_2^0 calculated without account taken of electrostriction**. Naturally, if the liquid is not in equilibrium, the ignoring of electrostriction will, generally speaking, result in an error.

* This distribution determines the elastic stresses—the so-called *electrostriction*—appearing in a body under the action of an electric field. This is where the name of the forces \mathbf{f}'' and the stress \mathbf{T}'' has been taken from.

** Strictly speaking, account must be taken of the fact that owing to electrostriction the density τ changes and together with it the permittivity ε of the liquid. As a result, the field of the charged ball A in the liquid and the Maxwellian stress \mathbf{T}' acting on it change. Calculations show, however, that these changes in the tension are absolutely negligible.

Thus, if we are interested not in the distribution of the ponderomotive forces over the volume of an arbitrary body A , but only in the resultant \mathbf{F} of these forces and their moment \mathbf{N} , then it will be sufficient for us to consider only the Maxwellian forces \mathbf{f}' and the Maxwellian stresses \mathbf{T}' , discarding the striction forces and stresses \mathbf{f}'' and \mathbf{T}'' , on condition that the body A is surrounded either by a vacuum or a dielectric medium in mechanical equilibrium.

4. Introducing into Eq. (2.115) the values of the components of the tensor $\mathbf{T} = \mathbf{T}' + \mathbf{T}''$, we can determine the components along the x -axis of the tensile force \mathbf{T}_n acting in an electric field on a unit area (stresses), the outward normal to which is directed along \mathbf{n} :

$$\begin{aligned} T_{xn} &= \frac{1}{4\pi} \left\{ \varepsilon E_x^2 - \frac{1}{2} \left(\varepsilon - \frac{\partial \varepsilon}{\partial \tau} \tau \right) E^2 \right\} \cos(\mathbf{n}, x) + \\ &+ \frac{\varepsilon}{4\pi} E_x E_y \cos(\mathbf{n}, y) + \frac{\varepsilon}{4\pi} E_x E_z \cos(\mathbf{n}, z) = \\ &= \frac{\varepsilon}{4\pi} E_x E_n - \frac{1}{8\pi} E^2 \left(\varepsilon - \frac{\partial \varepsilon}{\partial \tau} \tau \right) \cos(\mathbf{n}, x) \end{aligned}$$

The totality of this expression for T_{xn} and similar expressions for T_{yn} and T_{zn} in the vector form can be written as follows:

$$\mathbf{T}_n = \frac{\varepsilon}{4\pi} E_n \mathbf{E} - \frac{1}{8\pi} \mathbf{n} E^2 \left(\varepsilon - \frac{\partial \varepsilon}{\partial \tau} \tau \right) \quad (2.125)$$

In a particular case, if the element of area of the surface being considered is perpendicular to the field \mathbf{E} and its outward normal \mathbf{n} is parallel to \mathbf{E} , the tensile forces applied to it from outside is determined by the expression ($\mathbf{n} \parallel \mathbf{E}$):

$$\mathbf{T}_n = \frac{\varepsilon + \frac{\partial \varepsilon}{\partial \tau} \tau}{8\pi} E^2 \mathbf{n} \quad (2.126)$$

If the normal \mathbf{n} is perpendicular to the field, then

$$E_n = 0$$

and ($\mathbf{n} \perp \mathbf{E}$)

$$\mathbf{T}_n = - \frac{\varepsilon - \frac{\partial \varepsilon}{\partial \tau} \tau}{8\pi} E^2 \mathbf{n} \quad (2.127)$$

Finally, if the element dS has a certain intermediate direction, it can be resolved into two mutually perpendicular elements parallel and perpendicular to \mathbf{E} , the forces acting on these elements being determined from Eqs. (2.126) and (2.127). Hence, the system of

tensile forces in an electric field consists in the pull $\frac{\epsilon + \frac{\partial \epsilon}{\partial \tau} \tau}{8\pi} E^2$ along the direction of the field \mathbf{E} [Eq. (2.126)] and in the pressure (negative pull) $\frac{\epsilon - \frac{\partial \epsilon}{\partial \tau} \tau}{8\pi} E^2$ along a direction perpendicular to \mathbf{E} [Eq. (2.127)].

From the standpoint of the mechanical theory of a field, this pull and this pressure are nothing but elastic forces appearing in ether when it is deformed in an electric field. According to Eqs. (2.126) and (2.127), these forces act on all the sections of the ether (both in a vacuum and in material bodies) in which the field (i.e. the deformation of the ether) does not equal zero*.

Naturally, from the viewpoint of today's theory negating the existence of material ether (in the mechanical meaning of this word), the ponderomotive forces of an electric field can be applied only to electric charges and to the material bodies carrying these charges or, more exactly, consisting of electric charges (electrons and atomic nuclei). As has been proved, however, the resultant force acting on bodies in an arbitrary volume V can be formally represented in the form of the sum of the tensile stresses "acting" on the surface S of this volume (which can naturally pass through a vacuum and through material bodies). We can therefore operate with these stresses being sure that the final results are correct. The meaning of the concept of electromagnetic field stresses in principle will be revealed in Sec. 7.15 where we shall see that the equivalence between ponderomotive forces and electromagnetic tensile forces is violated in *variable* electromagnetic fields, and that the surplus of the sum of the stresses acting on the surface confining the volume V over the ponderomotive forces acting on the bodies in this volume determines the change in the momentum of the electromagnetic field in this volume. 5. The replacement of the ponderomotive forces with an equivalent system of tensile forces makes it much easier, particularly, to determine the forces applied to the surface of discontinuity of an electric field, i.e. to surfaces charged with free electricity, and to the interfaces between media having different polarizabilities. True, in deriving the system of tensions expressed by Eqs. (2.122) and (2.123), we proceeded from the results obtained in Sec. 2.13, in which such surfaces of discontinuity were assumed to be absent. By determining, however, the resultant force applied, for instance, to a charged layer having a finite thickness on the basis of Eqs. (2.122) and (2.123)

* We shall get a correct picture of these tensile forces if we imagine that material elastic filaments are stretched along the field lines, they being subjected to a pull of $\left(\epsilon + \frac{\partial \epsilon}{\partial \tau} \tau\right) (8\pi)^{-1} E^2$ and exerting a side pressure of $\left(\epsilon - \frac{\partial \epsilon}{\partial \tau} \tau\right) (8\pi)^{-1} E^2$ on one another.

and then passing over in the limit to an infinitely thin layer, we shall determine the magnitude of the force acting on a charged surface.

Let \mathbf{n} stand for a normal to the surface of the layer forming the boundary between medium 1 and medium 2. To avoid ambiguity, let us assume that \mathbf{n} is directed from the medium 1 to the medium 2. The element of area dS of the surface of the layer forming the boundary with the medium 1 will, according to Eq. (2.125), be acted upon by this medium with the force

$$\mathbf{T}'_{1n} dS = \frac{-\epsilon_1}{8\pi} (2E_{1n}\mathbf{E}_1 - E_1^2\mathbf{n}) dS \quad (2.128)$$

(for simplicity we replace the tensor \mathbf{T} with Maxwell's tensor \mathbf{T}'), while the element of area dS of the surface forming the boundary with the medium 2 will be acted upon by this medium with the force

$$\mathbf{T}'_{2n} dS = \frac{+\epsilon_2}{8\pi} (2E_{2n}\mathbf{E}_2 - E_2^2\mathbf{n}) dS \quad (2.129)$$

The signs in Eqs. (2.128) and (2.129) differ because, as stipulated, the direction of the *outward* normal to the layer coincides in the medium 2 with the direction of \mathbf{n} , and in the medium 1 with the opposite direction. The total force acting on an element of layer area of the surface dS thus equals

$$\begin{aligned} \mathbf{f}' dS &= (\mathbf{T}'_{1n} + \mathbf{T}'_{2n}) dS = \frac{1}{4\pi} (D_{2n}\mathbf{E}_2 - D_{1n}\mathbf{E}_1) dS - \\ &- \frac{1}{8\pi} (D_2E_2 - D_1E_1) \mathbf{n} dS \end{aligned}$$

This relationship must also obviously hold when passing over to the limiting case of an infinitely thin layer, i.e. to a surface. Thus, the resultant force acting on a unit area of an arbitrary surface is

$$\mathbf{f}' = \frac{1}{4\pi} (D_{2n}\mathbf{E}_2 - D_{1n}\mathbf{E}_1) - \frac{1}{8\pi} (D_2E_2 - D_1E_1) \mathbf{n} \quad (2.130)$$

where \mathbf{E}_1 , \mathbf{D}_1 and \mathbf{E}_2 , \mathbf{D}_2 are the values of the vectors \mathbf{E} and \mathbf{D} at the inner and outer (with respect to \mathbf{n}) sides of this surface, respectively.

It is quite obvious that for any surface which is not one of discontinuity, we have $\mathbf{E}_1 = \mathbf{E}_2$ and $\mathbf{D}_1 = \mathbf{D}_2$, so that the force \mathbf{f} becomes equal to zero.

The forces (2.130) applied to the surface of discontinuity manifest themselves in that under their action *elastic* stresses balancing them appear in bodies placed in the electric field. The sum of all the stresses (electrical and elastic) cannot be discontinuous and must be identical at both sides of any surface including charged surfaces or interfaces between two media. In the majority of cases, however, we are inter-

ested not in the stresses appearing in bodies under the action of an electric field, but in the resultant force acting in this field on a given body and determined not by a *jump* in the tensor of electric stress \mathbf{T} on the surface of this body S , but by the value of the components of the tensor \mathbf{T} at the *outer side of the surface of the body* S .

6. Let us now use the stress tensor \mathbf{T} to define more precisely the meaning of the conventional statement according to which a test charge q placed in a liquid or gaseous dielectric in which the intensity of the electric field is \mathbf{E}_0 is subjected to the force

$$\mathbf{F} = q\mathbf{E}_0 \quad (2.131)$$

Assume that the test charge q is a small charged body A of a metal or dielectric having an arbitrary shape. We shall denote by \mathbf{E}' the field of this body placed at the given point of the medium so that the true field when the test charge is present equals $\mathbf{E} = \mathbf{E}_0 + \mathbf{E}'$. The field \mathbf{E}' is induced, first, by the free charge q of the test body and, second, by the distribution of the bound or induced charges appearing in it under the action of the external field \mathbf{E}_0 . The resultant force acting on the test body A can be computed with the aid of Eq. (2.122); the tensor \mathbf{T}' , according to what has been proved, does not add anything to this force.

The components of the tensor \mathbf{T}' are quadratic functions of the components of the field \mathbf{E} . When calculating the components of \mathbf{T}' , we shall make two assumptions: (1) the body A is so small that over the entire length of its enclosing surface S the field \mathbf{E}_0 not disturbed by the presence of a test body and the permittivity of the medium ε may be considered constant, and (2) the charge q and the dimensions of the body A are so small that the field \mathbf{E}' it induces at points of the surface S is much smaller than \mathbf{E}_0 * so that when calculating \mathbf{T}' the terms quadratic with respect to \mathbf{E}' may be disregarded.

Assume that the x -axis coincides with the direction of the field \mathbf{E}_0 near the test body A . Hence, with the above assumptions, we get by simple calculations from Eq. (2.122) that

$$4\pi T'_{xx} = \varepsilon \left(\frac{E_0^2}{2} + E_0 E'_x \right),$$

$$4\pi T'_{xy} = \varepsilon E_0 E'_y \quad \text{and} \quad 4\pi T'_{xz} = \varepsilon E_0 E'_z$$

and after introduction of these expressions into Eqs. (2.115) and (2.112), we obtain

$$F_x = F'_x = \frac{\varepsilon}{8\pi} E_0^2 \oint \cos(\mathbf{n}, x) dS + \frac{1}{4\pi} E_0 \oint \varepsilon E'_n dS$$

* It should be noted that the dimensions of the surface S may considerably exceed those of the test body itself and are limited only by the first condition because the forces \mathbf{f}' do not act on sections of the homogeneous medium inside of S .

It can easily be seen that the first integral becomes equal to zero*, while the second one, on the basis of Eq. (2.19), equals $4\pi q$, so that $F_x = qE_0$.

Further, in the above conditions**, $F_y = F_z = 0$. Thus, the enumerated conditions do indeed ensure the correctness of Eq. (2.131).

Problem 20. A charge q is in a vacuum at a distance of z_0 from the surface of a homogeneous dielectric filling the half-space $z \leq 0$. Using the results obtained in solving the example at the end of Sec. 2.4, show that the force of attraction between the charge and the dielectric is

$$F = \frac{\epsilon - 1}{\epsilon + 1} \left(\frac{q}{2z_0} \right)^2$$

Note. This expression for the force of interaction naturally also remains true for a dielectric having finite dimensions if its transverse dimensions (more exactly the distance from the charge to the sections where the surface of the dielectric stops being flat or where the permittivity of the dielectric begins to change) and thickness are sufficiently great in comparison with z_0 .

* Compare with Sec. 4.15, in particular with Eq. (4.123), from which it follows that

$$\oint \cos(\mathbf{n}, x) dS = \oint \mathbf{i} dS = \mathbf{i} \oint dS = 0$$

** Indeed, taking into account that $\oint \cos(\mathbf{n}, y) dS = 0$, it is a simple matter to obtain the following expression for F_y :

$$F_y = F'_y = \frac{\epsilon}{4\pi} E_0 \oint \{ E'_y \cos(x, \mathbf{n}) - E'_x \cos(y, \mathbf{n}) \} dS$$

Introducing the unit vectors \mathbf{i} , \mathbf{j} and \mathbf{k} along the axes of coordinates, we can write

$$E'_y \cos(x, \mathbf{n}) - E'_x \cos(y, \mathbf{n}) = (\mathbf{E}'\mathbf{j})(\mathbf{in}) - (\mathbf{E}'\mathbf{i})(\mathbf{jn}) = [\mathbf{ij}][\mathbf{nE}']$$

Finally, on the basis of Eqs. (A.56) and (1.36), we get

$$\oint [\mathbf{nE}'] dS = \int \text{curl } \mathbf{E}' \cdot dV = 0$$

3

Steady Electric Current

3.1 Current in Metals. Ohm's and Joule's Laws. Voltage

1. According to the definition given in Sec. 1.5, conductors of electricity are bodies distinguished by the property that if the intensity of the electric field \mathbf{E} inside the conductor differs from zero, then an electric current, i.e. the motion of charges, appears in the conductor.

In the present book, we shall almost exclusively limit ourselves to the consideration of only one definite class of conductors, namely, *metals*. The flow of a current through metal conductors *is not attended by chemical processes* in the conductor*, whereas, for example, when a current flows through an electrolyte solution, electrolysis occurs, i.e. the liberation of ions of the electrolyte on the electrodes submerged in the solution.

This difference is explained by the fact that the carriers of the charges in electrolytes are ions, i.e. charged atoms or groups of atoms, while the charges in metals are transferred by "free" electrons detached from the atoms of the metal.

Without meanwhile diverting our attention to considering the physical mechanism owing to which a current flows through a metal, we shall begin with a treatment of the phenomenological theory of steady currents.

2. The fundamental law of a constant or steady current—*Ohm's law*, which is a generalization of *experimental* data, is usually formulated as follows:

$$I = \frac{\varphi_1 - \varphi_2}{R} \quad (3.1)$$

where I = current in the conductor

R = resistance of a definite section of this conductor

φ_1 and φ_2 = values of the potential at the beginning and end of the section (counting in the direction of the current).

* This also relates to the flow of a current through the so-called semi-conductors.

By the *current* is meant the quantity of electricity flowing through a cross section of the conductor in a unit time*, while the direction of the current is conventionally considered to coincide with the one in which *positive* charges ought to move under the action of a field. In other words, it is conventionally considered that a current flows from the greater potential to the smaller one ($\varphi_1 > \varphi_2$).

Consequently, in the absolute system of units, the dimension of current is

$$[I] = \left[\frac{\text{abs. units of electricity}}{s} \right] = M^{1/2}L^{3/2}T^{-2}$$

The absolute unit of current corresponds to the transfer of one absolute unit of electricity a second through a cross section of a conductor. In the practical system of units, the quantity of electricity is measured in coulombs, and the current is measured in amperes. By definition, a current of one ampere carries one coulomb a second through a cross section of a conductor:

$$\begin{aligned} 1 \text{ ampere} &= 1 \frac{\text{coulomb}}{\text{second}} = 3 \times 10^9 \frac{\text{abs. units of electricity}}{\text{second}} = \\ &= 3 \times 10^9 \text{ abs. units of current} \end{aligned}$$

3. As regards the resistance R , its dimension, as follows from Eq. 3.1), equals

$$[R] = \left[\frac{\varphi}{I} \right] = \frac{M^{1/2}L^{1/2}T^{-1}}{M^{1/2}L^{3/2}T^{-2}} = L^{-1}T$$

Thus, the dimension of resistance is inversely proportional to that of velocity.

In practical units, the potential is measured in volts, and the resistance in ohms. By definition, a conductor has a resistance of one ohm if upon a potential difference across its ends of one volt a current of one ampere flows through it:

$$1 \text{ ohm} = 1 \frac{\text{volt}}{\text{ampere}} = \frac{1}{300} \frac{\text{abs. unit of potential}}{3 \times 10^9 \text{ abs. units of current}}$$

$$\frac{1}{9 \times 10^{11}} \text{ abs. unit of resistance}$$

4. The potential difference $\varphi_1 - \varphi_2$ in Eq. (3.1), according to Eq. (1.40), can be expressed through the line integral of the field intensity

* This definition is unambiguous because the same quantity of electricity flows through any section of a conductor (if only the current is steady and the circuit has no branches— see Sec. 3.3).

\mathbf{E} taken from the initial to the final section of the portion of the conductor being considered:

$$\varphi_1 - \varphi_2 = \int_1^2 E_s ds \quad (3.2)$$

where ds is an element of length of the conductor.

The line integral of the electric field intensity between points 1 and 2 is called the *voltage* existing between (or across) these points and will be denoted by \mathcal{E}_{12} :

$$\mathcal{E}_{12} = \int_1^2 E_s ds \quad (3.3)$$

Care must be taken to avoid confusing the concepts of *voltage* \mathcal{E}_{12} and *field intensity* \mathbf{E} , moreover because sometimes these concepts are denoted by the same term "tension".

Introducing Eqs. (3.2) and (3.3) into (3.1), we get

$$IR = \int_1^2 E_s ds = \mathcal{E}_{12} \quad (3.4)$$

This form of Ohm's law for a constant electric field is equivalent to Eq. (3.1). It has the advantage, however, that it can also be applied to varying (quasistationary) currents, whereas for the field of these currents, as we shall see in Chapter 6, the concept of electric potential φ , and, therefore, Eq. (3.1), are not applicable.

5. The flow of a current is known to be inseparably linked with the *liberation of heat* (the heating of conductors) *in the circuit*.

The quantity of heat Q liberated by a current in a unit time in a section of a circuit can be determined as follows. If the current in a conductor is I , then in an element of time dt the number of units of electricity flowing through each cross section of the conductor will be $dq = I dt$; in particular, the number of units of electricity penetrating into a portion of a conductor being considered through its initial section 1 equals that leaving this portion of the conductor through its section 2 (Fig. 35). Since the distribution of the charges

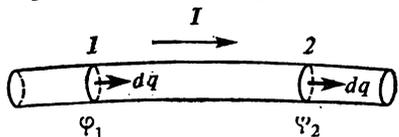


Fig. 35

in the conductor remains constant (a constant or steady current!), then the entire process is equivalent to the direct transfer of dq units of electricity from section 1 to section 2.

The work of the electric forces done in this transfer is

$$W = dq \int_1^2 E_s ds = I dt \int_1^2 E_s ds = I dt \mathcal{E}_{12} \quad (3.5)$$

where the line integral can be taken over the axis of the cylindrical conductor. According to the law of conservation of energy, the amount of energy equivalent to this work of the electric forces must be liberated in the form of a different kind of energy (for example in the form of heat). Hence, the energy liberated by a current is

$$Q dt = I dt \int_1^2 E_s ds$$

whence

$$\dot{Q} = I \int_1^2 E_s ds \quad (3.6)$$

Using Ohm's law (3.4), we get

$$Q = RI^2 \quad (3.7)$$

Finally, when the field E has the potential φ as for the field of steady currents, we can, according to Eq. (3.2), write this equation as follows:

$$Q = I(\varphi_1 - \varphi_2) \quad (3.8)$$

If a conductor is stationary and if no chemical reactions occur in it (electrolytes!), then this quantity of energy Q is liberated by the current in the form of *heat*. Thus, Eqs. (3.6) and (3.8) express the well-known *Joule law*.

We shall see already in Sec. 3.4 that the field of application of Eq. (3.7) is much broader than that of Eqs. (3.6) and (3.8), although within the limits of our present treatment all these equations are quite equivalent to one another. We shall see that in the presence of extraneous electromotive forces the equivalence of these equations is violated and that it is exactly Eq. (3.7) which may be used to determine the quantity of liberated heat.

6. The quantity Q equal to the amount of energy liberated in a unit time should obviously have the dimension of power. Indeed, $[Q] = [\varphi I] = (M^{1/2}L^{1/2}T^{-1})(M^{1/2}L^{3/2}T^{-2}) = ML^2T^{-3} = \frac{\text{work}}{\text{time}}$. Accordingly,

in the absolute system of units, Q is measured in ergs per second. In the practical system of units, Q is measured in watts: one watt is

the energy liberated by a current of one ampere when it passes a potential difference of one volt:

$$\begin{aligned} 1 \text{ watt} &= 1 \text{ volt-ampere} = \left(\frac{1}{300} \text{ abs. units of potential} \right) \times \\ &\times (3 \times 10^9 \text{ abs. units of current}) = \\ &= 10^7 \text{ abs. units of power} = 10^7 \text{ erg/s} \end{aligned}$$

In the practical system of units, work is measured in joules:

$$1 \text{ joule} = 10^7 \text{ ergs}$$

hence

$$1 \text{ watt} = \frac{\text{joule}}{\text{second}}$$

The energy Q liberated by a current can also be expressed in heat units, for instance in calories per second.

3.2 Current Density.

Differential Form of Ohm's and Joule's Laws

1. Apart from the current, the *current density* j also is very important. By definition, it equals the quantity of electricity flowing in one second *through a unit cross section* of a conductor at right angles to the current. In a homogeneous cylindrical conductor, the current is uniformly distributed over its cross section so that

$$j = \frac{I}{S} \quad (3.9)$$

where S is the cross-sectional area of the conductor.

In the general case, however, the current density j will not be the same over the entire cross section of the conductor so that by the current density at each given point of a conductor we must understand the limit of the ratio of the current dI flowing through the element dS of the conductor cross section *perpendicular to the direction of the current* to the magnitude of this element of cross-sectional area dS :

$$j = \lim_{dS \rightarrow 0} \frac{dI}{dS}$$

whence, conversely,

$$dI = j dS \quad (3.10)$$

If, finally, we consider the current density as a *vector* whose direction coincides with that of the current at a given point of a conductor, then with any direction of the element of area dS the relationship

$$dI = j_n dS$$

or

$$j_n = \frac{dI}{dS} \quad (3.11)$$

where j_n is the projection of the vector \mathbf{j} onto the outward normal \mathbf{n} to dS , and dI is the current flowing through dS , will hold. The correctness of this relationship follows from the fact that the component of the current density tangential to dS characterizes the flow of electricity *along* (and not *through*) the element of area dS (Fig. 36).

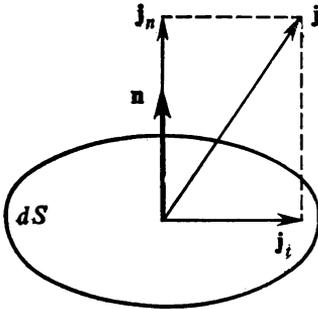


Fig. 36

It follows from Eq. (3.11), particularly, that either positive or negative values should be ascribed to the current dI flowing through the element of area dS depending on whether the current flows through dS in the direction of an arbitrarily chosen positive normal \mathbf{n} to this area or in the reverse direction.

2. Using the concept of current density, we can express fundamental equations of an electric current in the *differential* form which establishes the relationship between quantities relating to one definite point of a conductor, whereas Ohm's and Joule's laws in their integral form [Eqs. (3.1) and (3.7)] relate quantities characterizing different points (φ_1 and φ_2) or finite portions of a conductor (R).

Let us first deal with Ohm's law and consider a portion of a conductor that is homogeneous in composition and cylindrical in shape. The following relationship holds for it:

$$R = \frac{l}{S} \rho$$

where l = length of the portion of the conductor having the resistance R

S = cross-sectional area of this portion

ρ = resistivity, characterizing the substance of the conductor.

If we introduce the reciprocal of the resistivity ρ —the *conductivity* κ , i.e. $\kappa = 1/\rho$, instead of it, then we get

$$R = \frac{1}{S\kappa} \quad (3.12)$$

Using this expression in Eq. (3.4), we obtain

$$\frac{I}{S_{\kappa}} = \int_1^2 E_s ds$$

or in view of Eq. (3.9)

$$j = \frac{\kappa}{l} \int_1^2 E_s ds$$

For a steady current in a homogeneous cylindrical conductor, owing to the identity of the physical conditions along its entire length, the component of the field along the axis of the conductor E_s will evidently have a constant value so that

$$\int_1^2 E_s ds = E_s \int_1^2 ds = E_s l$$

and, consequently,

$$j = \kappa E_s$$

At each point of a conductor, the direction of the current coincides with that of the electric field* causing the charges to move. Hence, the vector of the current density must coincide in direction with the vector \mathbf{E} , and the last equation can finally be written in the form

$$\mathbf{j} = \kappa \mathbf{E} \quad (3.13)$$

This equation establishing that the current density in a conductor is proportional to the field intensity in it is the most general and simple formulation of Ohm's law. It can be called the *differential form of Ohm's law* (although it does not include any derivatives) because it establishes the relationship between quantities characterizing a *single definite point of a conductor*.

Although when deriving Eq. (3.13) we proceeded from a consideration of a homogeneous cylindrical conductor, in this differential form, however, Ohm's law may be applied to conductors having any

* Particularly, if a steady current is flowing through a homogeneous cylindrical conductor, then the vector \mathbf{E} in the conductor should be directed along its axis and, consequently, $E_s = E$ and $j = \kappa E$. Indeed, otherwise the component of the intensity \mathbf{E} perpendicular to this axis would induce a current parallel to it, i.e. movement of the charges from one side of the surface of the conductor to the other. This redistribution of the surface charges would continue until the field of these charges is compensated inside the conductor by a component of the external field perpendicular to its axis, i.e. until \mathbf{E} becomes parallel to the axis of the conductor.

shape, both homogeneous and heterogeneous [see, however, Eq. (3.23)].

Moreover, Eq. (3.13) also remains correct for varying electric fields and is thus one of the fundamental equations of electrodynamics.

3. Joule's law (3.7), being an integral law, can be transformed into a differential form similar to Ohm's law. For this purpose we shall introduce instead of Q the unit power of a current q , i.e. the quantity of heat liberated per second in a *unit volume* of a conductor:

$$q = \frac{Q}{V}$$

where V is the volume of the portion of the conductor in which the total quantity of heat Q is liberated*.

Let us again consider a homogeneous cylindrical conductor having the cross-sectional area S , the length l and the volume $V = Sl$. According to Eqs. (3.7) and (3.12), we get

$$q = \frac{Q}{V} = \frac{RI^2}{Sl} = \frac{1}{\kappa} \frac{I^2}{S^2}$$

whence on the basis of Eq. (3.9) we get

$$q = \frac{1}{\kappa} j^2 \quad (3.14)$$

or on the basis of Eq. (3.13)

$$q = \kappa E^2 = \mathbf{jE} \quad (3.15)$$

Equation (3.14) is the most general formulation of Joule's law applicable to any conductors regardless of the shape, homogeneity, etc., and, finally, regardless of whether we have to do with a steady or a variable current. As regards Eq. (3.15), we shall see in Sec. 3.5 that its range of application is somewhat narrower.

3.3 Conditions of Steadiness of Currents.

Continuity Equation.

Current Filaments

1. The electric field of steady currents, like an electrostatic field, is a potential field in the sense of Sec. 1.7. Particularly, the intensity vector \mathbf{E} of this field complies with condition (1.33) and can be expressed through the potential gradient:

$$\mathbf{E} = -\text{grad } \varphi$$

* If heat is liberated non-uniformly over the volume of a conductor, then, as usual, the value of q at each point of the conductor is determined by the relationship $q = \lim_{V \rightarrow 0} (Q/V)$.

Indeed, in the field of steady currents, the distribution of the charges in space must remain *stationary*, i.e. unchanging in time, because if any redistribution of the charges were to occur, the field intensity would inevitably change, and the current would stop being steady. But if the distribution of the charges is stationary, their field must be identical with the electrostatic field of the correspondingly distributed *fixed* charges; the circumstance that at a given point of space some of the elements of a charge owing to the presence of a current are replaced with others cannot effect the intensity of the electric field because the charge density at each point of space remains constant*. Hence, the *stationary field of steady currents*, like an electrostatic field, should be a potential field.

2. It follows from the stationary nature of the distribution of charges in the field of steady currents that the latter must either be closed or spread to infinity because otherwise charges would accumulate and disappear with time at the beginning (source) and termination (sink) of a current. For the same reason, an identical current should flow through different cross sections of a conductor (if only there are no branches of the conductor between these sections). Finally, at each point P of a circuit junction where two or in general n conductors that carry the currents I_i ($i = 1, 2, \dots, n$), respectively, come into contact, the so-called *Kirchhoff's first law* must be observed. According to it, at any circuit junction the algebraic sum of the currents must be zero:

$$\sum_{i=1}^n I_i = 0 \quad (3.16)$$

Otherwise electric charges will accumulate at the point P . For all the conductors meeting at P , the positive direction of the current must naturally be chosen in the same way, i.e. either coinciding with the direction toward the point P or with that away from it (Fig. 37).

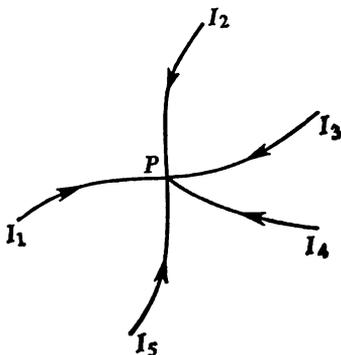


Fig. 37

* This statement is in essence one of the fundamental postulates of the theory of the electric field.

3. The most general condition for stationary currents and fields can be obtained as follows. According to Eq. (3.11), the integral $\oint j_n dS$ over an arbitrary closed surface S must equal the algebraic sum of the currents flowing through separate elements of the area of this surface dS , i.e. must equal the quantity of electricity flowing out in a unit time from the volume V confined by the surface S (if \mathbf{n} is an outward normal to S). On the other hand, according to the *law of conservation of electricity** which the theory of electricity is based on, the quantity of electricity passing out of the volume V in one second must equal $-\partial q/\partial t$, i.e. the decrease in the charge q inside of this volume during the same time**. We thus arrive at the equation

$$\oint j_n dS = -\frac{\partial q}{\partial t} \quad (3.17)$$

This very important equation, in accordance with the traditional terminology, is called a *continuity equation* and is a mathematical expression of the postulate of *conservation of the quantity of electricity*. We shall have to return to this equation on a later page. For the steady currents we are treating here, the distribution of the charges is stationary, i.e. $\partial q/\partial t = 0$ so that the continuity equation becomes

$$\oint j_n ds = 0 \quad (3.18)$$

4. If the volume V confined by the surface S contains no surfaces of discontinuity of the vector \mathbf{j} (such discontinuities, generally speaking, can occur only on interfaces of two different media), then Eq. (3.18) can be transformed with the aid of Gauss's theorem (A.17):

$$\oint j_n dS = \int \operatorname{div} \mathbf{j} dV = 0$$

* The law of conservation of electricity is often given the meaning that electric charges can only move in space, but cannot appear or vanish. Indeed, the seeming appearance of electric charges, for example upon the electrostatic charging of bodies by friction, consists only in the redistribution of elementary charges (electrons and ions) in space that previously existed in the bodies, but were so arranged in them that charges having opposite signs neutralized one another.

The conservation of the number of elementary charges, however, occurs only within the limits of conventional physical and chemical phenomena. The number of elementary charges does not remain constant in the field of phenomena relating to the physics of atomic nuclei and cosmic rays. For example, pairs of charges — an electron and a positron — may form at the expense of the energy of gamma-rays. Therefore, the law of conservation of electricity should be understood in the sense of the *constancy of the algebraic sum of the electric charges*, and not of the constancy of the sum of the charges of each sign taken separately.

** The sign of a *partial* derivative shows that in differentiation with respect to time the surface S and volume V are considered to be constant.

Owing to the arbitrary nature of the volume of integration V , it follows from the above equation that

$$\operatorname{div} \mathbf{j} = 0 \quad (3.19)$$

This equation is the most general expression of the fact that a steady current has no sources, i.e. that the *current lines* are always closed or pass away to infinity (compare with what was said about the lines of force of an electric field in Sec. 1.10)*. By current lines we must obviously understand the lines of the vector \mathbf{j} , i.e. lines, tangents to which coincide with the direction of the vector \mathbf{j} at the point of tangency.

The continuity of the current density vector may be violated on the interfaces of two different media. The component of this vector \mathbf{j} normal to the surface, however, must be identical at both sides of the surface of discontinuity because otherwise the quantity of electricity flowing in to one side of this surface would not be equal to the quantity of electricity flowing out from its other side. Hence

$$j_{1n} = j_{2n} \quad (3.20)$$

where \mathbf{j}_1 and \mathbf{j}_2 — current densities in the first and second media \mathbf{n} = normal to their interface.

Should a conductor border on a non-conducting medium, then $\mathbf{j} = 0$ in it, and, consequently, the current density component in the conductor normal to the surface must also equal zero:

$$j_n = 0 \quad (3.21)$$

5. Owing to the closed nature of steady currents, they can be resolved into a complex of infinitely thin closed (or passing away to infinity) *current filaments*. For this purpose, let us select an arbitrary element of area dS inside a conductor and conduct current lines through all the points of the contour of this element. The cylindrical surface formed by the totality of these lines will separate a so-called current filament from the volume of the conductor. Since no electric current evidently flows through the side surface of such a filament, the current dI in all the elements of area dS of each filament must be constant, i.e.

$$dI = \mathbf{j} dS = \text{const}$$

where dS should be understood to stand for an element of cross-sectional area of a current filament perpendicular to \mathbf{j} . Since, further,

* According to Sec. 1.10, there is a third possibility for lines having no sources: they can fill up finite portions of space. But in the absence of extraneous electromotive forces (see Sec. 3.4) the current lines in accordance with Eq. (1.13) coincide with the lines of force of a stationary electric field for which this possibility, according to Sec. 1.10, is excluded. It remains only to indicate that current lines can begin and terminate at points of indeterminacy of a field, where $\mathbf{E} = \mathbf{j} = 0$ (see the footnote on p. 63, Sec. 1.10).

the current lines, and, consequently, the surfaces of the current filaments cannot intersect anywhere (because at each point of space at which $\mathbf{j} \neq 0$ the direction of a current line is *unambiguously* determined by the direction of the vector \mathbf{j}), then each current filament should close on itself (i.e. be closed) or extend from infinity to infinity (see the footnote on p. 63).

6. With steady currents, as in electrostatics, the macroscopic density of the (free) charges inside *homogeneous* conductors equals zero because when $\kappa = \text{const}$ it follows from Eq. (3.19) that

$$\text{div } \mathbf{j} = \text{div } \kappa \mathbf{E} = \kappa \text{div } \mathbf{E} = 0, \text{ i.e.}$$

$$\frac{1}{4\pi} \text{div } \mathbf{E} = \rho = 0 \quad (3.22)$$

Problem 21. The space between the plates of a spherical capacitor (with the radii R_1 and R_2) is filled with a conducting medium having the conductivity κ . Find the current flowing through the capacitor if its plates are maintained at a constant potential difference $\varphi_2 - \varphi_1$ and show that the resistance of the spherical layer between the plates is

$$R = \frac{1}{4\pi\kappa} \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$$

Problem 22. Show that the current lines are refracted on the interface between two conductors, and

$$\tan \beta_1 : \tan \beta_2 = \kappa_1 : \kappa_2$$

where β_1 = angle between a current line in the first medium and a normal to the interface

κ_1 = conductivity of the first medium

β_2 and κ_2 = relevant quantities for the second medium (cf. Problem 16, Sec. 2.3).

Problem 23. Show that the normal component of the electric displacement at the surface of a conductor is also determined by the equation

$$D_n = 4\pi\sigma$$

if a steady current flows through the conductor, but that vector equation (2.26) stops being true in this case.

3.4 Extraneous Electromotive Forces. Quasilinear Currents. Kirchhoff's Second Law

1. A very significant distinction between the stationary field of a steady current and an electrostatic field is that a continuous expendi-

ture of energy is needed to maintain the former, whereas no conversions of energy occur in an electrostatic field. Indeed, we have seen that an electric current, i.e. the transfer of electricity along conductors under the action of the forces of an electric field, is attended by the work of these forces, and an amount of energy equivalent to this work is liberated in the form of the so-called Joule heat. Owing to the stationary nature of the field of steady currents, the entire energy liberated in the circuit of a current must continuously be compensated at the expense of other kinds of energy—mechanical (a dynamo); chemical (galvanic cells, accumulators), thermal (thermocouples) energy, etc. In other words, to maintain a steady current, it is necessary that *electromotive forces of a non-electrostatic origin* (induction, contact on the interfaces between different conductors, thermoelectric, etc.) act in definite sections of a current circuit. It is their work that compensates the expenditure of electric energy liberated in the form of Joule heat.

If all the e.m.f.'s acting in a circuit were forces of an electrostatic field, i.e. Coulomb forces, then under the action of these forces the positive charges of the conductors would flow from places having a greater potential to ones having smaller potentials, and the negative charges would flow in the opposite direction, which would result in levelling out of the potentials. All the conductors connected to one another would acquire the same potential, and the current would stop flowing. In other words, when only Coulomb forces are present a *stationary* field must be a *static* one. We thus arrive at a conclusion that in a certain respect reminds us of Earnshaw's theorem. As in Sec. 1.19, we have to introduce forces of a non-electrostatic origin that act on electric charges. The difference is only that in Sec. 1.19 we had to introduce such forces to take into consideration the possibility of *stable equilibrium* of a system of electric charges; now the existence of *steady* currents makes us do this.

Thus, we must assume that apart from electric forces of a stationary electric field, a certain field of forces of a non-electrostatic origin may also act on electric charges in conductors. For brevity's sake, we shall call these forces *extraneous* ones (relative to an electrostatic field) and denote the field intensity of the extraneous forces by \mathbf{E}^{ext} (or \mathbf{E}_{ext}).

In the *present chapter*, we shall consider that all e.m.f.'s of a non-electrostatic origin are extraneous forces. A very important class of these forces will be reduced in Chapter 6 to the forces of a varying electromagnetic field (*induction forces*), after which we shall no longer apply the term "extraneous" to forces of this class. Apart from them, however, there are also "*extraneous*" forces in the proper meaning of the word due to the physical and chemical non-uniformity of the conductors. Such are the forces appearing upon the contact of conductors having a different chemical composition (a galvanic cell, accumulator) or at different temperatures (a thermocouple),

when there is a gradient of concentration in an electrolyte solution (a concentration galvanic cell), etc. Naturally, the electron theory of matter is confronted with the problem of determining the mechanism causing all extraneous e.m.f.'s to appear and reducing them to the interaction of the electric charges in the atoms of non-uniform conductors. This problem, however, is beyond the scope of this book.

2. If, in accordance with Eq. (3.13), a current having the density

$$\mathbf{j} = \kappa \mathbf{E}$$

appears in a conductor under the action of the electrostatic field \mathbf{E} , then under the joint action of the field \mathbf{E} and the field of extraneous forces \mathbf{E}^{ext} a current having the density

$$\mathbf{j} = \kappa(\mathbf{E} + \mathbf{E}^{\text{ext}}) \quad (3.23)$$

should obviously appear. Equation (3.23) is a differential form of the *generalized Ohm's law* (for the case when extraneous e.m.f.'s are present), from which it is a simple matter to obtain the integral form of this law. In the present section, it will be sufficient for us to consider *quasilinear currents*.

3. We shall apply the term quasilinear (do not confuse it with line currents, which will be treated in the following chapter) to currents complying with the following conditions: in each portion of a conductor carrying a current the direction of its *axis* can be established so that at all points of any cross section of the conductor perpendicular to the axis all the physical quantities (\mathbf{j} , φ , κ , \mathbf{E} , \mathbf{E}^{ext} , etc.) may with sufficient accuracy be considered to be constant, and so that the current density vector \mathbf{j} will be parallel (or antiparallel) to this axis. We shall call such currents quasilinear because quite often the consideration of a conductor carrying a current may be replaced with the consideration of its axis, which we shall call the *contour* or *path of the current*.

4. Let us consider an arbitrary portion of a quasilinear current confined between cross sections 1 and 2 and first assume that there are no branches of the current circuit here. Let the area of a cross section of the conductor perpendicular to the axis be S . In general, S may be variable along the length of the conductor. Dividing Eq. (3.23) by κ , forming further the scalar product of this equation and an element of the conductor axis ds taken in the direction of the current \mathbf{j} , and integrating from cross section 1 to section 2, we get (since $\mathbf{j} ds = j ds$)

$$\int_1^2 \frac{j ds}{\kappa} = \int_1^2 E_s ds + \int_1^2 E_s^{\text{ext}} ds$$

Let us substitute I/S for j in the first integral and put I outside the integral as being a constant quantity. Further, the integral

$$\int_1^2 \frac{ds}{Sx} = R_{12}$$

is nothing but the resistance of the portion of the conductor between cross sections 1 and 2 because the integrand equals the resistance of an element of length of the conductor; particularly, for a homogeneous conductor having a constant cross section, R_{12} coincides with Eq. (3.12). Hence, finally,

$$IR_{12} = \int_1^2 E_s ds + \int_1^2 E_s^{\text{ext}} ds \quad (3.24)$$

which is the most general integral form of the *generalized Ohm's law*.

The voltage of the extraneous e.m.f.'s between points 1 and 2 [cf. Eq. (3.3)]

$$\mathcal{E}_{12}^{\text{ext}} = \int_1^2 E_s^{\text{ext}} ds \quad (3.25)$$

is often called simply the *electromotive force* applied between (across) these points and is abbreviated to e.m.f. Using Eqs. (3.3) and (3.25) in Eq. (3.24), we get

$$IR_{12} = \mathcal{E}_{12} + \mathcal{E}_{12}^{\text{ext}} \quad (3.26)$$

Hence, the product of the current and resistance of an arbitrary portion of a conductor equals the sum of the voltage and the extraneous e.m.f. applied to this portion.

If the electric field \mathbf{E} has the potential φ , as in the stationary field of steady currents, then according to Eq. (3.2) we can write Eq. (3.26) as follows:

$$IR_{12} = \varphi_1 - \varphi_2 + \mathcal{E}_{12}^{\text{ext}} \quad (3.27)$$

A particular case of this equation in the absence of extraneous e.m.f.'s is our initial expression (3.1) of the ungeneralized Ohm's law.

5. If a closed quasilinear current has no branches, then integrating in Eq. (3.24) over the entire length of this current, we get

$$IR = \oint E_s ds + \oint E_s^{\text{ext}} ds \quad (3.28)$$

where R is the total resistance of the closed conductor. If \mathbf{E} has a potential, then according to Eq. (1.33) the first integral vanishes so

that for a steady current Eq. (3.28) acquires the form

$$IR = \oint E_s^{\text{ext}} ds = \mathcal{E}^{\text{ext}} \tag{3.29}$$

where \mathcal{E}^{ext} is the total e.m.f. in the current circuit. Hence, an unbranched steady current equals the quotient obtained upon dividing the total extraneous e.m.f. in its circuit by the resistance of this circuit. Thus, in the absence of extraneous e.m.f.'s, a steady current must vanish, as we have already indicated at the beginning of this section.

Let us finally consider an arbitrary circuit of quasilinear currents with an arbitrary number of branches (Fig. 38) and compose an arbitrary closed loop L , for example loop 1231 , from separate portions of this circuit. Let I_{ik} ($i, k = 1, 2, 3$) be the current in portion ik . We shall consider the quantity I_{ik} to be positive if the current flows from point i to point k , and negative in the opposite case. Compiling for each portion of the circuit an equation similar to Eq. (3.27) and summing, we get

$$\sum I_{ik}R_{ik} = \sum (\varphi_i - \varphi_k) + \sum \mathcal{E}_{ik}^{\text{ext}}$$

Taking into account that

$$\sum (\varphi_i - \varphi_k) = (\varphi_1 - \varphi_2) + (\varphi_2 - \varphi_3) + (\varphi_3 - \varphi_1) = 0$$

we have

$$\sum I_{ik}R_{ik} = \sum \mathcal{E}_{ik}^{\text{ext}} \tag{3.30}$$

Thus, in any closed loop of currents, the algebraic sum of products of the kind $I_{ik}R_{ik}$ equals the sum of the extraneous e.m.f.'s applied to this loop. This statement is called *Kirchhoff's second law*. Equation (3.29) is obviously a particular case of applying this law to an unbranched current circuit.

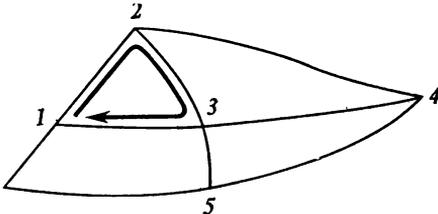


Fig. 38

6. It must be noted in conclusion that the existence of extraneous e.m.f.'s must naturally also be taken into account in electrostatics. For example, in a chemically or physically non-uniform conductor, the condition of electrostatic equilibrium consists not in the equality to zero of the intensity \mathbf{E} of the electric field inside the conductor, but in the equality

$$\mathbf{E} + \mathbf{E}^{\text{ext}} = 0 \text{ or } \mathbf{E} = -\mathbf{E}^{\text{ext}} \tag{3.31}$$

because only in this condition will there be no current in a conductor [cf. Eq. 3.23].

Problem 24. *A, B, and C are three consecutive stations on a telegraph line (Fig. 39). The telegraphist at A knows that the*

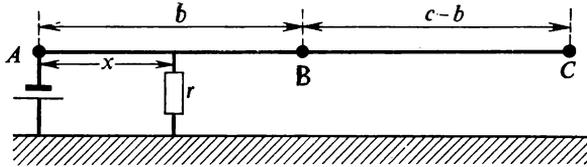


Fig. 39

insulation of the line between *A* and *B* has been damaged (which is equivalent to earthing of the line). By connecting a battery between the earth and his end of the line, he measures the resulting current in the line in three different conditions: (1) the line is earthed at station *C* and insulated at *B* (the current is *I*); (2) the line is earthed at *B* and insulated at *C* (the current is now *I'*); and (3) the line is insulated at both *B* and *C* (the current is *I''*). Determine the distance from *A* to the point where the line is damaged. It is assumed that the resistance of the earth and also that of the earthing at the stations may be disregarded.

3.5 Conversion of Energy in a Current Circuit. Contact E.M.F.'s

1. Multiplying Eq. (3.24) by the current *I* in the circuit and regrouping terms, we get

$$I \int_1^2 E_s ds = I^2 R_{12} - I \int_1^2 E_s^{\text{ext}} ds \quad (3.32)$$

The left-hand side of this equation, according to Eq. (3.5), equals the work done by the forces of the electric field in a unit time in portion 1-2 of the circuit. This work is equal to the difference between two terms, the first of them

$$Q = I^2 R_{12} \quad (3.33)$$

being quadratic relative to the current *I*. It is therefore independent of the direction of the current and is always positive. The second term

$$A = I \int_1^2 E_s^{\text{ext}} ds \quad (3.34)$$

is linear relative to I and changes its sign when the current changes its direction. The first term Q coincides with Eq. (3.7) for the Joule heat which we obtained in Sec. 3.1 on the assumption that there are no extraneous e.m.f.'s in a conductor.

When extraneous e.m.f.'s are present, the quantity Q quadratic relative to I also expresses the heat liberated by the current—the so-called Joule heat. The term A linear relative to I is obviously the work done in a unit time by the extraneous e.m.f.'s. Thus, Eq. (3.32) means that the Joule heat Q liberated by the current in circuit portion 1-2 equals the sum of the work done in this portion of the circuit, namely the work of the forces of the electric field $I \int_1^2 E_s ds$ and that of the extraneous e.m.f.'s.

The total quantity of heat liberated by the current in a given portion of a circuit, however, does not always coincide with the corresponding Joule heat Q . For instance, at the place of contact of two different conductors, in addition to the Joule heat, which depends only on the current and the resistance of the conductors, the so-called *Peltier heat* is also liberated. The latter depends on the extraneous e.m.f.'s, which are determined, in turn, by the chemical nature of the conductors, their temperature, etc. Similarly, when there is a temperature gradient in a conductor, apart from the Joule heat, the so-called *Thomson heat* is also liberated in it. Unlike the Joule heat, however, the Peltier and Thomson heats are *linear* functions of the current I and change their sign when the current changes its direction*. Therefore, the Joule heat Q can always be separated from the total quantity of heat liberated by a current; the Joule heat equals the half-sum of the heats liberated by a given current I when flowing in opposite directions.

It must be mentioned that in ordinary conditions (with not very small currents) the Peltier and Thomson heats are only an insignificant fraction of the Joule heat so that they may be in general ignored. Thus, Eq. (3.7) or (3.33) for the Joule heat also remains correct when extraneous e.m.f.'s are present, whereas Eqs. (3.6) and (3.8) express the work done by the forces of the electric field when a current flows; it equals the Joule heat only in the absence of extraneous e.m.f.'s. For example, it follows from Eqs. (3.24) and (3.2) that in the presence of extraneous e.m.f.'s Eq. (3.8) for the Joule heat must be replaced with the following one:

$$Q = I(\varphi_1 - \varphi_2) + I\mathcal{E}_{12}^{\text{ext}} \quad (3.35)$$

* A minus sign, for example, of the Peltier heat means that the corresponding amount of heat is *absorbed*, and not liberated, when a current flows through the place of contact of two conductors.

Equation (3.14) for the *unit* quantity of Joule heat (i.e. the Joule heat liberated in a unit time in a unit volume of a conductor)

$$q = \frac{1}{\kappa} j^2$$

was obtained from Eq. (3.7), which coincides with Eq. (3.33). Consequently, Eq. (3.14) also holds when extraneous forces are present. Instead of Eq. (3.15), from Eqs. (3.14) and (3.23) we get

$$q = \kappa (\mathbf{E} + \mathbf{E}^{\text{ext}})^2 = \mathbf{j} \cdot (\mathbf{E} + \mathbf{E}^{\text{ext}}) \quad (3.36)$$

which expresses the fact that the Joule heat liberated by a current in each element of the volume of a conductor equals the *sum* of the work done by the forces of the electric field and of that done by forces that are extraneous in this element of volume.

As we have already indicated at the beginning of Sec. 3.4, the total work of the Coulomb forces of the stationary field of steady currents should equal zero because otherwise the energy of this field would diminish and it could no longer be stationary. Hence, the *total* quantity of Joule heat liberated in the entire circuit of a current should equal the work of the extraneous e.m.f.'s. And, indeed, applying Eq. (3.32) to the entire length of an unbranched closed conductor and taking into account that in the field of steady currents \mathbf{E} has a potential, we get on the basis of Eq. (3.29)

$$Q = I \mathcal{E}_x^{\text{ext}} \quad (3.37)$$

which is a mathematical formulation of the statement we have just made.

3. To get an idea of the energy transformations in the circuit of a current, let us imagine that all the extraneous e.m.f.'s are concentrated in portion *a* of this circuit*. Hence, work of these e.m.f.'s will be done only in this portion *a*, while heat will be liberated in all the portions of the circuit. Since the total quantity of heat liberated** equals the work of the e.m.f.'s concentrated, according to our condition, in portion *a*, then from the energy viewpoint the part of the electric current consists in transferring the energy given up by the extraneous forces to the remote portions of the circuit.

Suppose, for example, that we have to do with an unbranched quasilinear conductor with points 1 and 2 being the boundaries of portions *a* and *b* (Fig. 40), and that the e.m.f. \mathcal{E}^{ext} concentrated in portion *a* is directed from 1 to 2:

$$\mathcal{E}^{\text{ext}} = \oint E_s^{\text{ext}} ds = \int_1^2 E_s^{\text{ext}} ds = \mathcal{E}_a^{\text{ext}} > 0$$

* We can imagine, for example, that an accumulator or a direct-current dynamo is connected in this portion of the circuit.

** For simplicity, we shall speak in the following only of the Joule heat, ignoring the Peltier and Thomson heats.

so that according to Eq. (3.27)

$$IR_a = \varphi_1 - \varphi_2 + \mathcal{E}_a^{\text{ext}}$$

where R_a is the resistance of portion a . Since, on the other hand, according to Eq. (3.29) we have

$$IR = I(R_a + R_b) = \mathcal{E}^{\text{ext}} = \mathcal{E}_a^{\text{ext}}$$

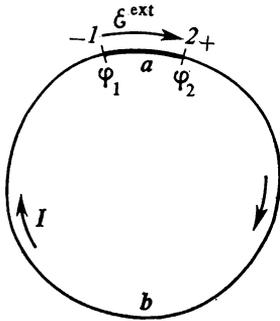


Fig. 40

then

$$\varphi_2 - \varphi_1 = \mathcal{E}_a^{\text{ext}} - IR_a = IR_b = \frac{\mathcal{E}_a^{\text{ext}} R_b}{R_a + R_b} > 0^* \quad (3.38)$$

Only Coulomb forces of the electric field act in the “external” portion b so that the positive charges will flow along b from the greater potential φ_2 to the smaller one φ_1 . This would result in levelling out of the potentials and discontinuing of the current if no extraneous e.m.f.’s acted on portion a . These e.m.f.’s send positive charges along a from 1 to 2, i.e. from the smaller potential φ_1 to the greater one φ_2 against the Coulomb forces of the electrostatic field acting in this portion. Since according to expression (3.38) we have $\mathcal{E}_a^{\text{ext}} > \varphi_2 - \varphi_1$, then these charges will indeed move along a “against” the potential difference $\varphi_2 - \varphi_1$ and in the direction of the e.m.f.

Thus, the e.m.f. continuously “pumps” charges along a from 1 to 2, whence they again flow along b to 1, etc. The work of the Coulomb forces of the electric field in portion a will be negative, and in b positive, while its sum equals zero (as in any motion of charges in a potential field along a closed path). The extraneous e.m.f.’s will do (positive) work only in portion a . An amount of heat will be liberated in a that is equivalent to the algebraic sum of the positive work of the e.m.f.’s and the negative work of the field forces; the

* Thus, the potential difference across the ends of the portion of the circuit which an extraneous e.m.f. is applied to is, generally speaking, less than this e.m.f. and may be considered equal to it only if the resistance R_a of this portion is vanishingly small in comparison with the “external” resistance R_b . The quantities $\varphi_2 - \varphi_1$ and $\mathcal{E}_a^{\text{ext}}$ will be exactly equal if the circuit is open ($R_b = \infty$).

surplus work of the e.m.f.'s over this amount of heat will be liberated by the current in portion *b*.

4. Let us finally consider the case when a current circuit has not one, but two portions *a* and *a'* in which extraneous e.m.f.'s $\mathcal{E}_a^{\text{ext}}$ and $\mathcal{E}_{a'}^{\text{ext}}$ directed opposite to each other are applied. If $\mathcal{E}_a^{\text{ext}} > \mathcal{E}_{a'}^{\text{ext}}$, then the current will flow as shown in Fig. 41. In portion *a*, the

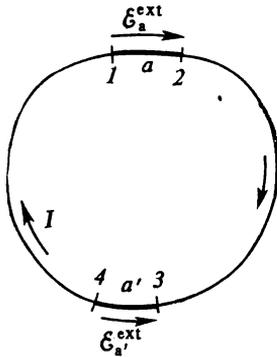


Fig. 41

direction of the current will coincide with that of the e.m.f. $\mathcal{E}_a^{\text{ext}}$, which will therefore do the *positive* work $I\mathcal{E}_a^{\text{ext}}$, whereas the e.m.f. $\mathcal{E}_{a'}^{\text{ext}}$ will do the *negative* work $-I\mathcal{E}_{a'}^{\text{ext}}$; finally, the quantity of heat liberated in the current circuit in a unit time will be

$$Q = I^2 R = I(\mathcal{E}_a^{\text{ext}} - \mathcal{E}_{a'}^{\text{ext}})$$

Thus, the work of the e.m.f. in portion *a* will be spent, first, for the liberation of the heat Q and, second, for overcoming the resistance to the current of the e.m.f. $\mathcal{E}_{a'}^{\text{ext}}$ in *a'*; hence, the current will do the positive work $I\mathcal{E}_{a'}^{\text{ext}}$ in it. In other words, energy will be transferred in the circuit from *a* to *a'* with Q lost with the liberated heat. This is how matters will be, for instance, if a dynamo or galvanic cell will be connected in *a* and a motor (whose rotation is attended by the appearance of an induced e.m.f. directed against the current—see Chap. 6) or an accumulator being charged by the current in *a'*.

5. We shall terminate this section with a few remarks of a general nature about the so-called *contact* e.m.f.'s. These "extraneous" e.m.f.'s appear in the boundary layer between contacting conductors having different chemical compositions; their magnitude depends on the chemical nature of the contacting conductors (and also on other physical conditions, for example the temperature), but does not depend on the shape and dimensions of the conductors. The thickness of the layer in which these contact e.m.f.'s act is so small that with sufficient accuracy we may consider them to be concentrated *on the surface* of contact of the conductors. Assuming that the resistance R_{12} to the current when it passes through the infinitely thin surface of contact of conductors 1 and 2 equals zero, we get from Eq.

(3.27) for two adjacent points at different sides of the contact surface

$$\mathcal{E}_{12}^{\text{ext}} = \varphi_2 - \varphi_1$$

Hence, the contact e.m.f. maintains between conductors *I* and 2 the contact potential difference (or jump) $\varphi_2 - \varphi_1$ equal to it*.

If we form a closed circuit from several conductors (*I*, 2, 3, etc.) of a different chemical nature connected in series, then the contact e.m.f.'s $\mathcal{E}_{12}^{\text{ext}}$, $\mathcal{E}_{23}^{\text{ext}}$, etc. will act on each surface of their contact (*I*, 2), (2, 3), etc. The current in the circuit, according to Kirchhoff's second law (3.30), will be determined by the sum of these e.m.f.'s.

It is general knowledge that all conductors can be divided into two classes. Conductors of the first kind, which includes all the metals, have the property that in any circuit formed only of these conductors the algebraic sum of the contact e.m.f.'s always equals zero if only all the parts of the circuit are at the same temperature (otherwise thermoelectromotive forces appear whose sum, generally speaking, differs from zero).

Consequently, a current in such a closed circuit cannot appear in the absence of e.m.f.'s of a different origin (for instance thermo-e.m.f.'s). It follows, particularly, that the contact e.m.f.'s between any three conductors of the first class are related by the expression

$$\mathcal{E}_{13}^{\text{ext}} = \mathcal{E}_{12}^{\text{ext}} + \mathcal{E}_{23}^{\text{ext}}$$

because upon forming a closed circuit from these conductors we should get

$$\mathcal{E}_{12}^{\text{ext}} + \mathcal{E}_{23}^{\text{ext}} + \mathcal{E}_{31}^{\text{ext}} = 0$$

and $\mathcal{E}_{13}^{\text{ext}}$ obviously equals $-\mathcal{E}_{31}^{\text{ext}}$.

These expressions give us the right to pay no attention to the contact e.m.f.'s between metal conductors when studying the currents in the circuits they form.

If, however, a circuit includes conductors of the second kind which, first of all, electrolytes belong to, then the sum of the contact e.m.f.'s, generally speaking, will differ from zero, and a current will appear in the circuit. This property of a conductor of the second kind underlies the design of galvanic cells and accumulators, which are a combination of conductors of the first and second kinds connected in series.

From the energy viewpoint, this difference between conductors of the first and second kinds consists in that when a current flows through a circuit of conductors of the first kind the total work of the contact e.m.f.'s equals zero, whereas when the circuit contains conductors of the second kind, this work generally differs from zero.

* As indicated by H. Helmholtz, this potential jump is due to the existence of an electrical double layer (Sec. 1.14) on the interface between the conductors. The very existence of a double layer, however, presupposes the presence of extraneous forces in the absence of which a double layer cannot exist: its opposite charges should combine and neutralize one another.

The work of contact e.m.f.'s is done at the expense of the *chemical* energy of the conductors of the second kind, the passage of a current along which is *always* attended by chemical reactions in them.

For example, a common lead-acid accumulator consists of two lead plates immersed in aqueous sulphuric acid. One of the plates is coated with a layer of lead peroxide PbO_2 . When the circuit which the accumulator is connected to is closed, the contact e.m.f.'s induce a current. It causes the substances of the plates, Pb and PbO_2 , to enter into a chemical reaction with the H_2SO_4 , as a result of which lead sulphate PbSO_4 appears on both plates. This reaction is connected with the liberation of chemical energy. In the conventional conditions of an experiment, it is completely liberated in the form of heat, while in an accumulator, part of it goes to maintain a current in a circuit. True, in the long run, this part of the chemical energy also transforms into Joule heat which is liberated, however, not only in the accumulator itself, but also in other portions of the circuit. Conversely, when charging an accumulator, current from an external source is passed through it in a direction *opposite* to the e.m.f. of the accumulator. Thus, this current does positive work in the accumulator. Apart from heating, it is used for the reverse chemical reaction:



that is attended by the *absorption* of energy and charges the accumulator.

As we have already mentioned at the beginning of this chapter, the circumstance that the passage of a current through electrolytes is attended by chemical reactions is due to charges being carried in them by ions, i.e. charged atoms or groups of atoms, while the carriers of a current in metals are not ions, but "free" electrons.

3.6 Fundamental Concepts of the Electron Theory of Metals. Tolman's Experiments

1. As already noted in Sec. 1.5, the simplest concept of the flow of a current through metal consists in the following.

If a metal is in the solid (or liquid) state, then its atoms are ionized, i.e. are decomposed into one or more electrons and a positive ion. These ions, which are arranged at the points of the crystal lattice and perform only slight oscillations about their positions of equilibrium, form the solid skeleton of a metal body. The so-called "free" electrons detached from the ions, on the other hand, chaotically travel in the interstices of the ions forming a special kind of an electron "gas".

In the absence of an external electric field, these electrons perform absolutely chaotic thermal motion. The appearance of a field, however,

leads to an increase in the motion of the electrons in the direction of the forces of the field acting on them, i.e. to the appearance of an electric current. Upon colliding in their motion with ions of the metal, the electrons give them up their surplus kinetic energy acquired under the action of the field forces. The result is an increase in the energy of the thermal motion (oscillations) of the ions, i.e. heating of the metal (the liberation of Joule heat).

2. From the standpoint of this notion of the mechanism of a current in metals, a number of phenomena observed in them acquire an exceedingly simple interpretation and explanation. Among them are, for instance, the thermionic phenomena mentioned in Sec. 1.11, which consist in the emission by incandescent metals of a stream of free (in the true meaning of the word) electrons into the surrounding space. At ordinary temperatures, the "free" electrons in a metal cannot escape from its surface because retaining forces directed into the metal act on them in its surface layer*. But when the metal is heated the velocity of the thermal motion of its electrons grows, so that at a sufficiently high temperature an appreciable part of these electrons acquire such a great stock of kinetic energy that they succeed in overcoming the retaining forces of the surface layer and escape from the metal. The nature of the quantitative dependence of the intensity of a thermionic beam of electrons on the temperature of the metal well agrees with this notion of the mechanism of thermionic phenomena.

All metals are known to be not only good conductors of electricity, but also good conductors of heat. From the viewpoint of the electron theory, this coincidence is explained not by chance, but by the presence of free electrons in metals. In the latter, unlike non-conductors, heat is transferred not only by collisions between the atoms, but also, and mainly, by the free electrons. Acquiring an additional kinetic energy in a heated section, the mobile electrons comparatively rapidly transfer it in their motion to adjoining sections of the body and thus considerably accelerate the process of heat conduction.

The most direct proof of the fact that electrons are indeed the carriers of a current in metals, however, was given by *Tolman's* experiments (see below). He measured the electric currents appearing in a metal when the latter was accelerated and due to the "free" electrons lagging behind the motion of the crystal lattice of the metal.

3. Assume that a unit volume of a metal contains n free electrons having the mass m and the charge e . Owing to the chaotic motion of the electrons in the absence of an external electric field, all the directions of the velocity of the electrons are equally probable, the electron gas is as a whole at rest relative to the positive ions of the lattice, and the mean density of the current equals zero. Assume

* The need to take into consideration such surface forces whose physical nature we shall not dwell on here was explained in Sec. 1.19.

further, however, that for some reason or other additional ordered motion of the electrons has appeared with a mean velocity \mathbf{u} relative to the lattice. When counting the mean density of the current \mathbf{j} it is evidently sufficient to take into account only this ordered motion, while the chaotic motion may be ignored completely.

All the electrons will pass through a unit cross-sectional area of the conductor perpendicular to the vector \mathbf{u} in a unit time that are at a distance less than or equal to u from it, i.e. all the electrons in a cylinder having a cross-sectional area of 1 and an altitude of u . Their number is nu , and their charge is enu . Hence [cf. Eq. (1.69)]

$$\mathbf{j} = en\mathbf{u} \quad (3.39)$$

4. To understand the essence of Tolman's experiments, let us imagine that a ring made of a metal wire is brought by an external force into non-uniform rotational motion about its axis, the linear velocity of the points of the ring being v^* . If the electrons in the metal were firmly bound to the atoms, they would travel with the same velocity v , and as a result no current would appear (because the motion of the electrons and the positive ions with the same velocity would create equal and oppositely directed currents). Conversely, if there were no interaction between the electrons and the lattice, then the motion of the lattice would not be transmitted to the electrons, the mean velocity of the electrons upon motion of the ring would remain equal to zero, and their mean velocity u relative to the lattice would equal $-v$. The current density in this case would be

$$\mathbf{j} = -env = en\mathbf{u}$$

i.e. it would be determined as previously by Eq. (3.39), although it would be due to the motion not of electrons, but of positive ions having the charge $-e$ ($e < 0$) with the velocity v .

Actually, an intermediate case will occur, the electrons will partly be carried along by the non-uniform motion of the lattice, and a varying current will appear in the conductor. The equation of motion of the electrons in a metal will be

$$m \frac{d}{dt} (\mathbf{v} + \mathbf{u}) = \mathbf{F} \quad (3.40)$$

where $\mathbf{v} + \mathbf{u}$ = total velocity of an electron

\mathbf{F} = force acting on it.

The force \mathbf{F} will consist of the "force of friction" between the electrons and the metal lattice characterized by the resistance of the metal R , and of the induced e.m.f. resisting any change of the current in the conductor and characterized by the self-inductance of

* If a wire is sufficiently thin, the dependence of the velocity v on the distance from a given point of the ring to its axis may be ignored.

the conductor L ("current inertia"*—see Chapter 6). It is significant that both these forces depend *only on the relative* (\mathbf{u}) and not on the total ($\mathbf{u} + \mathbf{v}$) velocity of the electrons, i.e. in view of Eq. (3.39) only on the current density in the conductor.

Thus, the force \mathbf{F} is a function of \mathbf{u} and, therefore, Eq. (3.40) can be written as follows:

$$m \frac{d\mathbf{u}}{dt} = \mathbf{F}(\mathbf{u}) - m \frac{d\mathbf{v}}{dt}$$

This equation differs from that of the motion of electrons in a conductor at rest in the addition to the force $\mathbf{F}(\mathbf{u})$ of the "force of inertia" — $m \frac{d\mathbf{v}}{dt}$ as follows from the general laws of the mechanics of relative motion. Therefore, instead of finding the form of the function $\mathbf{F}(\mathbf{u})$, we can directly use the general equation for varying currents, which we shall acquaint ourselves in detail with in Chapter 6:

$$\frac{1}{c^2} L \frac{dI}{dt} + RI = \oint E_s^{\text{ext}} ds$$

equating the extraneous force eE^{ext} acting on an electron to the force of inertia — $m \frac{dv}{dt}$ in this expression.

The tangential component dv/dt of the acceleration of points of the rotating ring equals the derivative of the numerical value of the linear velocity v ; hence

$$eE_s^{\text{ext}} = -m \frac{dv}{dt}$$

and

$$\frac{1}{c^2} L \frac{dI}{dt} + RI = -\frac{m}{e} \oint \frac{dv}{dt} ds = -\frac{ms}{e} \frac{dv}{dt}$$

where s is the length of the circumference of the ring. Integrating this equation with respect to time from $t = t_1$ to $t = t_2$ and assuming that the current I becomes equal to zero at the initial and final moments t_1 and t_2 of the time interval being considered, we get

$$R \int_{t_1}^{t_2} I dt = -\frac{ms}{e} \{v(t_2) - v(t_1)\} \quad (3.41)$$

* The inertia of currents is due not only to the inertia (mass) of the current carriers — electrons, taken into consideration by the first term of Eq. (3.40), but also to a considerably greater extent to the magnetic interaction of these current carriers with one another.

5. It is exactly this expression that Tolman used for determining the ratio of the charge to the mass e/m for the carriers of current in a metal. A round wire coil was brought into rotation about its vertical axis and then suddenly braked (at the moment t_1) and stopped [$v(t_2) = 0$] during a fraction of a second. The current I flowing during this interval $t_2 - t_1$ along the coil was measured by means of a stationary galvanometer connected by two wires to the ends of the coil.

By measuring the quantities $R \int_{t_1}^{t_2} I dt$, s , and $v(t_1)$ and taking into account all the side effects, we can use Eq. (3.41) to determine the ratio m/e for the carriers of a current in a metal.

The sign of this ratio proved that negative charges carry a current, and the numerical value of the ratio m/e in Tolman's experiments conducted in 1926 was found to equal

$$\frac{m}{e} = 4.58 \times 10^{-9} \text{ g/C} = 1.53 \times 10^{-18} \text{ abs. cgs units}$$

which in the order of magnitude agrees with the value obtained in measurements of free electrons in cathode rays:

$$\frac{m}{e} = 5.66 \times 10^{-9} \text{ g/C} = 1.90 \times 10^{-18} \text{ abs. cgs units}$$

3.7 Electron Theory of Electrical Conductivity.

Difficulties of the Classical Theory.

Sommerfeld's Theory

1. To determine the dependence of the electrical conductivity of a metal on other physical quantities characterizing its properties, we shall use Eq. (3.39):

$$j = enu$$

In the first approximation, we shall consider the "electron gas" in the metal to be an ideal gas, i.e. we shall consider that in the intervals between collisions with other electrons and ions the electrons move according to the laws of motion of material points subjected to the action of only the force of the external (macroscopic) field \mathbf{E} .

In the absence of the external field \mathbf{E} , the mean velocity of the electrons relative to the lattice obviously equals zero. Under the action of the field \mathbf{E} , the electrons acquire a certain additional velocity \mathbf{u} parallel to the force $e\mathbf{E}$ acting on them. This accumulation of the velocity \mathbf{u} parallel to the force $e\mathbf{E}$ occurs only during the free flight of an electron between two consecutive collisions of it with ions of

the lattice*. The direction and magnitude of the velocity of the electron change according to the laws of chance as a result of each such collision. Hence, directly after a collision, the mean value of \mathbf{u} equals zero, while directly before a collision

$$\mathbf{u} = \frac{e\mathbf{E}}{m} \tau$$

where $e\mathbf{E}/m$ equals the acceleration imparted by the force $e\mathbf{E}$ to an electron, and τ is the mean duration of the free flight of the electrons. Thus, the mean value of \mathbf{u} is

$$\bar{\mathbf{u}} = \frac{1}{2} \frac{e\mathbf{E}}{m} \tau \quad (3.42)$$

On the other hand, if λ is the mean free path of an electron, then

$$\tau = \frac{\lambda}{v} \quad (3.43)$$

where v is the mean velocity of the chaotic motion of the electrons in the absence of an external field, since in all practically interesting cases $v \gg u$, and therefore in calculating the mean *numerical* (but not vector) value of the velocity of an electron the additional velocity u may be disregarded.

Introducing Eq. (3.43) into (3.42) and the resulting equation into (3.39), we get

$$\mathbf{j} = \frac{e^2 n \lambda}{2mv} \mathbf{E}$$

Thus, the current density is proportional to the field intensity \mathbf{E} as required by Ohm's law (3.13)**; further, the proportionality factor, i.e. the conductivity of a metal κ , is equal to

$$\kappa = \frac{e^2 n \lambda}{2mv} \quad (3.44)$$

2. Let us assume that classical statistical mechanics may be applied to the electrons in a metal. According to its fundamental laws, the mean energy of the translational thermal motion of the molecules

* Collisions between free electrons do not affect their mean velocity because upon the collision of two bodies having an identical mass the vector sum of their velocities does not change (the law of conservation of momentum).

** It should be noted that the correctness of Ohm's law (and of Joule's law directly following from it) is closely related to the assumption that $u \ll v$, i.e. that the kinetic energy acquired by electrons under the action of the field \mathbf{E} is almost completely transferred to the ions of the metal upon collisions so that the mean velocity of the electrons when a field is present only insignificantly exceeds their velocity in the absence of a field. These conditions are not observed for an electronic current in a vacuum, and Ohm's law for this reason (and also due to the presence of a "space charge") cannot be applied at all (see Sec. 1.11).

of any gas depends only on the absolute temperature T , but not on the chemical nature and molecular mass of the gas, and equals

$$\frac{1}{2} mv^2 = \frac{3}{2} kT \quad (3.45)$$

where k is the Boltzmann constant. Applying this relationship to the electron gas in a metal, we get from Eq. (3.44)

$$\alpha = \frac{e^2 n \lambda}{2 \sqrt{3 k m T}} \quad (3.46)$$

Unfortunately, this equation cannot be verified by direct comparison with experimental results because we know neither the absolute value of the quantities n and λ nor the nature of their temperature dependence. This equation can be verified indirectly, however, if in addition to Eq. (3.46) we also consider other relationships established by theory between the unknown quantities n and λ on one hand and a number of experimentally measured quantities (the heat capacity of a metal, the thermoelectromotive force, etc.) on the other.

3. Because of the lack of space, we shall consider this matter only with respect to the heat capacity. If the mean kinetic energy of free electrons is determined by the classical formula (3.45), then the total kinetic energy of the unit volume of the electron gas in a metal is

$$n \cdot \frac{1}{2} mv^2 = \frac{3}{2} nkT$$

Hence, the heat capacity C_V (at constant volume) of a unit volume of the electron gas, i.e. the energy needed to raise its temperature by one kelvin, is

$$C_V = \frac{3}{2} nk = \frac{3}{2} \frac{nR}{N_A} \quad (3.47)$$

where we have used the known relationship between the Boltzmann constant k , Avogadro's constant N_A , and Clapeyron's gas constant R :

$$kN_A = R$$

What is directly measured is naturally the *total* heat capacity of a metal, i.e. the sum of the heat capacities of the ionic lattice and of the free electrons, and not each of these addends separately.

The heat capacity of the lattice can be assessed, however, both theoretically and on the basis of data relating to solid dielectrics whose heat capacity completely consists of that of the crystal lattice*.

* The heat capacity of a unit volume of a lattice at conventional temperatures approximately equals $C_V = 3n_a R/N_A$, where n_a is the number of atoms (or ions) in a unit volume.

When the heat capacity of the lattice is subtracted from the experimentally measured total heat capacity of metals, a relatively very small value remains for the heat capacity of the electrons. This value can be brought into agreement with Eq. (3.47) only for such small values of the density of the free electrons n when too small values are obtained for the conductivity by Eq. (3.47). More exactly, if we ascribe to the heat capacity of the electron gas in metals the highest value from among those agreeing with experimental results, then we can find the *upper* limit of the quantity n from the expression for C_V . Introducing it into Eq. (3.46) and knowing the value of κ from experiments, we can determine the *lower* limit of the free path of an electron λ , which, for example, for silver at a normal temperature is $\lambda \geq 5 \times 10^{-5}$ cm, while when $T = 14$ K we have $\lambda \geq 2 \times 10^{-3}$ cm. These values of the free path travelled by an electron between two consecutive collisions cannot in any way be brought into agreement with the fact, within the scope of the classical theory of electrons, that the order of magnitude of the distance between adjacent atoms in metals, as in other solids, is only 10^{-8} cm.

This contradiction is one of the most important objections to the classical electron theory of metals. If we also take into consideration the experimentally measured values of the contact and thermal e.m.f.'s, galvanomagnetic phenomena, etc., whose values, according to theory, are functions of the same unknown quantities n and λ , then for different groups of experimental data we get a number of absolutely different values for n and λ .

Thus, the classical theory of free electrons, which interprets the fundamental properties of metals and the phenomena occurring in them in a very simple way, is not able to give a consistent and non-contradictory *quantitative* description of these phenomena. When attempting to determine the fundamental constants of the theory, n and λ , from various phenomena, we get a number of values for these quantities that absolutely do not agree with one another. Although the idea of free electrons in a metal is undoubtedly just a first approximation to what actually takes place, these contradictions, however, cannot be related only to the account of the simplifications underlying the theoretical calculations.

4. The development of the quantum theory made it clear that the main difficulties of the electron theory of metals were not so much due to the simplification of the basic assumption on the existence of free electrons in metals as to the application to these electrons of classical statistical mechanics, or as it is briefly called, *classical statistics* [Eq. (3.45)!].

According to the quantum theory, an electron gas obeys not classical statistics, but the so-called Fermi-Dirac statistics. The conclusions of both statistics—quantum and classical—coincide for high temperatures and low densities of the gas. At low temperatures and great

densities, however, there occurs what is called *degeneracy* of the gas, i.e. deviation from the classical laws. Degeneracy of the gas sets in when the “degeneracy parameter”

$$A = \frac{nh^3}{2(2\pi mkT)^{3/2}} \quad (3.48)$$

becomes comparable in value with unity; in Eq. (3.48) the symbol h stands for the Planck constant [$h = 6.626\ 176(36) \times 10^{-34}$ J/Hz]. Thus, classical statistics may be applied to an electron gas only provided that $A \ll 1$. For an electron gas in metals, however, owing to its colossal density and the negligible mass of an electron, this condition is not complied with at all. Assuming, for example, that in monovalent metals (Ag, Au, Cu, Na, K, etc.) one free electron falls to each atom of the metal (i.e. that one electron is detached from each atom in the metal), we get for the density of the electrons n in these metals a number of the order of magnitude of 6×10^{22} cm $^{-3}$ and, consequently, for the parameter A a value of the order of magnitude of 2.2×10^3 (assuming that $m = 9.11 \times 10^{-28}$ g and $T = 300$ K).

According to the Fermi statistics, on the condition that $A \gg 1$, which is observed for the electrons in metals at all temperatures up to from 10 to 20 thousand kelvins, at all the temperatures at which solid metals can exist in general, the *mean kinetic energy of the free electrons* is not proportional to the absolute temperature T [as follows from the classical formula (3.45)], but *is virtually independent of the temperature and is determined unambiguously by the density of the electron gas*. Particularly, at the temperature of absolute zero, the kinetic energy of an electron is still quite considerable—absolute zero corresponds not to the absence of motion or the absence of the kinetic energy of the electrons, but only to a *minimum* store of this energy (the so-called *zero-point energy*), which cannot be taken away from a metal in any way. Physically, this will become absolutely clear if we recall that upon the cooling, for instance, of an isolated atom or molecule to absolute zero, the electrons in the atom nevertheless continue their motion about the atom’s nucleus. A crystal, in essence, is a sort of giant molecule.

5. According to Fermi statistics, the mean kinetic energy of the electrons in a metal (i.e. their mean energy provided that $A \gg 1$) is in a first, quite accurate approximation equal to

$$\frac{1}{2} mv^2 = \frac{3h^2}{40m} \left(\frac{3n}{\pi} \right)^{2/3} \quad (3.49)$$

[instead of Eq. (3.45)]. Using the value of v obtained from this equation in Eq. (3.44), we get the following expression instead of

Eq. (3.46)*:

$$\kappa = \sqrt{\frac{5}{3}} \left(\frac{\pi}{3} \right)^{1/3} \frac{e^2 n^{2/3}}{h} \lambda \quad (3.50)$$

Thus, the change in the conductivity κ with the temperature is determined, in essence, only by the temperature dependence of the mean free path of an electron λ (because the density of the electrons n virtually does not depend on T). This dependence of λ on T can be calculated on the basis of quantum mechanics (the wave nature of the electrons having the determining influence on the result of the calculations) and leads to a correct shape of the conductivity curve depending on the temperature: κ is inversely proportional to T at conventional and high temperatures and inversely proportional to T^5 at very low temperatures. In pure (deprived of impurities) metals, κ tends to infinity when T tends to zero. It must be noted that the mean free path of electrons was found to be hundreds of times greater than the distance between the atoms of a metal even at conventional temperatures.

The insignificant change in the mean energy of electrons with the temperature indicates a low heat capacity of the electron gas. Fermi statistics gives values for this heat capacity at ordinary temperatures that are about 50 to 70 times less than according to the classical formula (3.47)**. This eliminates the contradiction between the values of κ and C_V mentioned above, and this explains the circumstance that contrary to the expectations of the classical theory, the presence in metals of a large number of free electrons does not virtually affect their heat capacity. Nevertheless, a slight, very insignificant, fraction of all the electrons acquires a quite appreciable velocity when heated, and this is what explains (as in the classical theory) thermionic emission, which agrees quantitatively with experimental data.

The other difficulties of the classical theory are solved in a similar way, so that in general we can say that the theory of free electrons based on Fermi statistics explains the fundamental properties of metals quite satisfactorily.

6. The notion of the free motion of electrons in metals, however, is naturally only a first approximation to reality. This manifests itself, first, in the difficulties which the theory of free electrons encounters even in *qualitative* explanation of many phenomena such as thermoelectric phenomena and the Hall effect. Second, the mean free path of the electrons λ appears in all the *quantitative* conclusions of

* More accurate calculations lead to an expression that differs from Eq. (3.50) in substitution of the factor 2 for $\sqrt{5/3} = 1.29$.

** Equation (3.49) for the mean energy of electrons is only a first approximation. In the second approximation, a term proportional to T^2 is added to the right-hand side of this equation, so that the heat capacity of the electron gas differs from zero and is proportional to T .

the theory, while the magnitude of this path can be determined only when account is taken of the interaction of the electrons with the ionic lattice of a metal.

Still more significant is the *fundamental* inadequacy of the theory arising from the notions of the free or almost free motion of electrons in metals. There is no doubt that every electron in a metal resists colossal forces exerted by the electrons and ions surrounding it. Therefore, a consistent theory must explain first of all how and why, *notwithstanding these forces*, the motion of the electrons in a first approximation occurs as if they were free, the free path of the electrons reaching values that are hundreds of times greater than the distances between the atoms of a metal.

This problem, which the classical theory was absolutely powerless to solve, has been solved to a considerable extent (although not yet completely) by the modern quantum theory of metals. A treatment of this theory is beyond the scope of our course.

7. There is one phenomenon, however, whose mechanism only the quantum theory could explain completely. This is the phenomenon of *superconductivity*.

The resistance of all pure (deprived of impurities) metals when their temperature approaches absolute zero tends to zero (approximately proportional to T^5), but in some metals this change does not occur smoothly: at a certain quite definite temperature the resistance suddenly (in a jump) drops to zero or, at any rate, to an immeasurably small value. The abruptness of this jump is characterized by the fact that in some metals it occurs when the temperature changes by only 0.001 K.

The temperature of the jump is called the critical temperature T_{cr} , and the state of a metal below this temperature, characterized by the absence of resistance to a steady current, is called the superconductive state.

Superconductivity has been experimentally established to date in about 20 pure metals and in a number of alloys. The highest critical temperature among pure metals belongs to niobium ($T_{cr} \approx 9.2$ K), next come lead (7.26 K), lanthanum, mercury, tin, and so on.

Since the resistance of metals in the superconductive state equals zero, no Joule heat should be liberated in them, and currents, once having appeared, should remain for an indefinitely long time in the absence of an extraneous e.m.f. Indeed, quite a long time ago H. Kamerlingh Onnes generated induced currents in superconductors that lasted for days on end. As measurements showed, their intensity, if it did gradually diminish, did this at a rate not exceeding 1/80 000 of its magnitude an hour. It must be noted, however, that in high-frequency fields Joule heat is liberated in superconductors.

The superconductive state differs from the other states of a substance in still another feature: *a magnetic field does not penetrate into the*

depth of superconductors, i.e. the intensity and induction of a magnetic field inside superconductors equal zero*. This circumstance, in connection with the established reversibility of the transition of metals into the superconductive state at the critical temperature, served as the basis of the *thermodynamic theory of superconductivity* in which the transition of metals to the superconductive state is considered as a phase transition, and which made it possible to quantitatively relate the magnetic and thermal characteristics of superconductors (see also Supplement 1).

* More exactly, the magnetic field very rapidly diminishes exponentially from the surface into the depth of superconductors; in the customary experimental conditions the depth of penetration of the field into "good" superconductors has the order of magnitude of 10^{-6} to 10^{-5} cm.

The widespread consideration of superconductors as being similar to ideal diamagnetics, i.e. bodies whose permeability μ equals zero, is true only in two respects: first, the density $\mu H^2/8\pi$ of the energy of a magnetic field equals zero both in superconductors (in which $\mu \approx 1$, $H = 0$) and in ideal diamagnetics (in which $\mu = 0$ and $H \neq 0$); second, the *external* magnetic field is distorted in the same way when both a superconductor and an ideal diamagnetic of the same shape and volume are introduced into it.

4

Ponderomotive Interaction of Steady Currents and Their Magnetic Field (in the Absence of Magnetizing Media)

4.1 The Magnetic Field of Currents

1. It is general knowledge that ponderomotive (mechanical) forces of interaction appear between conductors carrying electric currents. These forces depend on the intensity of the currents and the arrangement of the conductors and can be directly measured using conventional ways of measuring mechanical forces. For the purpose of brevity, we shall call the forces of interaction of conductors carrying currents simply the forces of interaction of currents. The technical use of these forces is one of the most important tasks of electrical engineering (electric motors, various electrical measuring instruments, etc.).

In Chapters 1 and 2 we saw that our treatment of the forces of interaction of electric charges was exceedingly greatly simplified by introducing the concept of the electric *field* of these charges. The interaction of currents is appreciably more complicated than that of charges at rest, and, accordingly, the introduction of the concept of the field of currents into our treatment will facilitate the problem confronting us to a still greater extent. Therefore, from the very beginning we shall use the concept of the *field of currents**, i.e. we shall proceed from the following notion.

At all points of space surrounding an arbitrary current, there is *always* present a *field of forces* due to this current, no matter whether the existence of these forces manifests itself in their action on another current or does not manifest itself in any way in the absence of such a current. According to the historically established terminology, this field of forces is called the *magnetic field* of a current because permanent magnets create the same fields as electric currents. Thus, the problem of determining the interaction of currents breaks up into two simpler problems: (a) the *determination of the magnetic field* of an arbitrary current, and (b) the *determination of the forces acting in a given magnetic field* on a current placed in it.

* Naturally, the laws of interaction of steady currents can also be formulated without resorting to the concept of a field (see Sec. 4.2).

Within the confines of the science of steady currents, the concept of the magnetic field of these currents can be considered as a purely conditional one, introduced only to make the description of phenomena more convenient (the same relates to the concept of an electric field within the confines of electrostatics, cf. Sec. 1.2). When passing over to the studying of a varying electromagnetic field, however, we shall see that the concept of a field has a deep physical meaning, and that an electromagnetic field is an objective reality.

2. An electric field can be measured by introducing arbitrarily small test charges into its different points and determining the forces acting on these charges. Accordingly, isolated elements of steady currents ought to be used for measuring the magnetic field. This is impossible, however, because steady currents have to be closed (or pass away to infinity, see Sec. 3.3).

To circumvent this difficulty, we can rigidly fasten all the conductors forming the current circuit whose field is to be measured, letting only a small portion of the circuit (a “*current element*”) remain movable, and measure the force acting on this element* (for example, according to the force that must be applied to it to keep it in equilibrium). It is quite obvious that the movement of such a current element from one point of space to another involves the movement of the entire measuring circuit; it is therefore necessary to see during these measurements that these movements do not distort the current field being measured. Finally, only “current elements” of such small dimensions (cross-sectional area and length) are suitable for the measurements in which the field being measured may be considered constant along their length. In the following, we shall consider that all these conditions are observed.

3. As *experiments* show, the magnetic field at each point of space can be exhaustively characterized by a vector \mathbf{H} called the *magnetic field intensity*. A complex of experimental facts leads to the following expression for the force \mathbf{F} acting in a field characterized by the vector \mathbf{H} on a current element having the length ds along which a current I flows:

$$\mathbf{F} = \frac{I}{c} [ds \mathbf{H}] \quad (4.1)$$

Here c stands for a proportionality constant depending only on the units chosen. It is customary practice to write this constant in the denominator, and not in the numerator.

Thus, the force \mathbf{F} acting on the element ds depends quite appreciably on the *orientation* of this element: its magnitude is proportional to

* The part of such a movable current element is played, for instance, by the thread of a string (Einthoven) galvanometer whose deflection is determined by the force of the magnetic field induced by the current flowing through the galvanometer winding.

the sine of the angle between the direction of the field \mathbf{H} and that of the element ds ; the direction of the force, however, is perpendicular to the plane drawn through \mathbf{H} and ds , and the force is directed toward the point of a screw whose head is being turned from ds to \mathbf{H} (Fig. 42).

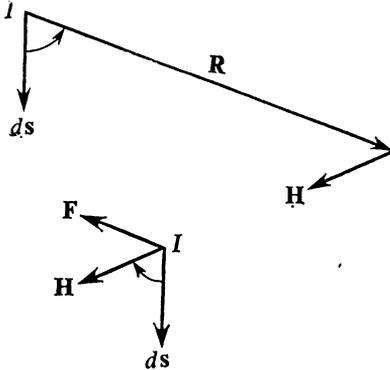


Fig. 42

Equation (4.1) can be considered as the *definition* of the concept “magnetic field intensity”. On its basis, we can measure the intensity of a magnetic field at any point of space as follows. We place the current element ds at a given point P and rotate it until it occupies a position in which the force \mathbf{F} acting on it becomes equal to zero. The magnetic field will evidently be either parallel or antiparallel to the direction of ds in this position. We next turn ds from this position through 90 degrees and measure the force \mathbf{F} acting on it in this new position. It is now a simple matter to determine the magnitude and direction of the vector \mathbf{H} .

4. It remains for us to solve the problem of how the intensity of the magnetic field at an arbitrary point of space depends on the characteristics of the current inducing this field (the position and shape of the current contour, its intensity, etc.). This problem would be exceedingly simplified if we could reduce it to that of the field induced by a separate *current element* and consider the field of an arbitrary system of currents as the superposition of the fields of the separate elements of these currents. Experiments show that the intensity of a field created by two currents is indeed equal to the sum of the intensities of the fields induced by each of these currents separately (this *principle of superposition* is violated only when a field contains ferromagnetics—see Chap. 5). Nevertheless, within the confines of the science of steady currents, the question of the field induced by a separate current element cannot be solved unambiguously because we cannot isolate a separate element of steady current whose circuit cannot fail to be closed. We shall consequently always have to do

with the resultant field of all the elements of a closed field, and our knowledge of its resultant will not be sufficient for the unambiguous determination of its components (see Sec. 4.2).

In a later chapter, however, we shall pass over to the studying of varying currents, which may also be unclosed. Finally, the electron theory, based on extensive experimental data, reduces the forces of interaction of currents to the interaction of moving electrons each of which is a current element in the exact meaning of this term. Thus, the mathematical procedure of decomposing finite currents into a combination of elementary ones corresponds in a known respect to modern physical notions that all currents consist in the motion of separate electrons (or ions).

Anticipating the results of studying varying currents and also the forces of interaction of moving elementary charges, we shall base all our reasoning on a definite law determining the magnetic field of a current element, considering this law to be established *experimentally*. It is called the *Biot-Savart law* and in the vector form can be written as follows:

$$\mathbf{H} = \frac{I}{cR^3} [ds \mathbf{R}] \quad (4.2)$$

where \mathbf{R} = distance from the current element $I ds$ inducing the magnetic field \mathbf{H} to the "point of observation" at which the intensity \mathbf{H} of this field is being determined

c = constant of proportionality depending only on the choice of the units. In all generally adopted systems of units these units are chosen so that the coefficients c in Eqs. (4.1) and (4.2) will be identical.

Thus, when moving away from the element $I ds$ along a definite half-line drawn from this element, the intensity of its field diminishes inversely proportional to the square of the distance R from ds ; upon movements over a sphere having a definite radius R with its centre at ds , the field changes like the sine of the angle between \mathbf{R} and ds . The direction of the field is at right angles to the plane drawn through \mathbf{R} and ds (see Fig. 42). If we imagine a spherical system of coordinates with its centre at ds and an axis directed along ds , then the direction of the field at each point of space P will be tangent to the polar circle passing through this point P . In other words, the lines of the vector \mathbf{H} , called *magnetic lines of force*, are circles threaded, as if onto an axis, on the straight line passing through the element ds . The direction of these magnetic force lines forms a right-handed system with the direction of the current in the element ds .

5. Equations (4.1) and (4.2) are the fundamental ones for the entire science of the magnetic field and of the interaction of steady currents; virtually all of the further matter in this chapter consists in a treatment of the conclusions following from these equations.

Particularly, the force acting on a closed current (current loop) I in a magnetic field \mathbf{H} is obviously equal to the sum of the forces acting on each of its elements, i.e. is

$$\mathbf{F} = \frac{I}{c} \oint [d\mathbf{s} \mathbf{H}] \quad (4.3)$$

The field intensity \mathbf{H} of a closed current I at an arbitrary point P , on the other hand, equals the sum of the fields induced by each of its elements, i.e. is

$$\mathbf{H} = \frac{I}{c} \oint \frac{[d\mathbf{s} \mathbf{R}]}{R^3} \quad (4.4)$$

where \mathbf{R} is a radius-vector conducted from the element $d\mathbf{s}$ to the "point of observation" P .

Problem 25. Show that the intensity of the magnetic field of an infinite rectilinear current at a distance of r from its axis is

$$H = \frac{2I}{cr} \quad (4.5)$$

and that the lines of force of this field are concentric circles whose plane is perpendicular to the current, the direction of the lines forming a right-handed system with the direction of the current.
Note. A steady current circuit is always closed. To consider an infinite rectilinear current means to consider a closed current circuit including a very long straight cylindrical portion and limit oneself to studying the field of the current near the middle section of this portion, ignoring the action of the remote portions of the current circuit.

Problem 26. A line current flows along a circle having the radius R_0 . Show that the field intensity at any point on the axis of this circle is directed along the axis, forms a right-handed system with the direction of the current, and equals

$$H = \frac{2\pi I}{c} \frac{R_0^2}{(R_0^2 + d^2)^{3/2}}$$

where d is the distance from the point on the axis being considered to the centre of the ring current.

4.2 Interaction of Current Elements. The Electromagnetic Constant

1. Let us consider two current elements $I_1 d\mathbf{s}_1$ and $I_2 d\mathbf{s}_2$ at a distance of \mathbf{R}_{12} from each other (we consider that \mathbf{R}_{12} is directed from $d\mathbf{s}_1$

to ds_2). The field induced by the first element at the place where the second one is, according to Eq. (4.2), equals

$$\mathbf{H}_{12} = \frac{I_1}{c} \frac{[ds_1 \mathbf{R}_{12}]}{R_{12}^3}$$

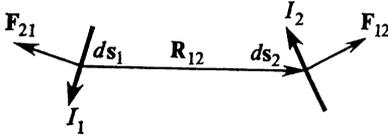


Fig. 43

and, consequently, the force \mathbf{F}_{12} acting on the second element from the side of the first one, according to Eq. (4.1), equals (see Fig. 43)

$$\mathbf{F}_{12} = \frac{I_1 I_2}{c^2 R_{12}^3} [ds_2 [ds_1 \mathbf{R}_{12}]] \quad (4.6)$$

The force \mathbf{F}_{21} which with the second element acts on the first one is similarly expressed by the equation

$$\mathbf{F}_{21} = \frac{I_2 I_1}{c^2 R_{21}^3} [ds_1 [ds_2 \mathbf{R}_{21}]] \quad (4.7)$$

In the above equations, $\mathbf{R}_{21} = -\mathbf{R}_{12}$. We can base all our reasoning not on Eqs. (4.1) and (4.2), but on this law of interaction of current elements, and then, introducing the concept of the magnetic field of currents, get the formulas of the preceding section from Eqs. (4.6) and (4.7).

2. In our preceding treatment, we left open the question of the dimension of the proportionality constant c in Eqs. (4.1) and (4.2) called the *electromagnetic constant*. If we adopt definite units of measurement for the length, mechanical force F , and current I , we thus unambiguously fix the dimension of the electromagnetic constant c because it follows from Eq. (4.6) that

$$[c^2] = \frac{[I^2]}{[F]} \quad (4.8)$$

We shall mainly use the so-called *Gaussian absolute units*. In this system of units, mechanical quantities (length, the force F , etc.) are measured in cgs units, while the current is measured in absolute *electrostatic* units in which, according to Sec. 3.1, $[I] = M^{1/2}L^{3/2}T^{-2}$. Hence, Eq. (4.8) becomes

$$[c^2] = \frac{ML^3T^{-4}}{MLT^{-2}} = \left(\frac{L}{T}\right)^2 \quad (4.9)$$

i.e. has the *dimension of velocity*.

Experimental studies have shown (see Sec. 4.18) that the numerical value of the electromagnetic constant c in these units equals 3×10^{10} cm/s.

We shall show in Chapter 7 that the coincidence of this value with the velocity of light in a vacuum is not accidental.

3. To understand the content of Eqs. (4.6) and (4.7), let us consider a number of particular cases. If ds_1 is *parallel* to ds_2 , then the forces

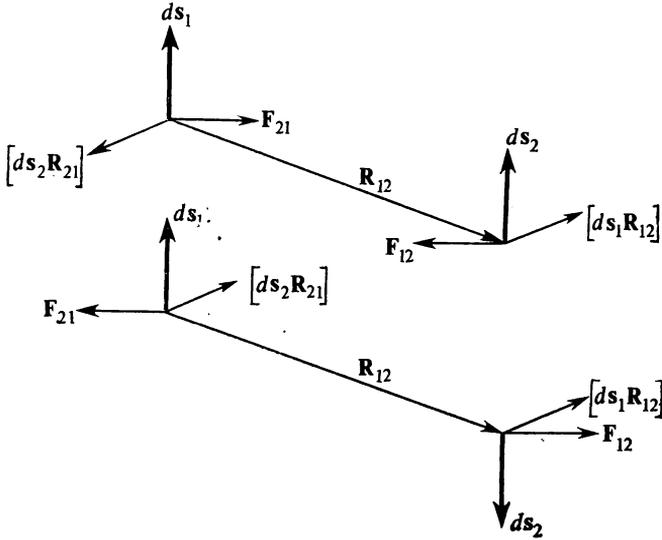


Fig. 44

of interaction tend to draw ds_1 and ds_2 together (*attraction*), if ds_1 and ds_2 are *antiparallel*, these forces will tend to move them apart (*repulsion*) (Fig. 44). Here $F_{12} = -F_{21}$. Even in this case, however, these forces do not comply with the principle of equality of action and reaction because their directions, generally speaking, are not on one straight line. The *violation of the principle of equality of action and reaction* manifests itself especially sharply when, for example, ds_1 is parallel to R_{12} , and ds_2 is perpendicular to R_{12} (Fig. 45). In this case, $[ds_1 R_{12}] = 0$, and therefore $F_{12} = 0$, whereas $[ds_2 R_{21}] \neq 0$ and $F_{21} \neq 0$; the element ds_1 is acted upon by a force exerted by the element ds_2 , but does not act on it itself.

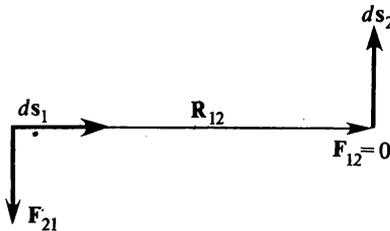


Fig. 45

For *steady currents*, however, which must necessarily be closed, this violation of Newton's third law is only connected with the

representation of the forces of interaction of currents as forces of the interaction of their elements in pairs. Indeed, as we shall show in Sec. 4.10, the forces of interaction of two *closed* currents or current loops comply with the principle of the equality of action and reaction (see also Problems 27 and 28 at the end of the section). In the general case of a varying electromagnetic field, however, we can and must generalize the concept of momentum so that the correctness of this principle will be ensured in all electromagnetic phenomena*.

4. We have already mentioned above that within the limits of studying *closed* steady currents, the force of interaction of the current *elements* cannot be determined unambiguously. Mathematically, this circumstance is expressed in the fact that if we modify the law of interaction of currents by adding a number of terms whose integral over any closed contour becomes equal to zero, then the total force exerted on an element by the *closed* current remains unchanged.

Particularly, it is a simple matter to see that the expression for the force \mathbf{F}_{12} can be modified as follows:

$$\mathbf{F}'_{12} = \mathbf{F}_{12} + \frac{I_1 I_2}{c^2} \left\{ \Phi(\mathbf{R}_{12})(d\mathbf{s}_1 \mathbf{R}_{12}) d\mathbf{s}_2 + \right. \\ \left. + d[\mathbf{R}_{12}(d\mathbf{s}_2 \mathbf{R}_{12}) f(\mathbf{R}_{12})] \right\} \quad (4.10)$$

where $\Phi(\mathbf{R}_{12})$ and $f(\mathbf{R}_{12})$ are arbitrary scalar functions of \mathbf{R}_{12} , and $d[\mathbf{R}_{12}(d\mathbf{s}_2 \mathbf{R}_{12})f(\mathbf{R}_{12})]$ stands for the increment (differential) of the expression in brackets when the origin of the radius-vector \mathbf{R}_{12} moves over the distance $d\mathbf{s}_1$ (so that $d\mathbf{R}_{12} = -d\mathbf{s}_1$). Indeed, upon integration over the contour of the current I_1 , the last term vanishes as the integral of a total differential over a closed path. Further, $\Phi(\mathbf{R}_{12}) \mathbf{R}_{12}$ can always be represented in the form of the gradient of a certain function $\varphi(\mathbf{R}_{12})$ [see Eqs. (A.7) and (A.8)]:

$$\Phi(\mathbf{R}_{12}) \mathbf{R}_{12} = \text{grad } \varphi(\mathbf{R}_{12})$$

Consequently,

$$\Phi(\mathbf{R}_{12}) (d\mathbf{s}_1 \mathbf{R}_{12}) = (d\mathbf{s}_1 \cdot \text{grad } \varphi(\mathbf{R}_{12})) = \frac{\partial \varphi(\mathbf{R}_{12})}{\partial s_1} ds_1$$

The integral of this total differential over the closed circuit of the current I_1 [and, therefore, the integral of the second term of Eq. (4.10)] also becomes equal to zero.

* In Sec. 7.15, we shall prove the generalized law of conservation of the total (mechanical and electromagnetic) momentum [Eq. (7.147)]. Since the law of conservation of momentum is equivalent to the law of equality of action and reaction, this will also prove the truth of the latter law in its generalized form.

Problem 27. Prove by direct integration on the basis of Eqs. (4.6) and (4.7) that the *resultant* of the forces exerted on a *closed* current by another closed current complies with the law of equality of action and reaction.

Problem 28. The law of the ponderomotive interaction of current elements was first formulated by A. Ampere. He proceeded from the assumption that the interaction of current elements must comply with Newton's third law and must be directed along the line joining them. The law found by Ampere using our notation is as follows:

$$\mathbf{F}_{12} = \frac{I_1 I_2}{c^2} \left\{ \frac{3}{R_{12}^5} (ds_1 \mathbf{R}_{12})(ds_2 \mathbf{R}_{12}) - \frac{2}{R_{12}^3} (ds_1 ds_2) \right\} \mathbf{R}_{12}$$

Show that the application of Ampere's formula to the calculation of the resultant force exerted on the element ds_2 by all the elements of the closed current I_1 gives the same result as the application of Eq. (4.6).

4.3 Transition from Line Currents to Currents Having a Finite Cross Section

1. In the preceding sections, we considered the elements of line currents. It is quite obvious that when determining the *field* induced by a current we may consider such currents to be *line* ones for which the dimensions of any cross section are sufficiently small in comparison with the distance from this section to point P of the field being considered. A current can naturally comply with this condition of linearity only when we limit ourselves to a consideration of points of a field considerably removed from it. When determining the *ponderomotive forces* acting on a current in an external magnetic field \mathbf{H} , however, the current may be considered to be a line one if the field \mathbf{H} does not change appreciably over any cross section of the current.

Thus, the formulas of Secs. 4.1 and 4.2 may be applied only if the conditions listed above are observed. For example, when $R \rightarrow 0$ the field intensity \mathbf{H} determined by Eq. (4.2) tends to infinity, i.e. becomes meaningless.

Quite insignificant transformations of the formulas of Secs. 4.1 and 4.2 are needed, however, to make their application possible with an arbitrary value of R and with an arbitrarily and rapidly changing from point to point intensity of the external field \mathbf{H} . For this purpose, it is sufficient to take advantage of the fact that in accordance with Sec. 3.3, a current having a finite cross section can be resolved into a complex of infinitely thin *current filaments*, and the formulas of Sec. 4.1 may be applied to the elements of these filaments.

The current dI flowing along a current filament, according to Eq. (3.10), is

$$dI = j dS$$

where j = current density

dS = cross-sectional area of the filament *perpendicular* to its axis.

Consequently,

$$dI ds = j dS ds = j dV$$

where ds and dV are the length and volume of an infinitely small portion of the filament, respectively. Since, finally, the axis of a current filament by definition coincides with the lines of current, then ds is parallel to \mathbf{j} , and

$$dI ds = \mathbf{j} dV \quad (4.11)$$

Thus, an *element of length* of each *current filament* whose complex forms a current I having a finite cross section is equivalent to an *element of volume* $\mathbf{j} dV$ of this current I .

Therefore, the intensity $d\mathbf{H}$ of the field of the volume element $\mathbf{j} dV$ of the current I , on the basis of Eqs. (4.2) and (4.11), is

$$d\mathbf{H} = \frac{dI}{cR^3} [ds \mathbf{R}] = \frac{[\mathbf{j}\mathbf{R}]}{cR^3} dV \quad (4.12)$$

The total intensity of the magnetic field of the entire closed current I will equal the sum of the intensities of the fields induced by its separate elements:

$$\mathbf{H} = \int d\mathbf{H} = \frac{1}{c} \int \frac{[\mathbf{j}\mathbf{R}]}{R^3} dV \quad (4.13)$$

where integration should be extended over the entire volume of the current (i.e. over the volume of the conductors through which the current flows), and \mathbf{R} is the distance from the current element dV to the point P of the field being considered.

Naturally, for line currents, Eq. (4.13) coincides with Eq. (4.4).

In an absolutely similar way, applying Eq. (4.1) to an element of a current filament having the volume dV and using Eq. (4.1), we find that the ponderomotive force acting on an element of volume of a conductor dV through which a current having the density \mathbf{j} flows is

$$\mathbf{F} = \frac{dI}{c} [ds \mathbf{H}] = \frac{1}{c} [\mathbf{j}\mathbf{H}] dV \quad (4.14)$$

where \mathbf{H} is the intensity of the magnetic field in the element dV . In other words, the volume *density of the ponderomotive forces* is

$$\mathbf{f} = \frac{1}{c} [\mathbf{j}\mathbf{H}] \quad (4.15)$$

2. In the following, we shall repeatedly have to pass over from considering line currents to currents having a finite cross section and vice versa. As follows from the above, particularly from a comparison of Eqs. (4.4) and (4.13), this transition is always equivalent to the replacement of integration over the length of a line current with integration over the volume of a current having a finite cross section:

$$I \oint \Phi ds \Leftrightarrow \oint \Phi \mathbf{j} dV \quad (4.16)$$

where Φ may be any scalar or vector position function. If a current complies with the conditions of linearity listed at the beginning of this section, then both expressions in formula (4.16) are equivalent to each other. Otherwise, they differ in their content, only the *volume* integral having a physical meaning.

It must also be noted that for a branched circuit only the right-hand side of Eq. (4.16) retains its form, whereas it must be taken into account in the left-hand side that the current in different branches of the circuit may be different.

Expression (4.16) may be conditionally written in a form corresponding to Eq. (4.11), substituting I for dI in the latter:

$$I ds \Leftrightarrow \mathbf{j} dV \quad (4.17)$$

This relationship, however, acquires a meaning in essence only when integration is conducted in the left-hand side over ds , and in the right-hand side over dV , as is explicitly indicated in Eq. (4.16).

3. Equations (4.13) and (4.14) may obviously be applied for all the points of the field of steady currents. Particularly, the field intensity \mathbf{H} they determine, unlike Eqs. (4.2) and (4.4), retains a finite value everywhere (if, naturally, the current density \mathbf{j} is finite everywhere, as follows from elementary physical considerations)*.

For the points of a field outside of currents ($R \neq 0$), this is obvious. To convince ourselves that the field \mathbf{H} within currents is finite, let us consider an arbitrary point P' inside a conductor carrying a current and encircle it with a sphere V' having a small, but nevertheless finite radius R_0 . The field induced at P by the points outside of the sphere V' is finite because these points are at a finite distance from P that is greater than R_0 . Hence, it is sufficient for us to prove that the field \mathbf{H}' induced by the currents inside the sphere V' is finite. It follows from Eq. (4.13) that

$$|\mathbf{H}'| \leq \frac{1}{c} \int_{V'} \frac{|\mathbf{j}\mathbf{R}|}{R^3} dV$$

* Attention should be attracted to the fact that it is sometimes convenient to use the notion of *surface currents* whose volume density j is infinite (cf. the surface charges in electrostatics). For surface currents see Sec. 4.8; the magnetic field of these currents has surfaces of discontinuity.

where $|\mathbf{H}'|$ and $|\mathbf{jR}|$ are the absolute values of the relevant vectors, and integration has been extended over the volume of the sphere V' . But

$$|\mathbf{jR}| \leq j_m R$$

where j_m stands for the maximum value of the current density inside the sphere V' . Consequently, $|\mathbf{H}'| \leq \frac{j_m}{c} \int_{V'} \frac{dV}{R^2}$.

Introducing the spherical coordinates R , θ , and α with the centre at P , we get

$$dV = R^2 \sin \theta \, d\theta \, d\alpha \, dR$$

and

$$|\mathbf{H}'| \leq \frac{j_m}{c} \int \frac{dV}{R^2} = \frac{j_m}{c} \int_0^{R_0} dR \int_0^\pi \sin \theta \, d\theta \int_0^{2\pi} d\alpha = \frac{4\pi R_0 j_m}{c}$$

Thus \mathbf{H}' is a finite quantity tending to zero when the radius of the sphere R_0 diminishes, Q.E.D.

It is also not difficult to prove the *continuity* of the vector \mathbf{H} , i.e. to prove that the difference between the values of the vector \mathbf{H} at adjacent points of the field P and P' tends to zero if the distance PP' tends to zero.

Assume that both points P and P' are inside of the sphere V' . When passing from P to P' , the field of the currents outside of V' changes continuously because these currents are at a finite distance from P and P' . As regards the field \mathbf{H}' of the currents inside V' , the intensity of this field, as has been proved, is less than $R_0(4\pi j_m/c)$. Consequently, the change in the value of \mathbf{H}' upon transition from P to P' cannot also be greater than this value. If PP' tends to zero, then R_0 can be selected with a value as small as is desired, whence the continuity of the vector \mathbf{H} follows.

Problem 29. A steady current flows through an infinite straight hollow circular cylinder parallel to its axis, and the current is uniformly distributed over the surface of the cylinder. Show that the field of the current inside the cylinder equals zero.

4.4 Lorentz Force

1. The element of volume dV of a conductor through which a current having the density \mathbf{j} flows is subjected in the magnetic field \mathbf{H} to the ponderomotive force equal to [see Eq. (4.14)]

$$\mathbf{F} = \frac{1}{c} [\mathbf{jH}] dV$$

This force differs from zero only when $\mathbf{j} \neq 0$, i.e. when electric charges flow in the conductor. From the viewpoint of the electron theory, all the ponderomotive forces acting on bodies in an electromagnetic field should in the long run consist of forces applied to the electric charges in these bodies. Accordingly, in the case being considered, we should also try to reduce the force \mathbf{F} acting on a current-carrying conductor to forces acting on the charges flowing in it.

For this purpose, we shall express the current density \mathbf{j} through the number of "free" electrons n in a unit volume, i.e. the number of electrons whose motion creates the current, and the mean velocity of these electrons \mathbf{u} ; by Eq. (3.39)

$$\mathbf{j} = en\mathbf{u}$$

The direction of the current \mathbf{j} is conventionally considered to coincide with the direction in which *positive* charges would flow if this current were created by their motion. Hence, the vector \mathbf{j} is directed oppositely to the mean velocity \mathbf{u} of motion of the *negative* electrons; consequently, if by e we understand the *absolute* value of a charge (for electrons $e < 0$), then we can write

$$\mathbf{j} = en\mathbf{u} \quad (4.18)$$

Using this equation in Eq. (4.14), we get

$$\mathbf{F} = \frac{en}{c} [\mathbf{uH}] dV \quad (4.19)$$

This is the resultant force acting on $n dV$ free electrons confined in an element of volume of a conductor and having the mean velocity \mathbf{u} . It is quite natural to assume that the force acting on each of these $n dV$ electrons is

$$\mathbf{F} = \frac{e}{c} [\mathbf{vH}] \quad (4.20)$$

where \mathbf{v} is the *actual* velocity of an electron. If there is no current, the electrons move chaotically, $\mathbf{u} = 0$, and the resultant of the forces applied to them equals zero. If the current differs from zero, then the resultant (4.19) of these forces imparts the corresponding increment of momentum to the electrons. When the electrons collide with the atoms (or ions) of the conductor the momentum increment is transmitted to the conductor and causes its displacement in the magnetic field (or tries to do this if the conductor is fixed). From the standpoint of the macroscopic theory, without delving into a consideration of the internal mechanism of the phenomena, this means that a ponderomotive force determined by Eq. (4.14) is acting on the conductor.

2. The results of studying the motion of free electrons (in the exact meaning of this word, for instance the electrons of cathode rays)

confirm the correctness of Eq. (4.20), which may be applied to (point) electric charges travelling with the velocity \mathbf{v} in an arbitrary (constant or varying) magnetic field \mathbf{H} . If we also take into account the force $e\mathbf{E}$ acting on a (point) charge in the electric field \mathbf{E} , then the total force acting on a charge e in an arbitrary electromagnetic field will be expressed by the formula

$$\mathbf{F} = q \left(\mathbf{E} + \frac{1}{c} [\mathbf{v}\mathbf{H}] \right) \quad (4.21)$$

This formula was proposed by H. Lorentz, and therefore the force \mathbf{F} is called the *Lorentz force*.

Assuming the correctness of Eq. (4.21), we may retain the *definition of the electric field intensity* \mathbf{E} given in Sec. 1.2, according to which \mathbf{E} equals the force acting on a single positive test charge placed in this field only if we add to this definition the reservation that the test charge must be *fixed* ($\mathbf{v} = 0$). In this case, Eq. (4.21) coincides with Eq. (1.5).

3. It is very significant that the *force* acting on a moving charge in a magnetic field is *perpendicular both to the direction of its motion* \mathbf{v} *and to the direction of the field* \mathbf{H} . Thus, this force only curves the path of a charge without changing the numerical value of its velocity, i.e. performs no mechanical work. This circumstance may seem to contradict the fact that the work done upon the motion of a current-carrying conductor in a magnetic field, generally speaking, differs from zero (an electric motor!). This seeming contradiction will be solved if we take into account that the motion of a conductor in a magnetic field is inevitably attended by phenomena of electromagnetic induction. This question will be treated in greater detail in Sec. 6.1.

4. We shall note in conclusion that the curving of the path of electric charges under the action of the force (4.20) should tell in the redistribution of the current over the cross section of the conductor when it is introduced into a magnetic field. This redistribution of the current does actually manifest itself in the so-called galvanomagnetic, thermomagnetic and related phenomena. We shall consider only one of these phenomena, the so-called Hall effect, as an example.

Let us assume that an electric current having the density \mathbf{j} flows through a homogeneous metal plate or ribbon whose width is a in the direction of the y -axis (Fig. 46). If we place the plate in a homogeneous magnetic field \mathbf{H} directed along the z -axis, then the electric charges whose motion the current is due to will, according to Eq. (4.19), be acted upon by the additional force

$$\mathbf{F} = \frac{q}{c} [\mathbf{u}\mathbf{H}]$$

where \mathbf{u} is the mean velocity of the current carriers. This additional force will divert the current in the direction of the x -axis; the stream

of electrons will “strike” the front edge of the plate and cause negative electric charges to accumulate there. This process will continue until the resultant force of the field of the negative charges that have accu-

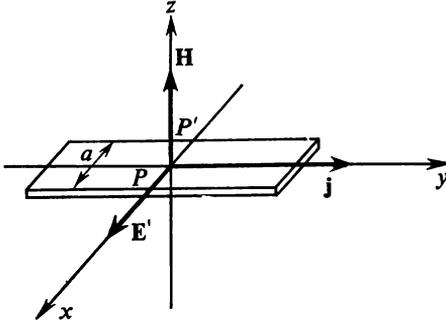


Fig. 46

mulated at the right-hand edge and of the field of the surplus of positive charges that have gathered at the left-hand edge balances the force F . Denoting this balancing field by E' , we get

$$qE' = -F = -\frac{q}{c} [uH]$$

Inspection of Fig. 46 shows that E' should be directed along the x -axis. Hence, the difference of potentials between the points P and P' on the same perpendicular to the axis of the plate will be

$$\varphi_{P'} - \varphi_P = E'_x a = \frac{1}{c} uaH$$

or, expressing the mean velocity of electrons u through the current density j with the aid of Eq. (4.18), we have

$$\varphi_{P'} - \varphi_P = \frac{jaH}{cqn} = RjaH \quad (4.22)$$

where the coefficient

$$R = \frac{1}{cqn} \quad (4.23)$$

is called the *Hall coefficient**.

* More accurate calculations taking into consideration the difference between the velocities of separate electrons and their mean velocity result only in an insignificant change in the numerical factor in Eq. (4.23):

$$R = \frac{3\pi}{8cqn}$$

if we apply classical statistics to electrons. If, however, we apply Fermi-Dirac statistics (Sec. 3.7) to electrons, then the result of these calculations coincides with Eq. (4.23).

The transverse potential difference $\varphi_P' - \varphi_P$ induced in a current-carrying conductor when a magnetic field is switched on can be measured directly in the relevant experiments. It is indeed found to be proportional to the quantities j , H , and a , as required by Eq. (4.22). The absolute value of this potential difference is very small. For example, when passing a current of 10 amperes through a gold band 1 cm wide and 0.1 mm thick in a magnetic field having an intensity of 10 000 gauss, this potential difference equals only seven microvolts. It is the most remarkable, however, that the sign of the Hall coefficient R , which according to Eq. (4.23) ought to be negative because negative electrons carry the current in metals ($e < 0$), actually differs for different metals. Thus, $R < 0$ for Au, Cu, Pt, Ag, Ni, and $R > 0$ for Fe, Co, Zn, Cd, Sb, etc. Thus, it seems as if positive charges carry the current in the second of these groups of metals instead of negative ones. This conclusion, however, contradicts all of our information on the nature of metals and for a long time was one of the main difficulties opposing the electron theory of metals. This contradiction is resolved quite satisfactorily by the quantum theory of metals.

Example. *The motion of electrons in a constant (static) magnetic field ($\mathbf{H} = \text{const}$). The equation of motion of an electron in a magnetic field, according to Eq. (4.20), states that*

$$\mathbf{m} \frac{d\mathbf{v}}{dt} = \mathbf{F} = \frac{e}{c} [\mathbf{v}\mathbf{H}]$$

where e is the charge of an electron.

Since the projection of the force \mathbf{F} onto the direction of the vector \mathbf{H} always equals zero, then the component v_H of the velocity of an electron in this direction will be constant. Since, on the other hand, the numerical value of the entire velocity v is also constant (because the acceleration is perpendicular to the velocity), then the *absolute* value of the component of the velocity v_{\perp} perpendicular to the vector \mathbf{H} must also be constant. Finally, the absolute value of the force \mathbf{F} must also be constant because

$$F = \frac{e}{c} |[\mathbf{v}\mathbf{H}]| = \frac{e}{c} v_{\perp} H = \text{const}$$

Thus, the motion of an electron can be resolved into two motion components: uniform motion in the direction of the field \mathbf{H} with the velocity v_H and motion in a plane perpendicular to \mathbf{H} with the acceleration

$$a = \frac{e}{cm} v_{\perp} H$$

that is constant in value and directed at right angles to its velocity v_{\perp} . But it is general knowledge that motion with a normal accele-

ration a constant in magnitude is uniform motion in a circle whose radius R can be found from the relationship

$$\frac{v_{\perp}^2}{R} = a = \frac{e}{cm} v_{\perp} H$$

consequently,

$$R = \frac{cmv_{\perp}}{eH} = \frac{cmv}{eH} \sin(\mathbf{v}, \mathbf{H}) \quad (4.24)$$

The combination of uniform translational motion with the velocity v_H and uniform circular motion with the velocity v_{\perp} in a plane perpendicular to \mathbf{H} is motion with a constant (in absolute value) velocity along a helix wound onto a right circular cylinder with the radius R whose axis is parallel to \mathbf{H} . The angle between the axis of the cylinder and a tangent to the helical trajectory of an electron is naturally constant and is determined by the initial conditions of motion. Particularly, if the initial velocity of an electron is perpendicular to \mathbf{H} , then its helical trajectory degenerates into a circle whose plane is perpendicular to \mathbf{H} .

Thus, in a constant uniform magnetic field, an electron describes, generally speaking, a helix whose axis coincides with the direction of the field. It should be noted that Eq. (4.24) makes it possible to determine the ratio e/m of the charge of an electron to its mass by measuring R , v , H , and $\sin(\mathbf{v}, \mathbf{H})$. This possibility is widely used in experimental physics for determining the value of the ratio e/m both for electrons and for other charged particles (alpha-rays, canal rays, etc.).

4.5 Vector Potential of a Magnetic Field

1. One of the basic equations of a magnetic field—Eq. (4.13)—can be transformed into a form more convenient for calculations.

The integrand of the right-hand side of this equation, according to Eq. (A.10), can be written as follows:

$$\frac{[\mathbf{j}\mathbf{R}]}{R^3} = - \left[\mathbf{j} \operatorname{grad}_a \left(\frac{1}{R} \right) \right] = \left[\operatorname{grad}_a \left(\frac{1}{R} \right) \cdot \mathbf{j} \right]$$

We must remind the reader here that in Eq. (4.13) \mathbf{R} stands for a radius-vector conducted from the current element $\mathbf{j} dV$ (the “source point”) to the “point of observation” P at which the magnetic field intensity is determined, and that the subscript a at the gradient sign means that when determining the gradient we consider the source point to be constant and the point of observation variable.

On the other hand, assuming in Eq. (A.43₃) that

$$\varphi = \frac{1}{R} \quad \text{and} \quad \mathbf{a} = \mathbf{j}$$

we get

$$\text{curl}_a \left(\frac{\mathbf{j}}{R} \right) = \left[\text{grad}_a \left(\frac{1}{R} \right) \cdot \mathbf{j} \right] + \frac{1}{R} \text{curl}_a \mathbf{j}$$

where the subscript a as previously denotes that in the formation of space derivatives only the point of observation P is considered to be variable. Since the value of the vector \mathbf{j} in the element dV (the source point) does not obviously depend on the movement of the point of observation P , then, consequently*

$$\text{curl}_a \mathbf{j} = 0$$

and

$$\text{curl}_a \left(\frac{\mathbf{j}}{R} \right) = \left[\text{grad}_a \left(\frac{1}{R} \right) \cdot \mathbf{j} \right] = \frac{[\mathbf{j}\mathbf{R}]}{R^3} \quad (4.25)$$

Introducing Eq. (4.25) into Eq. (4.13), we get

$$\mathbf{H} = \frac{1}{c} \int \text{curl}_a \left(\frac{\mathbf{j}}{R} \right) dV \quad (4.26)$$

Since in this expression, differentiation (the formation of a curl) is performed with respect to the coordinates of the point of observation, and integration is performed over the volume of the conductors through which a current flows, then the change in the sequence of these operations cannot affect the results of calculations. Consequently, we can write that

$$\mathbf{H} = \text{curl}_a \left(\frac{1}{c} \int \frac{\mathbf{j}}{R} dV \right) \quad (4.27)$$

If we introduce the symbol

$$\mathbf{A} = \frac{1}{c} \int \frac{\mathbf{j}}{R} dV \quad (4.28)$$

* We shall explain this by a transition to Cartesian coordinates. Let $x_a, y_a,$ and z_a be the coordinates of the point of observation P , and $x_q, y_q,$ and z_q the coordinates of the current element $\mathbf{j} dV$ so that $dV = dx_q dy_q dz_q$. The vector \mathbf{j} is a function only of $x_q, y_q,$ and z_q :

$$\mathbf{j} = \mathbf{j}(x_q, y_q, z_q)$$

whereas $R = \sqrt{(x_a - x_q)^2 + (y_a - y_q)^2 + (z_a - z_q)^2}$. When determining div_a and curl_a it is necessary to differentiate with respect to the coordinates of the point of observation so that, for instance,

$$\text{curl}_{ax} \mathbf{j} = \frac{\partial j_z(x_q, y_q, z_q)}{\partial y_a} - \frac{\partial j_y(x_q, y_q, z_q)}{\partial z_a} = 0$$

then this equation becomes

$$\mathbf{H} = \text{curl } \mathbf{A} \quad (4.29)$$

where the subscript a at the curl sign has been omitted as superfluous because with a given distribution of the currents the vector \mathbf{A} is a position function of only the point of observation. Thus, the intensity of the magnetic field can be represented in the form of the curl of a certain vector \mathbf{A} called the *vector potential* of the currents.

2. It should be noted that for *line* currents (i.e. at distances from the currents that are great in comparison with their cross-sectional dimensions) the expression for the vector potential can be transformed with the aid of expression (4.16) and acquires the following form:

$$\mathbf{A} = \frac{I}{c} \int_L \frac{ds}{R} \quad (4.30)$$

3. The introduction of the vector potential \mathbf{A} considerably facilitates the studying of the magnetic field of steady currents like the introduction of the scalar potential φ facilitates the studying of the electric field of a stationary system of electric charges. (We shall see in the following that the magnetic field of currents, like any vortex field, does not have a single-valued scalar potential.) The analogy between the part of the vector and the scalar potentials is especially clearly seen upon comparing similar formulas for an electrostatic and a magnetic fields:

$$\mathbf{E} = \int \frac{\rho \mathbf{R}}{R^3} dV, \quad \mathbf{H} = \frac{1}{c} \int \frac{[\mathbf{j} \mathbf{R}]}{R^3} dV$$

$$\varphi = \int \frac{\rho dV}{R}, \quad \mathbf{A} = \frac{1}{c} \int \frac{\mathbf{j} dV}{R}$$

$$\mathbf{E} = -\text{grad } \varphi, \quad \mathbf{H} = \text{curl } \mathbf{A}$$

It also follows from this comparison that the vector of the current density plays the same part for a magnetic field as the scalar of the charge density for an electric field.

4. Let us now pass over from integral relationships to differential equations for the vector potential.

If we introduce an arbitrary system of Cartesian coordinates $x, y,$ and $z,$ then Eq. (4.28) can be written in the following form:

$$A_x = \frac{1}{c} \int \frac{j_x dV}{R}, \quad A_y = \frac{1}{c} \int \frac{j_y dV}{R}, \quad A_z = \frac{1}{c} \int \frac{j_z dV}{R} \quad (4.31)$$

Each of these expressions for each of the components of the vector \mathbf{A} is absolutely similar to the expression for the scalar potential of

the electrostatic field:

$$\varphi = \int \frac{\rho dV}{R}$$

It was shown in Sec. 1.12 that the latter expression is the (integral) solution of Poisson's differential equation (1.65):

$$\nabla^2 \varphi = -4\pi\rho$$

Consequently, expressions (4.31) for the components of the vector potential are solutions of the following Poisson differential equations:

$$\nabla^2 A_x = -\frac{4\pi}{c} j_x, \quad \nabla^2 A_y = -\frac{4\pi}{c} j_y, \quad \nabla^2 A_z = -\frac{4\pi}{c} j_z \quad (4.32)$$

These three equations for the components of the vector \mathbf{A} , according to Eq. (A.41), are equivalent to the single vector equation

$$\nabla^2 \mathbf{A} = -\frac{4\pi}{c} \mathbf{j} \quad (4.33)$$

which is the required differential equation for the vector potential.

It remains for us to consider only the conditions in which integral expression (4.28) for the vector potential *unambiguously* follows from the differential equation (4.33). In Sec. 1.12 we studied the conditions in which the above integral expression for φ unambiguously follows from the Poisson equation. Repeating all the reasoning of Sec. 1.12 with the substitution of A_x , A_y , and A_z for φ and j_x/c , j_y/c , and j_z/c for ρ and assuming, in addition, that $\sigma = 0$, we shall find that the integral expression (4.28) for \mathbf{A} is the *only* solution of the differential equation (4.33) complying with the following conditions:

(a) both the vector \mathbf{A} itself and its space derivatives are finite and continuous throughout the entire space;

(b) at infinity each component A_k of the vector \mathbf{A} complies with conditions of the type of (1.85):

$$RA_k \text{ and } R^2 \nabla A_k \text{ remain finite when } R \rightarrow \infty \quad (4.34)$$

It is assumed here, naturally, that the density \mathbf{j} of the currents inducing the field is finite throughout the entire space and diminishes so rapidly with increasing R that the integral (4.28) converges. If we introduce into our consideration surface currents whose volume density \mathbf{j} is infinite, then Eq. (4.28) must be supplemented with another term (see Sec. 4.8).

5. We shall note in conclusion that the first derivatives of the vector \mathbf{A} with respect to the coordinates are related to one another by the expression

$$\operatorname{div} \mathbf{A} = 0 \quad (4.35)$$

Indeed, it follows from Eq. (4.28) that

$$\operatorname{div} \mathbf{A} = \frac{1}{c} \operatorname{div}_a \int \frac{\mathbf{j} dV}{R}$$

where the subscript a at the divergence sign indicates that space differentiation is performed with respect to the coordinates of the point of observation P . But, as indicated at the beginning of this section, the sequence of integration over the volume of the current and of differentiation with respect to the point of observation may be reversed. Hence,

$$\operatorname{div} \mathbf{A} = \frac{1}{c} \int \operatorname{div}_a \left(\frac{\mathbf{j}}{R} \right) dV$$

Applying Eq. (A.43₂), we can write that

$$\operatorname{div}_a \left(\frac{\mathbf{j}}{R} \right) = \frac{1}{R} \operatorname{div}_a \mathbf{j} + \mathbf{j} \operatorname{grad}_a \left(\frac{1}{R} \right) = \mathbf{j} \operatorname{grad}_a \left(\frac{1}{R} \right)$$

because the value of the vector \mathbf{j} does not depend on the coordinates of the point of observation P , owing to which $\operatorname{div}_a \mathbf{j} = 0$. On the other hand, using Eq. (A.10) and then again applying Eq. (A.43) we get

$$\mathbf{j} \operatorname{grad}_a \left(\frac{1}{R} \right) = -\mathbf{j} \operatorname{grad}_q \left(\frac{1}{R} \right) = -\operatorname{div}_q \left(\frac{\mathbf{j}}{R} \right) + \frac{1}{R} \operatorname{div}_q \mathbf{j}$$

The last term again equals zero because the divergence of the vector \mathbf{j} equals zero [Eq. (3.19)]. Hence, we finally get

$$\operatorname{div}_a \left(\frac{\mathbf{j}}{R} \right) = -\operatorname{div}_q \left(\frac{\mathbf{j}}{R} \right)$$

$$\operatorname{div} \mathbf{A} = -\frac{1}{c} \int \operatorname{div}_q \left(\frac{\mathbf{j}}{R} \right) dV$$

The last integral can be transformed according to the Gauss theorem because space differentiation under the integral sign is performed with respect to the same coordinates of the source points as integration (unlike the preceding expression for $\operatorname{div} \mathbf{A}$ in which differentiation under the integral sign was performed with respect to the coordinates of the point of observation). Consequently,

$$\operatorname{div} \mathbf{A} = -\frac{1}{c} \oint \left(\frac{j_n}{R} \right) dS$$

and integration must extend over the surface of all the conductors through which a current flows. But on the surface of the conductors, according to Eq. (3.21), we have

$$j_n = 0$$

therefore

$$\operatorname{div} \mathbf{A} = 0$$

Q.E.D.

4.6 Differential Equations of a Magnetic Field. Circulation of Magnetic Field Intensity

1. Since the divergence of any curl always equals zero [Eq. (A.42₂)], then a very important equation directly follows from Eq. (4.29), namely,

$$\operatorname{div} \mathbf{H} = 0 \quad (4.36)$$

On the other hand, forming a curl from both sides of the same equation (4.29), we get on the basis of Eq. (A.42₃)

$$\operatorname{curl} \mathbf{H} = \operatorname{curl} \operatorname{curl} \mathbf{A} = \operatorname{grad} \operatorname{div} \mathbf{A} - \nabla^2 \mathbf{A}$$

or, in view of Eq. (4.35),

$$\operatorname{curl} \mathbf{H} = -\nabla^2 \mathbf{A} \quad (4.37)$$

Using Eq. (4.33) for $\nabla^2 \mathbf{A}$ in Eq. (4.37), we finally get

$$\operatorname{curl} \mathbf{H} = \frac{4\pi}{c} \mathbf{j} \quad (4.38)$$

Equations (4.36) and (4.38) are a complete system of the fundamental differential equations of the magnetic field of steady currents (see Sec. 4.8).

2. Let us consider the circulation of the magnetic vector \mathbf{H} along an arbitrary closed line L . On the basis of the Stokes theorem (A.27) and Eq. (4.38), we can write

$$\oint_L H_s ds = \int_S \operatorname{curl}_n \mathbf{H} dS = \frac{4\pi}{c} \int_S j_n dS \quad (4.39)$$

The surface integrals can be extended over *any* surface resting on contour L (see Appendix, p. 643). It is not difficult, however, to also see directly that owing to the steady currents being *closed* (current loops), the value of these surface integrals depends only on the contour L of the integration surface S . Indeed, according to Eq. (3.11), the quantity $j_n dS$ equals the current dI flowing through an element of the surface dS in the direction of its positive normal. Consequently,

$$\oint_L H_s dS = \frac{4\pi}{c} \int_S j_n dS = \frac{4\pi}{c} \sum I \quad (4.40)$$

where $\sum I$ is the algebraic sum of the currents penetrating the contour L ; these currents should be considered positive or negative depending on whether their directions form a right-handed or a left-handed system with the direction of a positive circumvention of the contour L (Fig. 47).

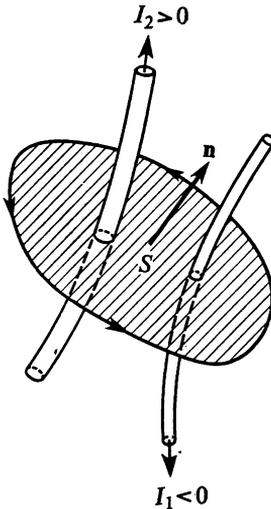


Fig. 47

If the two surfaces S and S' rest on the same contour L , then the totality of these surfaces forms a single closed surface confining a certain volume V . Owing to the closed nature of steady currents, the quantity of electricity $\sum I$ flowing through S into the volume V in a unit time must equal the quantity of electricity $\sum I$ flowing out of this volume through the surface S' . Thus, the quantity $\sum I$ will indeed be the same for both surfaces S and S' .

Hence, according to Eq. (4.40), the circulation of the magnetic field intensity vector along a curve not encircling currents equals zero, while its circulation along a curve encircling currents equals the sum of these currents (taken with the appropriate signs) multiplied by $4\pi/c$. This is one of the most important theorems in the theory of a magnetic field.

Problem 30. Show on the basis of Eq. (4.39) that the intensity of the field of a current I flowing through an infinite right circular cylinder having the radius r_0 is

$$H_e = \frac{2I}{cr} \quad (\text{when } r \geq r_0) \quad \text{and} \quad H_i = \frac{2Ir}{cr_0^2} \quad (\text{when } r \leq r_0)$$

where r is the distance from a point of the field to the axis of the current, the lines of force of the field being circles concentric to the current.

4.7 Potential Fields and Solenoidal Fields. Comparison of Differential Equations for an Electric and a Magnetic Fields

1. In Secs. 1.7 and 1.8, we proved that the necessary and sufficient condition for the field of an arbitrary vector \mathbf{a} to be a *potential* field is the equality to zero of the circulation of the vector along any closed contour [Eq. (1.33)]:

$$\oint \mathbf{a}_s ds = 0 \quad (4.41)$$

In this and only in this case may we introduce the single-valued scalar potential ψ of the vector \mathbf{a} determined by the relationship

$$\psi_1 - \psi_2 = \int_1^2 \mathbf{a}_s ds \quad (4.42)$$

and the vector \mathbf{a} , according to the results of Sec. 1.10, is equal to the gradient of this potential taken with the reverse sign*:

$$\mathbf{a} = -\text{grad } \psi \quad (4.43)$$

We showed further in Sec. 1.7 that a vector satisfying the integral condition (4.41) also satisfies the differential equation

$$\text{curl } \mathbf{a} = 0 \quad (4.44)$$

at all points of space**. Conversely, Eq. (4.41) follows from Eq. (4.44). Thus, conditions (4.41) and (4.44) are equivalent to each other, and, therefore, the equality of $\text{curl } \mathbf{a}$ to zero at all points of space is a necessary and sufficient condition for the field of the vector \mathbf{a} to have a single-valued scalar potential. This is why a potential field is also called a *vortex-free* or *non-circuital* field.

* Although the reasoning in Secs. 1.7, 1.8, and 1.10 relates directly to the vector of the electric field intensity, however, as indicated in these sections all this reasoning may be applied to any vector \mathbf{a} whose circulation along an *arbitrary* contour identically equals zero. Naturally, the quantity W encountered in these sections will no longer be, generally speaking, the work of the forces of the field; its physical meaning will depend on the meaning of the vector \mathbf{a} .

** Equation (4.44) can also be directly obtained from Eq. (4.43) on the basis of the known formula of vector analysis (A.42₁) according to which the curl of a gradient identically equals zero.

Particularly, the intensity of a constant electric field \mathbf{E} complies with condition (4.41) [see Eq. (1.33)] and therefore has the scalar potential φ , and also satisfies a differential equation of the type (4.44):

$$\text{curl } \mathbf{E} = 0 \quad (4.45)$$

[see Eq. (1.36)].

2. Let us compile a system of equations for the intensities of a magnetic and an electrostatic fields:

$$\text{div } \mathbf{E} = 4\pi\rho, \quad \text{div } \mathbf{H} = 0$$

$$\text{curl } \mathbf{E} = 0, \quad \text{curl } \mathbf{H} = \frac{4\pi}{c} \mathbf{j}$$

A magnetic field, unlike an electrostatic one, is a vortex field, and a purely vortex one in the sense that its divergence equals zero everywhere. Such fields are also called *solenoidal* ones. Consequently, a magnetic field has no scalar potential (at least no single-valued scalar potential—see below). Similar to how a potential electrostatic field is completely determined by setting the force of its sources (i.e. by setting its divergence as a function of the coordinates), so is a vortex magnetic field completely determined by the *power of its vortexes*, i.e. by a given curl of the field as a function of the coordinates. According to Eq. (4.38), the vortexes of a magnetic field are at the sections of the field through which currents flow and only in them. The vortex power of these curls (i.e. the numerical value of the curl) is proportional to the current density \mathbf{j} . In other words, the sections of a field through which currents flow can be called the *vortex space* of a magnetic field.

4.8 Boundary Conditions in the Magnetic Field of Currents.

Surface Currents. Surface Curl.

Field of an Infinite Solenoid

1. To establish the boundary conditions which the vector of a magnetic field on discontinuity surfaces must comply with, let us first assume that in all the conductors and, in particular, in thin current-conducting layers, if such exist, the volume density of the currents \mathbf{j} remains finite everywhere. We shall then make the thickness dl of these current-carrying layers tend to zero and require that the equations of a field (4.36), and (4.38) also remain true in these layers in the limiting case when $dl = 0$. The needed boundary conditions are determined unambiguously by this requirement.

Indeed, on the basis of this requirement, we get the boundary conditions for the normal component of the vector \mathbf{H} from Eq. (4.36) in accordance with Eq. (1.29):

$$\text{Div } \mathbf{H} = H_{2n} - H_{1n} = 0 \quad (4.46)$$

Integrating Eq. (4.38) over the surface S , on the other hand, we get, on the basis of the Stokes theorem (A.27), Eq. (4.39)

$$\oint_L H_s ds = \frac{4\pi}{c} \int j_n dS$$

Let us consider an arbitrary layer of the thickness dl separating medium 1 from medium 2 and consider the element of the normal cross section of this layer $dS = dl dt$ hatched in Fig. 48. We shall

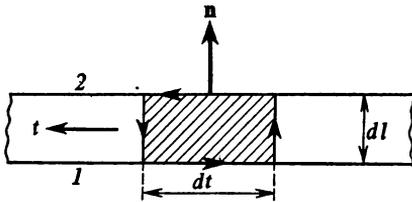


Fig. 48

apply integral equation (4.39) to this element $dl dt$. We select the positive direction around this section, for example, as shown in the figure.

If we begin to reduce the thickness of the layer dl so that it tends to zero, leaving the length of the section being considered dt unchanged, then the area of this section will also tend to zero. The left-hand side of Eq. (4.39) when $dl \rightarrow 0$ will become (up to infinitesimals of the second order)

$$\oint H_s ds = (H_{2t} - H_{1t}) dt$$

where H_1 and $H_2 =$ absolute values of the vector \mathbf{H} in the first and second media

$\mathbf{t} =$ unit vector tangent to the interface and lying in the plane of the layer section.

As regards the right-hand side of Eq. (4.39), it is proportional to the current flowing through the area element $dl dt$ and therefore will become equal to zero when $dl = 0$ if the volume density of the current \mathbf{j} is finite, as we have always assumed up to now. In a number of cases, however, if the currents are concentrated in a layer having a very small thickness, it is convenient to consider the limiting case of currents flowing through infinitely thin surfaces, i.e. *surface currents* (cf. space and surface electric charges).

2. By the *density \mathbf{i} of surface currents*, unlike the density \mathbf{j} of volume (space) currents, we shall mean the quantity of electricity flowing in a unit time through a *unit length of a segment* on the surface through which a current is flowing, and perpendicular to the direction of the current. If \mathbf{i} differs from zero, then the current flowing through

the hatched area $dt dl$ (Fig. 48) at the limit when $dl \rightarrow 0$ will obviously equal

$$\lim j_n dS = \lim j_n dt dl = i_N dt \quad (4.47)$$

where i_N is the component of the surface current density perpendicular to \mathbf{t} . We have introduced the subscript N here instead of n to let \mathbf{n} retain the meaning of a normal to the interface (\mathbf{n} is directed from medium 1 to medium 2). By \mathbf{N} we should understand a unit vector tangent to the surface and perpendicular to the tangent vector \mathbf{t} .

Using the expressions obtained in Eq. (4.39), after cancelling dt , we get the required boundary condition:

$$\frac{4\pi i_N}{c} = H_{2t} - H_{1t} \quad (4.48)$$

We can see from examination of Fig. 48, where according to the direction around the hatched area $dt dl$ which we have selected the vector \mathbf{N} should be directed toward the reader, that the mutually

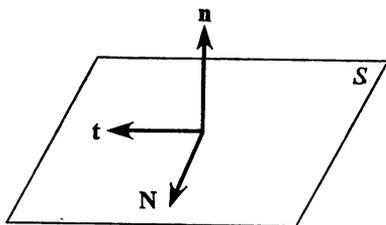


Fig. 49

perpendicular unit vectors \mathbf{t} , \mathbf{N} , and \mathbf{n} form a right-handed system (Fig. 49) so that

$$\mathbf{t} = [\mathbf{Nn}]$$

Consequently

$$H_t = \mathbf{Ht} = \mathbf{H}[\mathbf{Nn}] = \mathbf{N}[\mathbf{nH}] \quad (4.49)$$

$$i_N = \mathbf{Ni} \quad (4.50)$$

Introducing Eqs. (4.49) and (4.50) into Eq. (4.48), we get

$$\frac{4\pi \mathbf{Ni}}{c} = \mathbf{N}[\mathbf{n}(\mathbf{H}_2 - \mathbf{H}_1)]$$

Since \mathbf{N} can have an arbitrary direction in the interface plane, then

$$\frac{4\pi \mathbf{i}}{c} = [\mathbf{nH}_2] - [\mathbf{nH}_1] \quad (4.51)$$

This equation is exactly the boundary condition which the *tangential* components of the vector \mathbf{H} must comply with when there are

surface currents on the interface (because only these components are in the expression $[\mathbf{nH}]$).

In the absence of surface currents, this boundary condition becomes

$$[\mathbf{n}(\mathbf{H}_2 - \mathbf{H}_1)] = 0 \quad (4.52)$$

This equation means that the components of the field intensity \mathbf{H} tangent to an arbitrary surface are continuous, i.e. that in the absence of surface currents for any tangent direction \mathbf{t} we have

$$H_{2t} = H_{1t} \quad (4.53)$$

Equations (4.52) and (4.53) are obviously equivalent to each other.

3. It follows from the above that, generally speaking, if two *arbitrary* vectors \mathbf{a} and \mathbf{j} are related by the expression

$$\text{curl } \mathbf{a} = 4\pi\mathbf{j}$$

then for the discontinuity surfaces of these vectors the expression relating them acquires the form

$$[\mathbf{n}(\mathbf{a}_2 - \mathbf{a}_1)] = 4\pi\mathbf{i} \quad (4.54)$$

where \mathbf{i} is obtained from \mathbf{j} by a limit transition similar to Eq. (4.47). This is why the left-hand side of Eq. (4.54) is customarily called the *surface curl* of the vector \mathbf{a} and, unlike the conventional curl is designated by Curl with a *capital* letter C:

$$\text{Curl } \mathbf{a} = [\mathbf{n}(\mathbf{a}_2 - \mathbf{a}_1)] \quad (4.55)$$

(cf. the definition of the surface divergence in Sec. 1.6). Similar to Eq. (1.30), we can thus express what we have just formulated above in the following symbolic form:

$$\text{curl } \mathbf{a} = 4\pi\mathbf{j} \rightarrow \text{Curl } \mathbf{a} = 4\pi\mathbf{i} \quad (4.56)$$

4. Thus, the boundary conditions (4.46) and (4.51) for the magnetic field of steady currents can be written as follows:

$$\text{Div } \mathbf{H} = 0 \text{ and } \text{Curl } \mathbf{H} = \frac{4\pi}{c} \mathbf{i} \quad (4.57)$$

The boundary conditions (1.14) and (1.37) for a stationary electric field in a vacuum, proceeding from the equivalence of equations of the kind of (4.52) and (4.53), can be written as follows:

$$\text{Div } \mathbf{E} = 4\pi\sigma, \text{ Curl } \mathbf{E} = 0 \quad (4.58)$$

The equations

$$\left. \begin{aligned} \text{curl } \mathbf{H} &= \frac{4\pi}{c} \mathbf{j}, & \text{Curl } \mathbf{H} &= \frac{4\pi}{c} \mathbf{i} \\ \text{div } \mathbf{H} &= 0, & \text{Div } \mathbf{H} &= 0 \end{aligned} \right\} \quad (\text{B})$$

are a *complete system* of differential equations of the magnetic field of steady currents. In other words, a magnetic field is determined *unambiguously* by the system (B) if we know the distribution of the currents \mathbf{j} and \mathbf{i} and if at infinity the condition is observed that

$$HR^2 \text{ remains finite when } R \rightarrow \infty \quad (4.59)$$

which means that all the currents inducing the field are in a finite region of space [cf. Eqs. (1.85) and (4.55)]. Conversely, if the field intensity \mathbf{H} at each point of space is given, then the distribution of the currents \mathbf{i} and \mathbf{j} is unambiguously determined by the system (B).

The second statement is obvious. To prove the first one, let us assume that there are two solutions \mathbf{H} and \mathbf{H}' of the system (B) for the given values of \mathbf{j} and \mathbf{i} . Introducing both solutions into (B) and then subtracting the corresponding equations from one another, we get

$$\left. \begin{aligned} \text{curl } \mathbf{H}'' = 0, \quad \text{Curl } \mathbf{H}'' = 0 \\ \text{div } \mathbf{H}'' = 0, \quad \text{Div } \mathbf{H}'' = 0 \end{aligned} \right\} \quad (\text{B}')$$

where $\mathbf{H}'' = \mathbf{H} - \mathbf{H}'$. Further, assuming that $\mathbf{H}'' = \text{curl } \mathbf{A}''$, which is always possible because $\text{div } \mathbf{H}'' = 0$, we get the following train of equations on the basis of Eqs. (A.44) and (B'):

$$H''^2 = \mathbf{H}'' \text{ curl } \mathbf{A}'' = \text{div} [\mathbf{A}'' \mathbf{H}''] + \mathbf{A}'' \text{ curl } \mathbf{H}'' = \text{div} [\mathbf{A}'' \mathbf{H}'']$$

Consequently, the integral of H''^2 over the arbitrary volume confined by the surface S , on the basis of Eq. (A.17), will be

$$\int H''^2 dV = \int \text{div} [\mathbf{A}'' \mathbf{H}''] dV = \oint [\mathbf{A}'' \mathbf{H}'']_n dS \quad (4.60)$$

and the surface integral must be taken only over the boundary surface S because in the entire field, according to (B'), the vector \mathbf{H}'' and, therefore, the vector \mathbf{A}'' remain continuous.

If we now extend integration over the volume of the *total field* (p. 92), then on the basis of Eq. (4.59) the integral over the boundary surface S will vanish*. Consequently,

$$\int H''^2 dV = 0$$

whence it follows that $\mathbf{H}'' = \mathbf{H} - \mathbf{H}'$ vanishes at all points of the field. This proves the unambiguity of the solution of the system (B).

5. It should be noted that when surface currents are present, expressions (4.13) and (4.28) for the field intensity \mathbf{H} and for the vector

* Since \mathbf{H} is expressed through derivatives of the vector potential \mathbf{A} , then when condition (4.59) holds this auxiliary vector can always be chosen so that the following condition is observed at infinity:

$$AR \text{ remains finite when } R \rightarrow \infty \text{ [see also Eq. (4.34)].}$$

potential A become

$$\mathbf{H} = \frac{1}{c} \int \frac{[\mathbf{j}\mathbf{R}]}{R^3} dV + \frac{1}{c} \int \frac{[\mathbf{i}\mathbf{R}]}{R^3} dS \quad (4.61)$$

$$A = \frac{1}{c} \int \frac{\mathbf{j}}{R} dV + \frac{1}{c} \int \frac{\mathbf{i}}{R} dS \quad (4.62)$$

where the surface integrals must be extended over all the surfaces around which surface currents flow. It is easy to convince oneself in the correctness of these generalized expressions by means of a limit transition from space currents to surface ones.

In the following, we shall consider everywhere that surface currents are absent ($i = 0$) except where the opposite is explicitly indicated.

Example. *The magnetic field of an infinite cylindrical solenoid.* Assume that the current I circulates through a conductor wound in a helix around the surface of a cylinder whose cross-sectional area is S . Such a cylinder carrying a current is called a cylindrical solenoid.

Assume that there are n turns of the conductor per unit length of the cylinder. If the pitch of the helix is sufficiently small, then each turn of the solenoid may be approximately replaced with a closed annular current (a current loop) of the same intensity. If the cross-sectional area of the conductor is also small in comparison with that of the cylinder, we can consider approximately that a uniformly distributed *surface* current having the density

$$i = nI \quad (4.63)$$

circulates along an infinitely thin surface of the cylinder. The lines of this current are circles formed by intersecting the surface of the cylinder with planes perpendicular to its axis.

Let us assume that our solenoid is a cylinder having an infinite length. In this case, the field outside of the cylinder H_e equals zero, while the field inside it H_i is homogeneous and equals

$$H_i = \frac{4\pi}{c} i = \frac{4\pi}{c} nI, H_e = 0 \quad (4.64)$$

H_i being directed along the axis of the solenoid and forming a right-handed system with the direction of the current in the solenoid. Indeed, Eqs. (4.64) evidently satisfy the equations $\text{div } \mathbf{H} = 0$ and $\text{curl } \mathbf{H} = 0$ corresponding to the absence of space currents. Further, \mathbf{H} is everywhere parallel to the surface of the solenoid, i.e. to the surface of discontinuity of the field, and therefore $\text{Div } \mathbf{H} = 0$. Finally, it can be easily seen that with the indicated direction of H_i the equations (4.64) also comply with the equation

$\text{Curl } \mathbf{H} = \frac{4\pi}{c} \mathbf{i}$ on the surface of the solenoid. Thus, Eqs. (4.64) satisfy all the equations of the system (B) and, owing to the completeness of this system, are the only solution of the problem*. Equations (4.64) may obviously also be approximately applied to the field of a *finite* solenoid on its portions whose distance from the ends of the solenoid is great both in comparison with that to the nearest portions of the solenoid and in comparison with a cross section of the solenoid. It is essentially only this case that is borne in mind when the field of an "infinite" solenoid is mentioned.

Problem 31. Solve Problem 29 on the basis of the differential equations (B) of a field and, in addition, show that the field outside of a hollow cylinder coincides with the field of a line current of the same magnitude flowing along the axis of the cylinder.

4.9 Ponderomotive Forces Acting on a Current Loop in a Magnetic Field. Potential Function of a Current in an External Magnetic Field

1. In determining the mechanical forces acting on a closed current or current loop I in an external magnetic field, we shall first limit ourselves to the case when the field does not change to any appreciable extent over any cross section of the current I so that this current may be considered as a *line* one (p. 226).

Let us first pose the problem of determining the work done by the ponderomotive forces of the magnetic field \mathbf{H} upon the arbitrary displacement of a contour of the current I . This displacement, in general, may be connected with deformation of the current contour.

Assume that each element ds of the contour L of the current I performs a certain arbitrary infinitely small displacement q that does not naturally violate the integrity of the contour (Fig. 50), *the current I remaining constant during this displacement* (a virtual displacement). The work done by the forces of the magnetic field in this displacement of the element ds will, according to Eq. (4.14), equal

$$\mathbf{qF} = \frac{I}{c} \mathbf{q} [ds \mathbf{H}]$$

* Generally speaking the above proof of the unambiguity of the system (B) cannot be applied directly to the field of an *infinite* solenoid because in this case the currents are not concentrated in a finite portion of space, \mathbf{H} does not vanish at infinity and we cannot assert that the quantities \mathbf{H}'' and \mathbf{A}'' in Eq. (4.60) will vanish at infinity. If, however, we consider the field of an infinite solenoid as the limiting case of the field of a very long finite solenoid (and it is only such a consideration that has a physical meaning), then the unambiguity of the field in the limiting case of an infinite solenoid also follows from the proof of the unambiguity of the field of a very long, but finite, solenoid.

while the total work δW connected with the displacement of all the elements of the current contour will be

$$\delta W = \oint \mathbf{F} \mathbf{q} = \frac{I}{c} \oint \mathbf{q} [ds \mathbf{H}] = \frac{I}{c} \oint \mathbf{H} [\mathbf{q} ds]$$

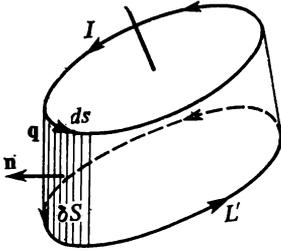


Fig. 50

But

$$[\mathbf{q} ds] = \delta S$$

where δS is the element of area covered by the element of the contour ds upon its displacement \mathbf{q} (Fig. 50). The sequence of the multipliers ds and \mathbf{q} in the expression for δS has been selected so that the direction of the vector δS (i.e. the direction of a *positive* normal \mathbf{n} to the element δS) forms a right-handed system with the direction of the current in the contour L' . Hence

$$\delta W = \frac{I}{c} \int_{\Delta} \mathbf{H} \delta S = \frac{I}{c} \int_{\Delta} H_n \delta S$$

where integration should be extended over all the elements δS of the surface Δ covered by the contour of the current L when its points move over the distance \mathbf{q} to the position L' .

Let us denote by Φ the flux of the magnetic vector or, more briefly, the *magnetic flux* through the current contour L (i.e. through the arbitrary surface area S resting on this contour):

$$\Phi = \int_S H_n dS \tag{4.65}$$

where \mathbf{n} is a *positive* normal to S forming a right-handed system with the direction of the current. The magnitude of this flux depends only on the arrangement of the contour L , but not on the shape of the surface S because, according to Eqs. (4.29) and (A.27), we have

$$\Phi = \int_S H_n dS = \int_S \text{curl}_n \mathbf{A} dS = \oint_L A_s dS \tag{4.66}$$

Thus, the magnetic flux Φ through the contour L equals the circulation of the vector potential \mathbf{A} along this contour.

Using the designation (4.65), we can write

$$\int_{\Delta} H_n dS = \delta\Phi$$

because the increment in the magnetic flux through the contour of the current obviously equals the magnetic flux through the surface Δ circumscribed by the contour upon its displacement. Consequently

$$\delta W = \frac{I}{c} \delta\Phi \quad (4.67)$$

We thus arrive at the following very simple result: the work of the ponderomotive forces of a magnetic field upon an arbitrary displacement of the current equals the increment in the magnetic flux through the contour of this current multiplied by I/c . This means, particularly, that such displacements in the current when the magnitude of the magnetic flux through its contour does not change are not connected with the work of the magnetic field.

2. If we introduce the notation

$$U = -\frac{I}{c} \Phi \quad (4.68)$$

then Eq. (4.67) will become

$$\delta W = -(\delta U)_I \quad (4.69)$$

where the subscript I of δU indicates that in determining the increment of the function U the currents I should be considered constant.

Hence, the work of the ponderomotive forces of a magnetic field equals the reduction in the function U which thus plays the part of a *potential or force function* of a current in a magnetic field. Particularly, we can easily see on the basis of Eq. (4.69), using methods of reasoning which are conventional in analytical mechanics, that if the function U is expressed depending on "generalized" coordinates q_i characterizing the position of the contour of a current, then the "generalized" (in the meaning implied by analytic mechanics) ponderomotive force Θ_i acting on the current in the direction of any one of these coordinates q_i will equal

$$\Theta_i = -\frac{\partial U}{\partial q_i} \quad (4.70)$$

These properties of the potential or force function U may induce us to identify it with the *potential energy* of a magnetic field. Such a conclusion, however, would be *unsubstantiated* because, as we shall see in the following chapter, the displacements of a conductor in a

magnetic field are attended not only by the work of the ponderomotive forces of this field, but also by the work of the electromotive forces induced by the field in the moving conductor. As a result, the change in the energy of a magnetic field upon the displacement of conductors cannot be determined according to the work of only the ponderomotive forces of the field.

Therefore, even if we do call U the potential “energy” sometimes for convenience’s sake, this is only in the sense that the ponderomotive forces of a magnetic field are related to U in the same way as the forces of a *conservative* field of forces are related to the potential energy of this field.

Although the potential force function U does not equal the energy of a magnetic field, nevertheless the introduction of this function appreciably facilitates the studying of the ponderomotive forces acting in a magnetic field on current loops because this eliminates the need in each separate case to perform complicated summation of the forces acting on separate current elements.

Particularly, it is easy to see by conventional well-known reasoning on the basis of Eqs. (4.69) and (4.70) that the *stable* equilibrium of a contour of a steady current corresponds to the *minimum of the potential function* U , i.e. according to Eq. (4.68), to the *maximum of the magnetic flux* Φ .

3. The formulas we have obtained for line currents **can** easily be generalized for space currents, i.e. for cases when we may not ignore the change in the magnetic field intensity over the current cross section. For this purpose, we shall first introduce Eq. (4.66) into Eq. (4.68):

$$U = -\frac{I}{c} \oint \mathbf{A} ds \quad (4.71)$$

and then pass over to space currents in the equation obtained in accordance with formula (4.16):

$$U = -\frac{1}{c} \int \mathbf{A} \mathbf{j} dV \quad (4.72)$$

This is the required expression of Eq. (4.68).

Equations (4.71) and (4.72) can be interpreted in the sense (with all the reservations we have just made) that every current element $I ds$ (or $\mathbf{j} dV$) has the potential “energy” $-\frac{I}{c} \mathbf{A} ds$ (or $-\frac{1}{c} \mathbf{A} \mathbf{j} dV$) and that the potential function U of a current loop equals the sum of the “energies” of its separate elements.

Example. *A rectangular loop in a homogeneous magnetic field.*
 Let us consider an arbitrary flat contour (a loop) having the cross-sectional area S through which the current I flows. Assume that the loop is placed in a homogeneous magnetic field \mathbf{H} and is

secured so that it can rotate about its axis perpendicular to the field \mathbf{H} (Fig. 51).

Let θ be the angle between \mathbf{H} and a positive normal \mathbf{n} to the loop, i.e. a normal forming a right-handed system with the direction

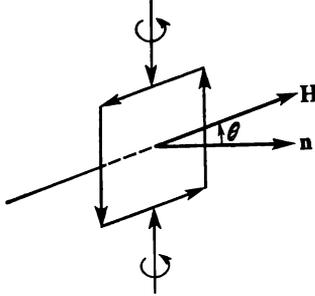


Fig. 51

of the current in the loop. Hence, the magnetic flux through the loop will be

$$\Phi = HS \cos \theta$$

and the potential function of the loop will be

$$U = -\frac{1}{c} ISH \cos \theta \quad (4.73)$$

The generalized force corresponding to the generalized coordinate θ , as is well known (see the example in Sec. 1.18), is nothing but the moment N of the forces applied to the loop that tends to turn it about its axis of rotation:

$$N = -\frac{\partial U}{\partial \theta} = -\frac{1}{c} ISH \sin \theta$$

The equilibrium positions of the loop correspond to $N = 0$, i.e. to $\theta = 0$ and $\theta = \pi$. The first of these positions corresponds to the minimum, and the second to the maximum of the potential function U , and, consequently, only the first position of equilibrium is stable. Hence it follows that the ponderomotive forces of a magnetic field tend to turn the plane of the current so that a positive normal to it would coincide with the direction of the magnetic field \mathbf{H} . Particularly, two interacting current contours will tend to arrange themselves with their planes parallel to each other and with the currents flowing in the same direction.

4.10 Ponderomotive Interaction of Currents. Mutual Induction

1. Let us now consider the interaction of two line current loops I_1 and I_2 flowing through contours L_1 and L_2 . Let \mathbf{H}_1 and \mathbf{A}_1 be the

values of the intensity and vector potential of the field of the first loop, and \mathbf{H}_2 and \mathbf{A}_2 , the corresponding quantities for the second loop. Further, let us denote by Φ_{12} the magnetic flux of the first loop through the contour of the second one:

$$\Phi_{12} = \int_{S_2} H_{1n} dS_2 = \oint_{L_2} A_{1s} ds_2 = \oint_{L_2} \mathbf{A}_1 \cdot d\mathbf{s}_2 \quad (4.74)$$

where S_2 is the surface resting on the contour L_2 , and ds_2 is an element of length of this contour [cf. Eq. (4.66)]. We shall denote the magnetic flux sent by the second loop through the contour of the first one, accordingly, by Φ_{21} :

$$\Phi_{21} = \int_{S_1} H_{2n} dS_1 = \oint_{L_1} \mathbf{A}_2 \cdot d\mathbf{s}_1 \quad (4.75)$$

Using in Eq. (4.74) the value of the vector potential of the line current I_1 from Eq. (4.30), i.e.

$$\mathbf{A}_1 = \frac{I_1}{c} \oint_{L_1} \frac{d\mathbf{s}_1}{R}$$

we get

$$\Phi_{12} = \frac{I_1}{c} \oint_{L_1} \oint_{L_2} \frac{d\mathbf{s}_1 \cdot d\mathbf{s}_2}{R} \quad (4.76)$$

where integration must be performed over both contours L_1 and L_2 . The scalar product of each element of length ds_1 of the contour L_1 and each element ds_2 must be divided by the distance R between these elements ds_1 and ds_2 . In an absolutely similar way, we find

$$\Phi_{21} = \frac{I_2}{c} \oint_{L_2} \oint_{L_1} \frac{d\mathbf{s}_2 \cdot d\mathbf{s}_1}{R} \quad (4.77)$$

2. The double integral in Eqs. (4.76) and (4.77) is generally designated by L_{12} or L_{21} :

$$L_{12} = L_{21} = \oint_{L_1} \oint_{L_2} \frac{d\mathbf{s}_1 \cdot d\mathbf{s}_2}{R} \quad (4.78)$$

and is called the *mutual inductance* of the contours L_1 and L_2 (the meaning of this name will be revealed in Chap. 5 when we establish the relationship between the quantity L_{12} and the induction interaction of currents). Introducing this designation in Eqs. (4.76) and (4.77), we get

$$\Phi_{12} = \frac{1}{c} I_1 L_{12} \quad \text{and} \quad \Phi_{21} = \frac{1}{c} I_2 L_{21} \quad (4.79)$$

The mutual inductance is naturally a *purely geometrical quantity* depending only on the configuration and mutual arrangement of the contours L_1 and L_2 and on the choice of the direction of positive circumvention of each of these contours*.

On the basis of Eqs. (4.79), however, we can say that the mutual inductance of the contours L_1 and L_2 numerically equals the magnetic flux sent through one of this contours (for example L_2) by a current having the intensity c circulating through the other contour L_1 :

$$\Phi_{12} = L_{12} \text{ (when } I_1 = c \text{)} \quad (4.80)$$

3. According to Eqs. (4.68) and (4.79), the potential function U_{12} of the current I_2 in the field of the current I_1 equals

$$U_{12} = -\frac{I_2}{c} \Phi_{12} = -\frac{1}{c^2} L_{12} I_1 I_2 \quad (4.81)$$

The potential function U_{21} of the current I_1 in the field of the current I_2 will also be expressed in exactly the same way:

$$U_{21} = -\frac{I_1}{c} \Phi_{21} = -\frac{1}{c^2} L_{21} I_2 I_1 \quad (4.82)$$

The quantity U_{12} (or the quantity U_{21} equal to it) plays the part of the *mutual* potential energy of the currents I_1 and I_2 in the sense that the work of the *ponderomotive* forces of interaction of these current loops upon the motion of either of them or of both simultaneously equals the decrease in the function U_{12} . Particularly, the generalized ponderomotive forces of interaction of these currents Θ_i , according to Eq. (4.70), equal the derivatives of U_{12} with respect to the corresponding generalized coordinates q_i taken with the opposite sign. Since according to Eq. (4.69) *the currents I_1 and I_2 should be considered constant* when determining the work done by these forces according to the change in the value of U_{12} , then [cf. Eqs. (4.69) and (4.68)]

$$\delta W = -(\delta U_{12})_I = \frac{1}{c} I_1 I_2 \delta L_{12} \quad (4.83)$$

$$\Theta = -\frac{(\partial U_{12})_I}{\partial q_i} = \frac{1}{c^2} I_1 I_2 \frac{\partial L_{12}}{\partial q_i} \quad (4.84)$$

4. It follows from the above formulas, incidentally, that the mechanical interaction of *current loops* (unlike the interaction of current elements—see Sec. 4.2) complies with the principle of the *equality of action and counteraction* because the forces acting on each of the

* When the direction of one of the currents (for example I_1) is reversed and that of the other current is left unchanged, the sign of the quantity L_{12} becomes the opposite because the direction of the vector ds_1 changes.

interacting loops are determined by derivatives of the same function $U_{12} = U_{21}$, which depends only on the relative arrangement of the two contours.

We shall use a simple example to explain this statement. Let $q = l$ be the distance between the centres of two parallel ring currents L_1 and L_2 . The forces F_1 and F_2 acting respectively on the contours L_1 and L_2 in the direction of a growth in the distance l equal

$$F_1 = - \frac{\partial U_{21}}{\partial l}$$

$$F_2 = - \frac{\partial U_{12}}{\partial l} = F_1$$

If $-\partial U_{21}/\partial l > 0$, then the forces F_1 and F_2 tend to increase the distance l , i.e. consist in the mutual repulsion of contours L_1 and L_2 . Otherwise, they consist in the attraction of these contours. It is important, however, that in both cases the forces F_1 and F_2 are numerically equal and opposite in direction, i.e. comply with Newton's third law.

Similarly, if $q = \alpha$ equals the angle between the planes of the two contours L_1 and L_2 , then the generalized force $N = -\partial U/\partial \alpha$ is the moment of the couple of forces tending to increase the angle α . As in the preceding case, it is easy to see that the moments of the couples N_1 and N_2 applied to L_1 and L_2 are numerically equal and opposite in direction.

5. If the currents I_1 and I_2 cannot be considered line ones, i.e. if the field of one of these currents appreciably changes along the cross section of the other current, then the mutual potential "energy" of the currents U can be determined on the basis of Eq. (4.72). If \mathbf{j}_1 is the current density in the first current of the volume V_1 , and \mathbf{j}_2 is the current density in the second current of the volume V_2 , then according to Eq. (4.72), the potential function U_{12} of the current I_2 in the field of the current I_1 will equal

$$U_{12} = - \frac{1}{c} \int_{V_2} \mathbf{A}_1 \mathbf{j}_2 dV \quad (4.85)$$

while the potential function U_{21} of the current I_1 in the field of the current I_2 will equal

$$U_{21} = - \frac{1}{c} \int_{V_1} \mathbf{A}_2 \mathbf{j}_1 dV \quad (4.86)$$

Using in these equations expressions (4.28) for the vector potentials \mathbf{A}_1 and \mathbf{A}_2 of the currents \mathbf{j}_1 and \mathbf{j}_2 ,

$$\mathbf{A}_1 = \frac{1}{c} \int_{V_1} \frac{\mathbf{j}_1 dV}{R} \quad \text{and} \quad \mathbf{A}_2 = \frac{1}{c} \int_{V_2} \frac{\mathbf{j}_2 dV}{R}$$

we get

$$U_{12} = -\frac{1}{c^2} \iint_{V_1 V_2} \frac{\mathbf{j}_1 \mathbf{j}_2 dV_1 dV_2}{R} = U_{21} \quad (4.87)$$

where R is the distance between the element dV_1 of the volume V_1 in which the current density is \mathbf{j}_1 and the element dV_2 of the volume V_2 in which the current density is \mathbf{j}_2 .

Example. Two identical contours L_1 and L_2 have the shape of squares whose side is a . The sides of the two squares are parallel to one another, while the centres of the squares are at a distance d from each other on a straight line perpendicular to their planes. Find the mutual inductance L_{12} of these squares and the force F with which these contours are attracted if identically directed currents I_1 and I_2 flow through them.

In the double integral of Eq. (4.78)

$$L_{12} = \oint_{L_1} \oint_{L_2} \frac{ds_1 ds_2}{R}$$

all the terms relating to mutually perpendicular pairs of elements ds_1 and ds_2 equal zero. For this reason in the case being considered, the expression for L_{12} consists in the sum of the integrals relating to pairs of parallel sides of the squares L_1 and L_2 .

For two parallel straight lines with a length of a and at a distance of h from each other (Fig. 52), we have

$$\begin{aligned} L(a, h) &= \iint \frac{ds_1 ds_2}{R} = \int_{-a/2}^{+a/2} \int_{-a/2}^{+a/2} \frac{dx_1 dx_2}{\sqrt{(x_1 - x_2)^2 + h^2}} = \\ &= \int_{-a/2}^{+a/2} dx_2 [\log \{x_1 - x_2 + \\ &\quad + \sqrt{(x_1 - x_2)^2 + h^2}\}] \Big|_{x_1 = -a/2}^{x_1 = a/2} \end{aligned}$$

where x_1 and x_2 are the running coordinates of both straight lines counted from their middles. Integration by parts of each

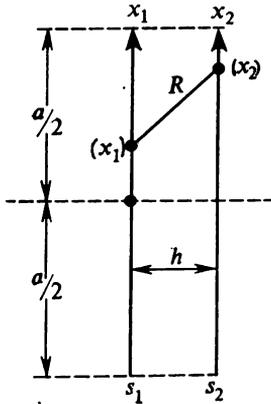


Fig. 52

of the terms of the difference obtained after substituting for x_1 its values $\pm a/2$ yields

$$\begin{aligned}
 & \int_{-a/2}^{+a/2} dx_2 \log \left\{ \pm \frac{a}{2} - x_2 + \sqrt{\left(\pm \frac{a}{2} - x_2\right)^2 + h^2} \right\} = \\
 & = \left[x_2 \log \left\{ \pm \frac{a}{2} - x_2 + \sqrt{\left(\pm \frac{a}{2} - x_2\right)^2 + h^2} \right\} \right]_{x_2=-a/2}^{x_2=a/2} + \\
 & + \left[\sqrt{\left(\pm \frac{a}{2} - x_2\right)^2 + h^2} \right]_{x_2=-a/2}^{x_2=a/2} \pm \\
 & \pm \frac{a}{2} \left[\log \left\{ x_2 \pm \frac{a}{2} + \sqrt{\left(x_2 \mp \frac{a}{2}\right)^2 + h^2} \right\} \right]_{x_2=-a/2}^{x_2=a/2}
 \end{aligned}$$

Substituting the value of $\pm a/2$ for x_2 , we get a total of $3 \times 4 = 12$ terms for $L(a, h)$, which after cancelling and transformations diminish to three:

$$\begin{aligned}
 L(a, h) &= 2h - 2\sqrt{a^2 + h^2} + a \log \frac{\sqrt{a^2 + h^2} + a}{\sqrt{a^2 + h^2} - a} = \\
 &= 2h - 2\sqrt{a^2 + h^2} + 2a \log \frac{a + \sqrt{a^2 + h^2}}{h}
 \end{aligned}$$

Combining the parallel sides of the squares L_1 and L_2 in pairs, we get four pairs of segments for which the distance h equals d_1

and four other pairs for which $h = \sqrt{a^2 + d^2}$. Parallel currents flow through the first four pairs, and oppositely directed currents through the second four. Thus, the mutual inductance of these squares is

$$L_{12} = 4L(a, d) - 4L(a, \sqrt{a^2 + d^2})$$

or

$$L_{12} = 8 \left(d - 2\sqrt{a^2 + d^2} + \sqrt{2a^2 + d^2} + a \log \frac{a + \sqrt{a^2 + d^2}}{d} - a \log \frac{a + \sqrt{2a^2 + d^2}}{\sqrt{a^2 + d^2}} \right)$$

To determine the force F of attraction between the squares, it is sufficient to use Eq. (4.84), assuming that the generalized coordinate q_i in it equals d because the force F acts in a direction opposite to the coordinate d , i.e. tends to diminish the distance between the squares d . Hence,

$$F = -\frac{\partial U}{\partial q} = \frac{\partial U}{\partial d} = -\frac{1}{c} I_1 I_2 \frac{\partial L_{12}}{\partial d}$$

After differentiation and some algebraic transformations, we obtain

$$F = \frac{8}{c^2} I_1 I_2 \left(\frac{a^2 + 2d^2}{d\sqrt{a^2 + d^2}} - \frac{d\sqrt{2a^2 + d^2}}{a^2 + d^2} - 1 \right) \quad (4.88)$$

Other examples on the calculation of the mutual inductance and also of the self-inductance will be given in Sec. 6.6.

4.11 Self-Inductance.

Total Potential Function of a System of Currents

1. We shall now pass over to a treatment of the ponderomotive forces of interaction of elements of the same current (for example the current I_1) and try to determine the potential function U_{11} of these forces, which with the corresponding reservations can be called the intrinsic potential "energy" of I_1 .

Naturally, when considering the interaction of adjacent elements of current, we can no longer consider this current to be a line one and must proceed not from formula (4.71), but from (4.72). Supplying

\mathbf{A} and \mathbf{j} in this formula with the subscript 1 to avoid ambiguity, we get

$$U_{11} = -\frac{1}{2c} \int_{V_1} \mathbf{A}_1 \mathbf{j}_1 dV \quad (4.89)$$

where the integral is preceded by the factor $1/2c$ instead of $1/c$, unlike Eq. (4.72) and the equations of Sec. 4.10. The appearance of the factor $\frac{1}{2}$ is explained by the fact that the interaction of each pair of current elements $\mathbf{j}_1 dV$ and $\mathbf{j}'_1 dV'$ is taken into account twice in the integral of Eq. (4.89): it is a constituent of both $\mathbf{A}_1 \mathbf{j}_1 dV$ and $\mathbf{A}'_1 \mathbf{j}'_1 dV'$ because the value of \mathbf{A}_1 in the volume element dV includes the field of the element $\mathbf{j}'_1 dV'$. This circumstance can be seen directly if we use in Eq. (4.89) the expression (4.28) for the vector potential \mathbf{A}_1 :

$$\mathbf{A}_1 = \frac{1}{c} \int_{V_1} \frac{\mathbf{j}'_1 dV'}{R}$$

whence Eq. (4.89) becomes

$$U_{11} = -\frac{1}{2c^2} \int_{V_1} \int_{V_1} \frac{\mathbf{j}_1 \mathbf{j}'_1 dV dV'}{R} \quad (4.90)$$

where integration both over dV and over dV' must be extended over the entire volume V_1 of the current I_1 , i.e., in other words, the sum of the integrands for all possible combinations of elements of the volume V_1 in pairs must be taken. Here, obviously, the two combinations of the elements dV and dV' differing only in the sequence of the multipliers must nevertheless be considered different; R is naturally the distance between the elements dV and dV' .

2. Since the distribution of a current over the cross section of a conductor depends only on the geometrical and physical properties of the conductor and not on the current in it, then the current density in each element of the volume of the conductor is proportional to I_1 , i.e.

$$U_{11} = -\frac{1}{2c^2} \int_{V_1} \int_{V_1} \frac{\mathbf{j}_1 \mathbf{j}'_1 dV dV'}{R} = -\frac{1}{2c^2} L_{11} I_1^2 \quad (4.91)$$

where L_{11} is the so-called *self-inductance* of the conductor. It depends only on the geometrical *configuration* of the conductor (if it is homogeneous, otherwise L_{11} also depends on the ratio between the electric conductivities of the separate elements of volume of the conductor), *but not on the current in it.*

The following expression for the self-inductance is obtained from Eq. (4.91)*:

$$L_{11} = \frac{1}{I^2} \int \int_{V_1, V_1} \frac{j_1 j_1' dV dV'}{R} \quad (4.92)$$

It must be noted that the following expression for the mutual inductance of two volume currents I_1 and I_2 is obtained in an absolutely similar way from Eqs. (4.81) and (4.87):

$$L_{12} = L_{21} = \frac{1}{I_1 I_2} \int \int_{V_1, V_2} \frac{j_1 j_2 dV_1 dV_2}{R} \quad (4.93)$$

It transforms into Eq. (4.78) if these currents can be considered line ones.

3. The intrinsic potential "energy" of a current U_{11} could also be determined by dividing the current I_1 into a combination of infinitely thin current filaments dI_1 , expressing the potential function of each filament with the aid of Eq. (4.68),

$$dU = - \frac{dI_1}{c} \Phi$$

where Φ stands for the magnetic flux sent by the entire current I_1 through the contour of a given filament and, finally, summing dU over all the filaments:

$$U_{11} = - \frac{1}{2c} \sum dI_1 \Phi \quad (4.94)$$

The factor $\frac{1}{2}$ has to be introduced before the summation sign because the interaction of each pair of current filaments is taken into account two times in the sum.

If we denote by Φ_{11} the *mean* value of the magnetic flux through separate current filaments, then Eq. (4.94) can be written in a form similar to Eq. (4.81):

$$U_{11} = - \frac{1}{2c} \Phi_{11} \sum dI_1 = - \frac{I_1}{2c} \Phi_{11} \quad (4.95)$$

* We must note the absolutely wrong indication encountered in some courses of physics that to determine the self-inductance L_{11} of the contour L_1 it is sufficient to identify the contour L_2 with the contour L_1 in Eq. (4.78), i.e. that

$$L_{11} = \oint_{L_1} \oint_{L_1} \frac{ds_1 ds_2}{R}$$

It is simple to see that this expression becomes equal to infinity, i.e. has no meaning.

The quantity Φ_{11} can be called the (mean) magnetic flux sent by the current I_1 through its own contour. In essence, it is more correct to say that the value of Φ_{11} by definition must satisfy Eq. (4.95) and can be determined by comparing this equation with Eq. (4.91):

$$\Phi_{11} = \frac{1}{c} I_1 L_{11} \tag{4.96}$$

This relationship completely conforms with Eq. (4.79). Hence, we can say that the self-inductance L_{11} of an arbitrary closed conductor L_1 numerically equals the magnetic flux Φ_{11} sent through the contour of this conductor by the current c circulating through it:

$$\Phi_{11} = L_{11} \text{ when } I_1 = c \tag{4.97}$$

It must be remembered, however, that Φ_{11} is the *mean* value of the magnetic flux through the contours of the separate filaments into which the current I_1 can be resolved, and that both Φ_{11} and L_{11} appreciably depend on the shape and *dimensions of the cross section* of the conductor L_1 . For an infinitely thin line contour, however, the quantities Φ_{11} and L_{11} become equal to infinity, i.e. lose their meaning.

4. Returning to a system of two currents, we shall note that the total potential “energy” U of this system evidently equals the sum of their mutual “energy” U_{12} ($= U_{21}$) and the intrinsic potential “energies” U_{11} and U_{22} of each of them:

$$U = U_{11} + U_{12} + U_{22} = -\frac{1}{c^2} \left(\frac{1}{2} L_{11} I_1^2 + L_{12} I_1 I_2 + \frac{1}{2} L_{22} I_2^2 \right) \tag{4.98}$$

Since $L_{12} = L_{21}$, we can also write

$$U = -\frac{1}{2c^2} (L_{11} I_1^2 + L_{12} I_1 I_2 + L_{21} I_2 I_1 + L_{22} I_2^2) = -\frac{1}{2c^2} \sum_{i,k} L_{ik} I_i I_k \tag{4.99}$$

The last expression can also be used for a system of an arbitrary number (for example n) of currents if only we extend summation over all the possible pairs of subscripts i and k ($i, k = 1, 2, \dots, n$).

5. The total potential function of a system of currents can be directly expressed through the density of the currents and the vector potential of the field of the currents. For this purpose, we shall first write Eqs. (4.85) and (4.86) in the symmetrical form: owing to the equality of U_{12} and U_{21} [see Eq. (4.87)], we have

$$U_{12} = \frac{1}{2} (U_{12} + U_{21}) = -\frac{1}{2c} \int_{V_2} \mathbf{A}_1 \mathbf{j}_2 dV - \frac{1}{2c} \int_V \mathbf{A}_2 \mathbf{j}_1 dV$$

Further, taking into account Eq. (4.89), we get

$$\begin{aligned} U &= U_{11} + U_{12} + U_{22} = \\ &= -\frac{1}{2c} \int_{V_1} \mathbf{A}_1 \mathbf{j}_1 dV - \frac{1}{2c} \int_{V_1} \mathbf{A}_2 \mathbf{j}_1 dV - \frac{1}{2c} \int_{V_2} \mathbf{A}_1 \mathbf{j}_2 dV - \\ &\quad - \frac{1}{2c} \int_{V_2} \mathbf{A}_2 \mathbf{j}_2 dV \end{aligned}$$

or, since $\mathbf{A}_1 + \mathbf{A}_2 = \mathbf{A}$, where \mathbf{A} is the vector potential of the resultant field of both currents,

$$\begin{aligned} U &= -\frac{1}{2c} \int_{V_1} \mathbf{A} \mathbf{j}_1 dV - \frac{1}{2c} \int_{V_2} \mathbf{A} \mathbf{j}_2 dV = \\ &= -\frac{1}{2c} \int \mathbf{A} \mathbf{j} dV \end{aligned} \quad (4.100)$$

The last integral should obviously be extended over the volume of both currents I_1 and I_2 . If there are no other currents in the field, then we can extend integration over the volume of the entire field because outside of the currents $j = 0$, and the corresponding terms of the integral become equal to zero.

As we have already indicated in connection with Eq. (4.72), the expression for the potential function U can if desired be interpreted in the sense that each element of volume of a current has the potential "energy" $-\frac{1}{c} \mathbf{A} \mathbf{j} dV$ in a magnetic field.

4.12 Magnetic Lines of Force

1. A description of the properties of a magnetic field, like those of an electric field, is often facilitated to a very great extent by introducing the concept of the so-called *lines of force* (or *field lines*) of this field. According to definition, magnetic lines of force are lines, the direction of tangents to which at each point of the field coincides with the direction of the field intensity \mathbf{H} at the same point*. The differential equation of these lines will obviously have the form [cf. Eq. (1.60)]

$$\frac{dx}{H_x} = \frac{dy}{H_y} = \frac{dz}{H_z} \quad (4.101)$$

* Since a magnetic field has no potential, magnetic lines of force unlike the lines of an electrostatic field cannot be defined as orthogonal trajectories of equipotential surfaces.

Magnetic lines of force, like their electric counterparts, are customarily drawn so that at any portion of a field the number of lines intersecting a unit area of surface perpendicular to them will be proportional to the numerical value of the field intensity on this area. As we shall see on a later page, however, this requirement cannot always be observed.

2. On the basis of Eq. (1.11),

$$\oint_S E_n dS = 4\pi \sum_i q_i$$

we arrived at the following conclusion in Sec. 1.10: electric lines of force can begin or terminate only at the points of a field where there are electric charges*. Applying Gauss's theorem (A.17) to the magnetic flux vector, on the basis of Eq. (4.36) we get

$$\oint_S H_n dS = \int_V \operatorname{div} \mathbf{H} dV = 0 \quad (4.102)$$

Thus, unlike the flux of the electric vector \mathbf{E} , the flux of the magnetic vector \mathbf{H} through an *arbitrary* closed surface always equals zero. This is a mathematical expression of the fact that there are *no magnetic charges* similar to electric charges: *a magnetic field is induced not by magnetic charges, but by the motion of electric charges* (i.e. currents). On the basis of this circumstance and of a comparison of Eq. (4.102) with Eq. (1.11), it is not difficult to see from the reasoning in Sec. 1.10 that *magnetic lines of force cannot begin or terminate* at any points of a field.

3. The conclusion is usually arrived at from this circumstance that magnetic lines of force unlike electric ones must be closed lines or extend from infinity to infinity.

Indeed, both these cases are possible. According to the results obtained in solving Problem 25 in Sec. 4.1, the lines of force in the field of an infinite straight current are circles perpendicular to the current with their centre on the current axis. On the other hand (see Problem 26), the direction of the magnetic vector \mathbf{H} in the field of a ring current at all points on the axis of the current coincides with the direction of this axis. Thus, the axis of a ring current coincides with the line of force passing from infinity to infinity. The drawing in Fig. 53 is a cross section of a ring current by a meridian plane (i.e. a plane perpendicular to the plane of the current and passing through its centre). The lines of force of this current are shown by dotted lines in the drawing.

A third case is also possible, however, which attention is not always given to, namely, a line of force may have neither a beginning

* See, however, the reservation on points of indeterminacy in the footnote on p. 63.

nor an end and at the same time not be closed and not pass from infinity to infinity. This case occurs if the line of force fills a certain

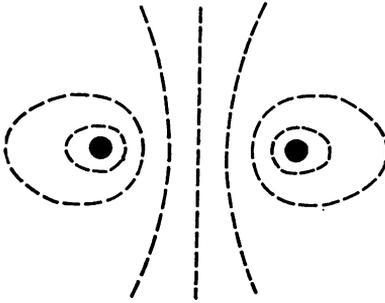


Fig. 53

surface and, using a mathematical term, tightly fills it everywhere. It will be the simplest to explain this on a concrete example.

4. Let us consider the field of two currents—a flat ring current I_1 and an infinite straight current I_2 flowing along the axis of the current I_1 (Fig. 54). If only the current I_1 existed, then the lines of force of the field \mathbf{H}_1 of this current would be in meridian planes and would have the form shown in Fig. 53. Let us consider one of these lines, shown in Fig. 54 by a dotted circle. The totality of all such lines that can be obtained by rotating a meridian plane about the axis I_2 forms the surface S of a ring or torus (Fig. 55). The force lines of the field \mathbf{H}_2 of the straight current I_2 are concentric circles. Hence, at each point of the surface S both \mathbf{H}_1 and \mathbf{H}_2 are tangent to this surface. Consequently, the vector of the intensity of the resultant field $\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2$ is also tangent to it. This means that each line of force

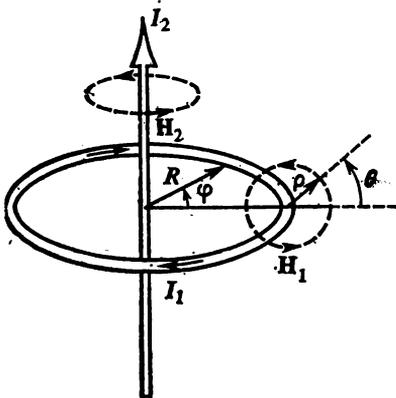


Fig. 54

of the field \mathbf{H} passing through any point of the surface S must lie on this surface with *all* its points. This line will obviously be a *helix* on the surface of the torus S (Fig. 55). The pitch of this helix will depend on the ratio of the currents I_1 and I_2 and on the position and

shape of the surface S . It is quite evident that only with a certain definite selection of these conditions will this helix be closed. In general, however, upon continuation of the helix, its new turns will

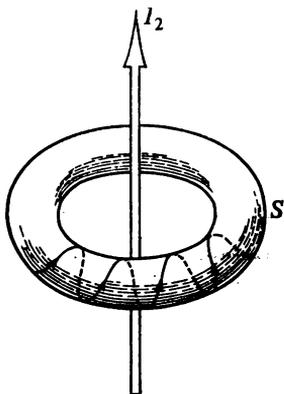


Fig. 55

be *between* the previous ones. Upon unlimited continuation of the helix, it will approach as close as is desired to any point through which it has already passed, but will never return to it a second time. This means that while remaining unclosed, the helix will tightly fill the surface of the torus S everywhere.

5. To strictly prove the possibility of existence of unclosed lines of force, let us introduce the orthogonal curvilinear coordinates φ (the azimuth of a meridian plane) and θ (the polar angle in the meridian plane with the apex on the intersection of this plane with the axis of the ring—Fig. 54) on the surface of the torus S .

The intensity of the fields H_1 and H_2 on the surface of the torus is a function of only the angle θ , the vector H_1 extending in the direction of growth or diminishing of this angle, and the vector H_2 in the direction of growth (or diminishing) of the angle φ . Assume that $\rho(\varphi)$ is the distance from the given point on the surface S to the centre line of the torus, and $R(\theta)$ is the distance from it to the vertical axis of the current I_2 . It is not difficult to see that an element of length of the line on S is expressed by the equation

$$ds^2 = r^2(\theta) d\theta^2 + R^2(\theta) d\varphi^2$$

where

$$r^2(\theta) = \rho^2(\theta) + \left(\frac{d\rho}{d\theta}\right)^2$$

Accordingly, the differential equation of the lines of forces [cf. Eq. (4.101)] on the surface S becomes

$$\frac{r(\theta) d\theta}{H_1(\theta)} = \frac{R(\theta) d\varphi}{H_2(\theta)}$$

Taking into account that H_1 and H_2 are proportional to the currents I_1 and I_2 and integrating, we get

$$\varphi - \varphi_0 = \int_{\theta_0}^{\theta} \frac{H_2(\theta) r(\theta)}{H_1(\theta) R(\theta)} d\theta = \frac{I_2}{I_1} \{F(\theta) - F(\theta_0)\}$$

where $F(\theta)$ is a certain function of the angle θ not depending on I_1 and I_2 .

For a helical line to be closed, i.e. for it to return to its initial point, it is essential that a *whole* number m of its revolutions about the vertical axis correspond to a *whole* number n of revolutions of the helix about the torus. In other words, it is essential that we be able to find two such integers n and m that a growth of the angle φ by $2\pi m$ will correspond to a growth of the angle θ by $2\pi n$:

$$\varphi - \varphi_0 = 2\pi m = \frac{I_2}{I_1} \{F(2\pi n + \theta_0) - F(\theta_0)\} \quad (4.103)$$

Let us now take into account that $F(\theta)$ is the integral of a periodic function of the angle θ with the period 2π . The integral of a periodic function in the general case is known to be the sum of a periodic function and a linear one*. Hence,

$$F(\theta) = C\theta + \Phi(\theta) \quad (4.104)$$

where C is a constant, and $\Phi(\theta)$ is a function with the period 2π . Therefore,

$$F(2\pi n + \theta) = 2\pi nC + C\theta + \Phi(\theta) \quad (4.105)$$

Introducing Eqs. (4.105) and (4.104) into Eq. (4.103), we get the condition for the lines of force to be closed on the surface of a torus S :

$$m = \frac{I_2}{I_1} Cn \quad (4.106)$$

Here the coefficient C does not depend on I_1 and I_2 . It is obvious that two *integers* n and m satisfying this condition can be found only if the quantity $(I_2/I_1) C$ is a *rational* number (an integer or fraction). This will occur only with a definite ratio between the currents I_1 and I_2 . Generally speaking, $(I_2/I_1) C$ will be an irrational quantity and, consequently, the lines of force on the surface of the torus S being considered will be unclosed. In this case too, however, we can

* Because a periodic function may include a constant term whose integral is a linear function of an independent variable.

always find an integer n so that $(I_2/I_1) Cn$ will differ as little as desired from an integer m . This means that an unclosed line of force after a sufficient number of revolutions will arbitrarily approach any point of the field that it has already passed. We can show in a similar way that this line after a sufficient number of revolutions will approach any preset point of the surface S as close as desired, and this means, by definition, that it will tightly fill this surface everywhere.

6. The existence of unclosed magnetic lines of force tightly filling a certain surface S everywhere obviously makes it impossible to give an exact graphical picture of a field with the aid of these lines. Particularly, it is far from always possible to comply with the requirement that the number of lines intersecting a unit area perpendicular to them be proportional to the intensity of the field on this area. For example, in the case just considered, the same unclosed line will pass through any finite area intersecting the surface of the ring S an infinite number of times*.

Taking the proper care, however, the use of the concept of lines of force, although being approximate, is nevertheless a convenient and illustrative way of describing a magnetic field.

7. According to Eq. (4.40), the circulation of the magnetic field intensity vector along a curve that does not envelop the currents equals zero, while circulation along a curve that does envelop the currents equals the sum of the forces of the enveloped currents (taken with the appropriate signs) multiplied by $4\pi/c$. The circulation of the vector \mathbf{H} along a line of force cannot equal zero (since an element of length of the line of force ds and the vector \mathbf{H} are parallel, the quantity $H_s ds$ is essentially positive). Consequently, each *closed magnetic line of force should envelop* at least one of the conductors carrying a *current*. Moreover, the unclosed lines of force tightly filling a surface S (if only they do not extend from infinity to infinity) should also *wind around the currents*. Indeed, the integral of the vector \mathbf{H} over an almost closed turn of such a line is essentially positive. Hence, the circulation of \mathbf{H} along the closed contour obtained from this turn by adding an arbitrarily small segment to close it differs from zero. Therefore, this contour should be pierced by the current.

* Hence there also follows the conditional nature of the concept of *tubes of force* (field tubes), i.e. filaments of a small but nevertheless *finite* cross section whose surface is formed by lines of force. If these lines are not closed, a field tube when extended over a sufficient length must *intersect itself* an infinite number of times (because its cross section is finite while the lines of force forming the surface of the tube when extended over a sufficient length will approach any points that have already been passed as many times as desired and as close as desired).

4.13 Topology * of a Vortex (Magnetic) Field. Conditional Barriers

1. To gain an understanding of the geometrical or, more correctly, the topological properties of a magnetic field, let us first consider the case when a field contains only one closed conductor through which the current I flows. The vortex of the magnetic field $\text{curl } \mathbf{H}$ differs from zero only inside the conductor carrying the current. Therefore the space occupied by the current can be called a vortex space or, owing to the closed nature of the current, a *vortex ring*.

From the purely topological viewpoint, all closed lines or contours (both coinciding with magnetic lines of force and not coinciding with them), if only they are *outside* of a vortex space, fall into two classes depending on whether they do not envelop or envelop a vortex ring. If we mentally *separate a vortex space from a field* or agree to consider it impermeable, then contours of the different classes cannot be made to coincide with one another by continuous deformation without violating their intactness. Any two contours belonging to the same class, however, can always be made to coincide by *continuous* deformation.

Further, contours of the first class by continuous deformation can be contracted to a point (i.e. to an infinitely small length). Contours of the second class, however, cannot be contracted to a point without intersecting the vortex space.

A space in which there are closed lines or contours that cannot be contracted to a point is called a *multiple-connected* space. The number of essentially different classes of contours determines the number of connections (the connection order) of the space. Thus, the space remaining after a vortex ring has been separated from it is a double-

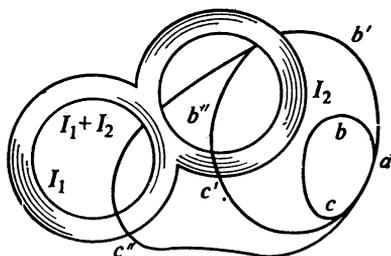


Fig. 56

connected space (two essentially different classes of contours); if two rings (two current loops) which, particularly, may be in contact with each other are separated from a space (Fig. 56), then the

* Topology (*analysis situs*) is a branch of mathematics studying the properties of figures and geometrical images remaining unchanged upon the continuous deformation of these figures.

remaining space will be a triple-connected one because it contains three classes of contours that cannot be made to coincide with one another (abc , $ab'c'$, and $ab'b''c''$ in Fig. 56; a contour such as $b''c''c'$ can be divided into two contours such as $ab'c'$ and $ab'b''c''$)*.

These topological properties of multiple-connected spaces are closely related to the physical properties of a magnetic field of steady currents because from the physical point of view the contours in the field of currents also fall into classes depending on the value of the circulation of the vector \mathbf{H} along these contours. For example, for one loop current, the circulation of \mathbf{H} along contours of the first class that can be contracted to a point equals zero, whereas the circulation along lines of the second class enveloping the current equals $\pm 4\pi I/c$ (the sign depends on how the direction of circumventing the contour is chosen). There are no intermediate values of the circulation. In exactly the same way, it is simple to see for two or more currents that the division of the lines into classes according to the physical feature (the magnitude of the circulation) coincides with their division according to the topological feature. Particularly, all the lines of force of a magnetic field (both closed and open) must envelop a current, i.e. a vortex space.

2. The relationship between the topological and physical properties of a field, however, is not limited to what has been said above and can be followed much further.

It is known that any multiple-connected space can be transformed into a simply connected one by introducing the appropriate partitions or barriers into it. For example, if we cover an opening of a vortex ring with an impermeable barrier, then the drawing of a second-class contour enveloping this ring will become impossible, and the space

* If a contour is intricate, for instance loop-shaped (Fig. 57), then it must be divided first into two or more simple contours, each of which by continuous



Fig. 57

deformation can be given the shape of a circle. In exactly the same way, if a contour of the second class envelops a vortex space not once, but several times, it can be divided into a number of contours each of which envelops this space once.

will become simply connected. In exactly the same way, a triple-connected space (Fig. 56) will become a simply connected one if we cover openings of each of the rings in it with impermeable barriers.

It must be noted that the shape and position of these barriers remain arbitrary to a considerable extent; it is only important that the contour of each barrier should rest on the surface of the corresponding vortex ring.

Let us now consider the physical properties of a magnetic field that has become simply connected owing to the mental introduction into it of such conditional barriers. The curl of the vector \mathbf{H} at all points of this field equals zero (we consider as previously that the vortex space has been separated from the field). The circulation of the vector \mathbf{H} along any contour possible in it (i.e. not intersecting a conditional barrier) equals zero. Hence (see Sec. 4.7), in this simply connected field, we can *unambiguously* determine the *scalar* potential ψ of the magnetic field, assuming by analogy with the potential φ of an electric field that

$$\psi_1 - \psi_2 = \int_1^2 H_s ds \quad [\text{cf. Eq. (1.40)}] \quad (4.107)$$

$$\mathbf{H} = -\text{grad } \psi \quad [\text{cf. Eq. (1.59)}] \quad (4.108)$$

According to Eq. (4.36), for all the points of the space being considered, we have

$$\text{div } \mathbf{H} = \nabla^2 \psi = 0 \quad (4.109)$$

so that the magnetic field which has become a potential one owing to the separation of the vortex space and the introduction of conditional barriers is found to be deprived of volume sources.

Owing to the continuity of the vector \mathbf{H} , the potential ψ and its spatial derivatives will also be continuous throughout the entire field except for points adjoining the conditional barriers. Indeed, let us consider two infinitely close points P and P' separated by a conditional barrier S (Fig. 58). The potential difference of these points, according to Eq. (4.107), will be

$$\psi - \psi' = \int_P^{P'} H_s ds$$

The path of integration L should not naturally intersect the barrier. Since the points P and P' are infinitely close to each other, this path L (i.e. PP') differs only to an infinitely small extent from the closed line $PP'P$ and, consequently, according to Eq. (4.40)

$$\psi - \psi' = \oint_{PP'P} H_s ds = \frac{4\pi I}{c} \quad (4.110)$$

Thus, *conditional barriers* are surfaces of discontinuity of the magnetic potential, which undergoes on them a jump of $4\pi I/c$. This jump will be positive ($\psi > \psi'$) if the circulation along the contour

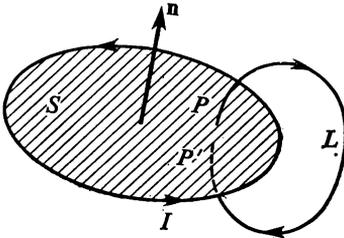


Fig. 58

$PP'P$ is positive, i.e. as can be seen from the figure, if the infinitely small vector $P'P$ forms a *right-handed* system with the direction of the current I .

3. When studying the potential electrostatic field, we acquainted ourselves with surfaces of discontinuity of the potential in a Coulomb field of electric charges and saw that physically these discontinuity surfaces correspond to double layers of charges. By analogy with an electrostatic field, we can also consider a magnetic field of currents that has become potential owing to the separation from it of a vortex space and the introduction of conditional barriers as a field of double layers of *fictitious magnetic charges* m interacting according to Coulomb's law:

$$F = C \frac{mm'}{R^2} \text{ [cf. Eq. (1.1)]} \tag{4.111}$$

Here F is the ponderomotive force of interaction of the magnetic charges m and m' at a distance of R from each other, charges of the same sign repulsing each other, and of opposite signs attracting each other, and C is a proportionality constant depending on what units are used. If we equate this constant to unity, assuming that

$$F = \frac{mm'}{R^2} \tag{4.112}$$

then we shall thus introduce the *absolute unit of the quantity of magnetism*: the unit of magnetism is such a quantity of magnetism that repulses an equal quantity of magnetism at a distance of 1 cm from it with a force of one dyne.

Further, we can introduce the intensity of the field of a fictitious magnetic charge,

$$H = \frac{m}{R^2} \tag{4.113}$$

as the force acting on a test positive magnetic charge equal to unity, and so on, in complete analogy with an electrostatic field.

4. An appreciable distinction of a field of fictitious magnetic charges from an electrostatic one consists, however, in that according to Eq. (4.109) this field is deprived of space (and also point) sources, and in that all the magnetic charges are in *double layers* on the surfaces of discontinuity of the magnetic potential ψ coinciding with the conditional partitions introduced above. Thus, magnetic charges of opposite signs are *inseparable from each other*, and we may consider that elementary magnetic charges are joined up in pairs into solid *magnetic dipoles* (elementary magnets).

4.14 Magnetic Sheets. Their Equivalence to Currents

1. Thus, the consideration of the magnetic field of currents can be replaced with the consideration of the equivalent field of fictitious magnetic dipoles forming magnetic double layers or, as they are customarily called, *magnetic shells* or *sheets*. The surface of these sheets must coincide with the conditional barriers which we introduced in Sec. 4.13. Consequently, the contours of the sheets must coincide with the contours of the currents. For the field of these sheets outside of the vortex space occupied by currents and outside of points on the sheet itself (where the field intensity of the sheet becomes infinite, cf. Sec. 1.14) to be identical with the field of currents, it is sufficient* that the jump in the magnetic potential on the surface of the sheet be equal to $4\pi I/c$ [Eq. (4.110)]. The potential jump $\psi - \psi'$ on the surface of a double layer (sheet), according to Eqs. (1.91) and (1.95), equals

$$\psi - \psi' = 4\pi\sigma l = 4\pi\tau$$

where τ = strength of the sheet (double layer)

l = its width

σ = surface density of the charge of each of its layers.

Consequently, the strength of a magnetic sheet equivalent to the current must be assumed equal to

$$\tau = \sigma_m l = \frac{I}{c} \quad (4.114)$$

where σ_m stands for the density of the fictitious magnetic charges on the surface of the sheet. Thus, a sheet equivalent to the current must be *homogeneous*, i.e. its strength τ should be constant along its entire length. It is easy, finally, to see that the magnetic dipoles forming the sheet should be arranged so that the positive direction

* Because the potential of the field of a double layer is unambiguously (up to an insignificant additive constant) determined by giving the position of the layer and the potential jump at each of its points [cf. Eqs. (1.94) and (1.95)].

of all these dipoles (from $-m$ to $+m$) will form a *right-handed* system with the direction of the current (Fig. 59).

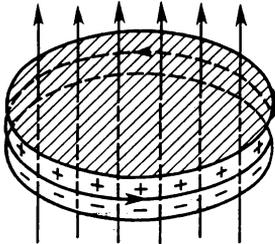


Fig. 59

2. The potential ψ of a homogeneous double layer, according to Eq. (1.94), may be written in the following form:

$$\psi = \tau\Omega \tag{4.115}$$

where Ω is the solid angle at which the contour of the double layer is seen from a point of observation having the potential ψ ; the sign of the angle Ω is considered to coincide with that of the charges of the side of the layer which is seen from the point of observation.

Thus, as when introducing the conditional barriers, we again see that only the position of the contour of a sheet is of significance; it must coincide with the contour of the current. In all other respects, the shape and the position of the sheet remain arbitrary. The conditional nature of the replacement of a current with an equivalent sheet manifests itself especially clearly in this arbitrariness of the shape of a magnetic sheet. By changing the shape of the sheet, we can make it pass through any point of space that we wish and thus “create” a jump in the magnetic potential at this point. This is naturally possible only because the very concept of the potential of a magnetic field does not in essence have a real physical meaning and can be unambiguously defined only *after* the introduction of conditional barriers (sheets) into the field of currents. If we would nevertheless want to determine the magnetic potential with the aid of Eq. (4.107) without introducing these barriers (sheets), then this potential will be found to be a *multiple-valued* function of a point.

Indeed, let us ascribe the potential ψ_0 to a point P . We shall then move away from P along a certain line L determining the potential of the points of this line with the aid of relationship (4.107). Let the line L be closed so that upon moving along it in the same direction all the time, we shall again return to our initial point P . The increment of the potential upon a complete circumvention of the contour L , according to Eq. (4.107), will equal the circulation of the vector \mathbf{H} along L , i.e. according to Eq. (4.40) will equal $\pm 4\pi I/c$, if the contour L envelops the current I once, and $\pm 4\pi In/c$ if it envelops the current

n times. Thus, returning to the initial point P , we shall “find” at it (i.e. shall have to ascribe to it) the potential

$$\psi = \psi_0 \pm \frac{4\pi I}{c} n \quad (4.116)$$

which, generally speaking (when $n \neq 0$) differs from the previous potential ψ_0 and depends on the value of n , i.e. on the position and shape of the contour L . And this means that the magnetic potential is a *multiple-valued* position function and that this function can be made a single-valued one only artificially by introducing conditional barriers (magnetic sheets) into a multiple-connected field of currents.

3. The equivalence of the magnetic field of sheets and line currents, unlike the way of proving this we have chosen, can also be established by direct calculation.

Let us compare a current loop I having the contour L with the field of a magnetic sheet of the strength $\tau = I/c$ having the surface S resting on the contour L . The component of the field intensity of this sheet in any direction \mathbf{m} , according to Eqs. (4.108) and (4.115), will be

$$H_m = -\text{grad}_m \psi = -\tau \frac{d\Omega}{dm} = -\frac{I}{c} \frac{d\Omega}{dm}$$

Here $d\Omega/dm$ is the ratio of the change $d\Omega$ in the solid angle Ω when the point of observation P moves over the segment dm to its length dm . This change $d\Omega$ is obviously equal to the change which the angle Ω would have if P remains fixed and the contour L moves in the opposite direction over the segment $-dm$. Upon this displacement, each element of length ds of the contour L covers the area $dS = [-dm ds]$, which will be seen from the point P at the angle [cf. Eq. (1.7)]

$$d\omega = \frac{\mathbf{R} dS}{R^3} = \frac{[-dm ds] \mathbf{R}}{R^3}$$

A glance at Fig. 60 will show that the sequence of the multipliers dm and ds in the expression for dS has been chosen correctly, i.e.

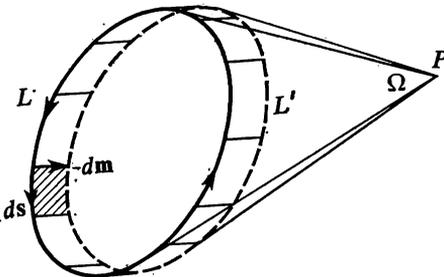


Fig. 60

that the angle $d\omega$ will be positive if from the point P we see the positive side of the element dS of the magnetic sheet confined between the initial and the displaced contours L and L' , and negative in the opposite case (we remind our reader that \mathbf{R} coincides with the direction from dS to P).

Using the known rules of transformation from vector algebra, we get

$$d\omega = \frac{-dm [ds \mathbf{R}]}{R^3} = \frac{-dm [ds \mathbf{R}]_m}{R^3}$$

The total change in the solid angle at which the entire contour L is seen equals

$$d\Omega = \int d\omega = -dm \oint_L \frac{[ds \mathbf{R}]_m}{R^3}$$

hence,

$$H_m = -\frac{I}{c} \frac{d\Omega}{dm} = \frac{I}{c} \oint \frac{[ds \mathbf{R}]_m}{R^3}$$

Since this equation holds for any choice of the direction of \mathbf{m} , then the vector of the field intensity of the magnetic sheet must equal

$$\mathbf{H} = \frac{I}{c} \oint \frac{[ds \mathbf{R}]}{R^3}$$

i.e. must coincide with the field intensity of a line current I [see Eq. (4.4)], Q.E.D.

4. We shall note once more in conclusion that everything said above relates only to the field of a vortex space (i.e. outside of currents), while inside of this space the concept of magnetic potential loses all meaning. Further, the equivalence of currents and magnetic sheets occurs, in essence, only for line currents, i.e. currents whose distance from the points of the field being considered is great in comparison with the cross section of the current. Otherwise, the field of the current depends on the current distribution over the cross section of the vortex ring (i.e. the conductor carrying the current), whereas the field of a sheet depends only on the position of its line contour. True, a current having a finite cross section can always be resolved into a combination of infinitely thin current filaments each of which can be replaced with an equivalent sheet having an infinitely small width, but it is not expedient to resort to such intricate constructions.

5. It remains for us to show that the replacement of line currents with equivalent magnetic sheets is permissible not only when determining the field of currents, but also when determining the *pondero-*

motive forces acting on currents. Naturally, a current may be replaced with an equivalent sheet only in determining the forces acting on currents in an external magnetic field, but not when determining the interaction of elements of the same current.

Let us consider the line current I of the contour L in an external magnetic field having the intensity \mathbf{H} . According to Eqs. (4.65) and (4.68), the potential function U of the current I in this field will equal

$$U = -\frac{I}{c} \Phi = -\frac{I}{c} \int H_n dS$$

where S is a surface resting on the contour L , but remaining arbitrary in the remaining respects. Let us assume that the field \mathbf{H} is induced by currents that do not intersect this surface S . In this case, according to the results of the preceding section, the field \mathbf{H} over the entire length of the surface S may be considered as a potential field having a certain magnetic potential ψ :

$$\mathbf{H} = -\text{grad } \psi$$

Let us now replace the current I with an equivalent magnetic sheet coinciding with the surface S . The potential energy of the magnetic charge dm of an element of the surface of the sheet dS (here $dm = \pm \sigma_m dS$) in the external field \mathbf{H} , by analogy with an electrostatic field, is determined by the expression

$$\psi dm = \pm \psi \sigma_m dS$$

while the total potential energy of all the charges of the sheet will be

$$U_{\text{sh}} = \int (\sigma_m \psi_+ - \sigma_m \psi_-) dS$$

where ψ_- and ψ_+ are the values of the potential of the external field on the negative and positive surfaces of the magnetic layer (magnetic sheet), respectively, i.e. the values of ψ at points at a distance from each other equal to the width of the sheet l . Hence,

$$\psi_+ - \psi_- = l \text{ grad } \psi = -H_n l$$

where \mathbf{n} is the direction of a positive normal to the sheet directed from its negative side to its positive one. Consequently,

$$U_{\text{sh}} = - \int H_n \sigma_m l dS = - \int H_n \tau dS \quad (4.117)$$

or in view of Eq. (4.114)

$$U_{\text{sh}} = -\frac{I}{c} \int H_n dS \quad (4.118)$$

Thus, the potential energy U_{sh} of a magnetic sheet in an external magnetic field \mathbf{H} does indeed equal the potential function U of the

current I equivalent to the sheet. Consequently, the ponderomotive forces acting in this field on a current equal the forces acting on a magnetic sheet equivalent to the current, Q.E.D.

Example. *Replacement of a solenoid with a magnet.* Assume that the current I circulates along a cylindrical solenoid (Sec. 4.8, p. 247) having n turns of the conductor per unit length. If the pitch of the helical line is small enough, then each turn of the solenoid may be approximately replaced with a *closed* ring-shaped current loop of the same intensity I fitted onto the cylinder. Let us replace these current loops with flat magnetic sheets having the strength

$$\tau = \sigma_m l = \frac{I}{c}$$

If the distance $1/n$ between adjacent turns of the solenoid is sufficiently small in comparison with the distance from the points of the field being considered to the solenoid, then we can approximately assume that the current is distributed over the surface of the cylinder uniformly and continuously. Accordingly, the "infinitely small" width of a magnetic sheet l can be assumed equal to $1/n$. In this case, adjacent sheets will be in contact with their oppositely charged surfaces. Therefore, their charges will neutralize one another except for the charges of the external surfaces of the two extreme sheets coinciding with the bases of the cylindrical solenoid. These bases will be uniformly covered with magnetic charges of opposite signs having the density

$$\sigma_m = \frac{\tau}{l} = n\tau = \frac{nI}{c} \quad (4.119)$$

Consequently, the field of a solenoid approximately coincides with that of a cylindrical magnet having the same length and the same cross section whose bases are uniformly covered with magnetic charges of the indicated density. It is easy to see that the direction of the moment of this magnet (from the negative or south pole to the positive or north one) should form a *right-handed* system with the direction of the current in the solenoid (Fig. 61). Only the *external* field of such a magnet, however, is equivalent to the field of a current; *inside* the magnet the field \mathbf{H} is directed *oppositely* to the field of the current (from N to S). (We shall also consider this question from a somewhat different viewpoint in Chap. 5.) This is what should be expected because we filled the entire inner space of the solenoid with magnetic sheets of a finite thickness. Only the external field of a sheet, however, is equivalent to the field of a current, while inside the sheet its field is directed oppositely to the field of the current which it is "equi-

valent" to. We did not have to give attention to this in our preceding treatment because we considered only infinitely thin sheets

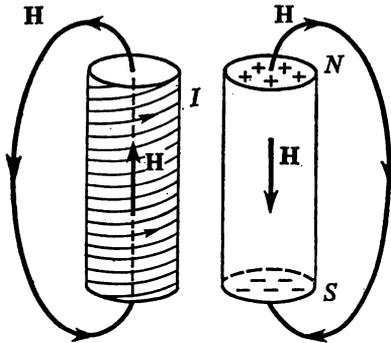


Fig. 61

inside of which the very concept of a field loses any meaning because its intensity H becomes equal to infinity (cf. Sec. 1.14).

4.15 Magnetic Moment of a Current. Elementary Currents and Magnetic Dipoles

1. We saw in Sec. 2.1 that the electric field of a neutral system of charges (i.e. a system of charges whose algebraic sum equals zero) at great distances from the system is determined (with an accuracy to the terms diminishing more rapidly with increasing distance) by a single parameter—the electric moment of the system \mathbf{p} —and that the forces acting on this system of charges in an external electric field are determined by the same parameter. Similarly, an arbitrary closed current in known conditions is also characterized by the single parameter $\boldsymbol{\mu}$ called the *magnetic moment of the current*.

We shall use the term *elementary current* to denote a current loop meeting the following requirements: (1) the dimensions of the current contour are vanishingly small in comparison with the points of the field in which we consider its field, and (2) over the entire length of a current loop the values of the quantities characterizing the external field (more exactly, the value of the intensity of this field \mathbf{H} and the value of the space derivatives of this intensity $\nabla\mathbf{H}$) may be considered constant. It is quite obvious that in definite conditions any current loop can be considered as elementary.

According to the results of the preceding section, any steady current is equivalent to a magnetic sheet both in the active (the field induced by it) and the passive (the forces acting on it) respects. As regards an *elementary current*, it is *equivalent* to the simplest magnetic sheet—a magnetic dipole.

Indeed, let us consider a magnetic sheet or a magnetic double layer having a strength of $\tau = I/c$ and equivalent to the given current I . Each pair of opposite elements dS of the double layer having the charge $dm = \pm \sigma_m dS$ is a magnetic dipole with the moment

$$\mathbf{l} dm = \mathbf{l} \sigma_m dS = \tau dS$$

(the vector dS is directed along the positive axis \mathbf{l} of this dipole). If the conditions are observed in which the current in question may be considered as elementary, then both when determining the field induced by the double layer equivalent to the current and when determining the forces acting on it we may disregard the difference in the position of separate elements of the double layer and replace the entire layer with a single magnetic dipole having the moment

$$\boldsymbol{\mu} = \int \tau dS = \frac{I}{c} \int dS \quad (4.120)$$

Let \mathbf{S} stand for the *vector value of the area* of a sheet, i.e. for the vector sum of the elements dS of this sheet:

$$\mathbf{S} = \int dS \quad (4.121)$$

Since a sheet equivalent to the current rests on the contour of this current, then \mathbf{S} is the vector value of the area enveloped by the current. On the basis of Eq. (4.121), Eq. (4.120) becomes

$$\boldsymbol{\mu} = \frac{I}{c} \mathbf{S} \quad (4.122)$$

Thus, an elementary current is equivalent to a magnetic dipole whose moment determined by Eq. (4.122) is also called the *magnetic moment of the current*.

2. It should be noted that the absolute value $|\mathbf{S}|$ of the vector quantity \mathbf{S} of the arbitrary surface S , generally speaking, is less than the area of this surface; only for a *flat* surface do these two numerical values equal each other. Particularly, the vector quantity of an arbitrary *closed* surface is identically equal to zero, which can easily be seen by considering the projection of a closed surface onto an arbitrary plane*. Hence it follows that the vector quantities \mathbf{S}_1 and \mathbf{S}_2 of two arbitrary surfaces resting on the same contour L can differ from each other only in their sign (which depends on the choice of the directions of the outward normals to them). Indeed, a combination of two such

* For example, for a spherical surface, the projections of the northern and southern hemispheres onto the plane of the equator numerically equal each other, but are opposite in sign.

surfaces (with the proper choice of the directions of the normals) forms a single closed surface so that

$$S_1 + S_2 = \oint dS = 0 \quad (4.123)$$

3. Thus, the value of the magnetic moment of a current [Eq. (4.122)] does not depend on our arbitrary choice of the shape of the surface S resting on the contour L of this current. The direction of a normal to this surface S and, consequently, the direction of the magnetic moment of the current $\boldsymbol{\mu}$ should form a right-handed system with the direction of the current in the contour L . Indeed, a normal to the double layer equivalent to the given current, on one hand, forms a right-handed system with the direction of the current (see Sec. 4.14, in particular Fig. 59), and on the other, according to Eq. (4.122), determines the direction of the vector $\boldsymbol{\mu}$.

4. By analogy with Eq. (1.49), the scalar potential of the field of a magnetic dipole obviously equals

$$\psi = \frac{\boldsymbol{\mu}\mathbf{R}}{R^3}$$

and the intensity of this field is

$$\mathbf{H} = -\text{grad} \left(\frac{\boldsymbol{\mu}\mathbf{R}}{R^3} \right) = \frac{3(\boldsymbol{\mu}\mathbf{R})\mathbf{R}}{R^5} - \frac{\boldsymbol{\mu}}{R^3} \quad (4.124)$$

[cf. Eq. (1.61)].

As has been proved, the field intensity of a current having the moment $\boldsymbol{\mu}$ at distances from the current considerably exceeding its dimensions will have the same value.

Further, the potential energy of a magnetic dipole in an external field \mathbf{H} by analogy with Eq. (1.103) is

$$U = -\boldsymbol{\mu}\mathbf{H} \quad (4.125)$$

while the resultant force \mathbf{F} and the resultant moment \mathbf{N} of the forces applied to it by analogy with Eqs. (1.127) and (1.130) equal

$$\mathbf{F} = -\nabla U = \nabla(\boldsymbol{\mu}\mathbf{H}) \quad (4.126)$$

$$\mathbf{N} = [\boldsymbol{\mu}\mathbf{H}] \quad (4.127)$$

Owing to the equality of the forces acting on an elementary current and on a magnetic dipole having the same moment $\boldsymbol{\mu}^*$, these formulas can also be applied to an elementary current.

* This naturally relates not to the forces acting on separate current elements, but to the resultant force and the resultant moment of all these forces. In this sense, we can say that a magnetic dipole is equivalent to a solid elementary current, i.e. to a current flowing along a solid undeforming conductor because a system of forces applied to a solid can be replaced with their resultant and a resultant couple.

5. The last statement, however, requires certain reservations.

When proving in Sec. 4.14 that the potential energy U_{sh} of a magnetic sheet in the external field \mathbf{H} equals the potential function of the current equivalent to the sheet, we followed the assumption that the external field \mathbf{H} is induced by currents that do not intersect the surface of the sheet (i.e. that do not intersect the contour of the current equivalent to the sheet). Only when this condition is observed is it possible to consider the external field on the surface of the sheet as a potential field having the magnetic potential ψ , and only when this condition is observed is it possible in general to speak about the potential energy of a magnetic sheet in an external field. To eliminate this restriction on sources of an external field, we must calculate the forces acting on a magnetic sheet and on a current independently. We shall begin with the latter.

The applicability of Eqs. (4.65) and (4.68)

$$U = -\frac{1}{c} I\Phi = -\frac{1}{c} I \int H_n dS = -\frac{1}{c} I \int \mathbf{H} dS$$

determining the potential function of a current is not bound by the above restriction. For an elementary current, the intensity of the external field, by definition, may be considered as constant over the entire current and be put outside the integral. On the basis of Eqs. (4.121) and (4.122), we get

$$U = -\frac{1}{c} IH \int dS = -\frac{1}{c} IHS = -\mu H$$

Thus, Eq. (4.125) and, therefore, Eqs. (4.126) and (4.127) can be applied to an elementary current without any additional restrictions.

Matters are different with a magnetic dipole. In Sec. 1.17 by the direct calculation of the forces acting on the charges of an electric dipole in an arbitrary electric field \mathbf{E} , we found the following expression for the resultant of these forces [Eq. (1.118)]:

$$\mathbf{F} = \mathbf{p} \nabla \cdot \mathbf{E}$$

Then in Sec. 1.18 we showed that

$$\mathbf{F} = \mathbf{p} \nabla \cdot \mathbf{E} = \nabla (\mathbf{pE}) - [\mathbf{p} \text{curl } \mathbf{E}]$$

Thus, only provided that $\text{curl } \mathbf{E} = 0$, i.e. only in a potential electric field is the force acting on a dipole expressed by the gradient of its potential energy taken with the opposite sign, i.e. $U = -\mathbf{pE}$. Accordingly, the force acting on a magnetic dipole in an arbitrary external field \mathbf{H} equals

$$\mathbf{F} = \mu \nabla \cdot \mathbf{H} = \nabla (\mu \mathbf{H}) - [\mu \text{curl } \mathbf{H}] \tag{4.128}$$

This expression coincides with Eq. (4.126) only when $\text{curl } \mathbf{H} = 0$, i.e. only on condition that at the places where the dipole is, the density of the currents inducing the external field equals zero [cf. Eq. (4.38)].

Thus, the last condition is also the condition for the identity of the forces acting on an elementary current and on a magnetic dipole equivalent to it.

This condition, however, is superfluous for the identity of the moment of the forces \mathbf{N} acting on the current and on the dipole; this identity is always ensured, which follows from a comparison of Eq. (4.127) and Eq. (1.119) applicable to a dipole in an arbitrary external field.

4.16 Direct Determination of the Field of Elementary Currents and the Forces Acting on Them

1. In the preceding section, we determined the field of elementary currents and the forces acting on them in an external field on the basis of the equivalence of current loops and magnetic sheets proved in Sec. 4.14.

Now we shall give the direct derivation of these expressions without resorting to the notion of fictitious magnetic charges, and characterizing a magnetic field not by the scalar potential ψ , but by the vector potential \mathbf{A} . Such a method of derivation, first, is more consistent, and, second, it allows us to consider not line currents, as in Sec. 4.15, but space ones whose replacement with magnetic sheets, if not being impossible, would nevertheless be exceedingly cumbersome.

We shall need the results of this section only in Secs. 5.2, 5.7, and 7.8.

2. Let us consider the vector potential \mathbf{A} of an arbitrary system of currents circulating in the arbitrary volume V :

$$\mathbf{A} = \frac{1}{c} \int_V \frac{\mathbf{j} dV}{R'}$$

where R' stands for the distance from the current element $\mathbf{j} dV$ to the point of observation P at which the value of \mathbf{A} is determined.

Let us select inside the system of currents being considered an arbitrary point O which we shall conditionally call the centre of the currents. Let \mathbf{R}_0 and \mathbf{R} be respectively the distances from the point of observation P and the current element $\mathbf{j} dV$ to the centre O so that

$$\mathbf{R}' = \mathbf{R}_0 - \mathbf{R} \text{ and } R'^2 = R_0^2 - 2\mathbf{R}_0\mathbf{R} + R^2$$

(see Fig. 27, where \mathbf{R}'_i and \mathbf{R}_i correspond to our present values \mathbf{R}' and \mathbf{R}).

If $R_0 \gg R$, i.e. if the distance from the point of observation P to the system of currents considerably exceeds the dimensions of this system, then in the expansion

$$\frac{1}{R'} = \frac{1}{R_0} \left(1 - \frac{2\mathbf{R}_0\mathbf{R}}{R_0^2} + \frac{R^2}{R_0^2} \right)^{-\frac{1}{2}} = \frac{1}{R_0} + \frac{\mathbf{R}_0\mathbf{R}}{R_0^3} + \dots \quad (4.129)$$

the first two terms written out will be quite sufficient. Using this expansion in the expression for \mathbf{A} , we get

$$\mathbf{A} = \frac{1}{cR_0} \int \mathbf{j} dV + \frac{1}{cR_0^3} \int \mathbf{j} (\mathbf{R}_0\mathbf{R}) dV \quad (4.130)$$

It is easy to see that*

$$\mathbf{j} (\mathbf{R}_0\mathbf{R}) = \frac{1}{2} [[\mathbf{R}\mathbf{j}] \mathbf{R}_0] + \frac{1}{2} \{ \mathbf{j} (\mathbf{R}\mathbf{R}_0) + \mathbf{R} (\mathbf{j}\mathbf{R}_0) \}$$

therefore,

$$\mathbf{A} = \frac{1}{cR_0} \int \mathbf{j} dV + \frac{1}{2cR_0^3} \left[\int [\mathbf{R}\mathbf{j}] dV \cdot \mathbf{R}_0 \right] + \frac{1}{2cR_0^3} \mathbf{K} \quad (4.131)$$

where we have put the vector \mathbf{R}_0 not depending on the position of the element of integration dV outside the integral, and where we have denoted by \mathbf{K} the integral.

$$\mathbf{K} = \int \{ \mathbf{j} (\mathbf{R}\mathbf{R}_0) + \mathbf{R} (\mathbf{j}\mathbf{R}_0) \} dV$$

To transform this integral, let us multiply it by an arbitrary but constant vector \mathbf{a} and take advantage of the fact that according to the formulas of vector analysis (A.11) and (A.43₂)

$$\begin{aligned} (\mathbf{a}\mathbf{j}) (\mathbf{R}\mathbf{R}_0) + (\mathbf{a}\mathbf{R}) (\mathbf{j}\mathbf{R}_0) &= \mathbf{j} \text{grad} \{ (\mathbf{a}\mathbf{R}) (\mathbf{R}\mathbf{R}_0) \} = \\ &= \text{div} \{ \mathbf{j} (\mathbf{a}\mathbf{R}) (\mathbf{R}\mathbf{R}_0) \} - (\mathbf{a}\mathbf{R}) (\mathbf{R}\mathbf{R}_0) \text{div} \mathbf{j} \end{aligned}$$

where the vector \mathbf{R}_0 is considered to be constant upon differentiation.

Since we are treating steady currents, then according to Eq. (3.19) we have $\text{div} \mathbf{j} = 0$ and, therefore, on the basis of Gauss's theorem (A.17), we get

$$\mathbf{a}\mathbf{K} = \int_V \text{div} \{ \mathbf{j} (\mathbf{a}\mathbf{R}) (\mathbf{R}\mathbf{R}_0) \} dV = \oint_S j_n (\mathbf{a}\mathbf{R}) (\mathbf{R}\mathbf{R}_0) dS$$

* This expansion corresponds to the expansion of the tensor $U_{ik} = j_i R_k$ into the sum of the symmetrical and antisymmetrical tensors:

$$U_{ik} = \frac{1}{2} (j_i R_k - j_k R_i) + \frac{1}{2} (j_i R_k + j_k R_i)$$

where the surface integral is taken over the surface S of the volume of integration V . Since this equality is observed upon any selection of the constant vector \mathbf{a} , it follows from it that

$$\mathbf{K} = \oint j_n \mathbf{R} (\mathbf{R} \mathbf{R}_0) dS \quad (4.132)$$

Let us, finally, introduce the symbol

$$\boldsymbol{\mu} = \frac{1}{2c} \int_V [\mathbf{R} \mathbf{j}] dV \quad (4.133)$$

The vector $\boldsymbol{\mu}$ determined by this equation is called the magnetic moment of the currents circulating in the volume V , or simply the magnetic moment of the volume V^* . Introducing the expressions for \mathbf{K} and $\boldsymbol{\mu}$ into Eq. (4.131), we finally get

$$\mathbf{A} = \frac{1}{cR_0} \int_V \mathbf{j} dV + \frac{[\boldsymbol{\mu} \mathbf{R}_0]}{R_0^3} + \frac{1}{2cR_0^3} \oint_S j_n \mathbf{R} (\mathbf{R} \mathbf{R}_0) dS \quad (4.134)$$

3. The ratio of the last two terms to the first term in the right-hand side of Eq. (4.134) in its order of magnitude, generally speaking, equals l/R_0 , where l stands for the lateral dimensions of the volume V . Therefore for distances R_0 that are great in comparison with l , these being the only ones for which Eq. (4.134) holds, the last two terms in this equation, in general, are small in comparison with the first one, so that we can approximately limit ourselves to the first term

$$\mathbf{A} = \frac{1}{cR_0} \int_V \mathbf{j} dV$$

whose meaning is obvious.

If, however

$$\int_V \mathbf{j} dV = 0 \quad (4.135)$$

then the value of \mathbf{A} is determined by the last two terms of Eq. (4.134). This is how matters are, particularly, for an arbitrary *system of current loops* confined within the volume V , i.e. if no currents flow through the surface S confining the volume V :

$$j_n = 0 \text{ on the surface } S \quad (4.136)$$

We shall prove that for steady currents Eq. (4.135) does indeed follow from condition (4.136). In the latter condition, the system of

* We shall show at the end of this section that for line currents this definition of the magnetic moment coincides with the one given in Sec. 4.15.

steady currents being considered can be resolved into a combination of closed current filaments completely inside the volume V . Integration over the volume of each filament, according to Eq. (4.16), may be replaced with integration over its length:

$$\int \mathbf{j} dV \rightarrow I \oint ds$$

The last integral over the closed contour of a filament obviously equals zero, whence the correctness of Eq. (4.135) follows.

When condition (4.136) is observed, not only the first term of Eq. (4.134) vanishes, but also its last term. Therefore, the vector potential of a closed system of currents at great distances from it equals

$$\mathbf{A} = \frac{[\boldsymbol{\mu}\mathbf{R}]}{R^3} \tag{4.137}$$

Here we have discarded the subscript 0 in \mathbf{R}_0 so that unlike our previous treatment \mathbf{R} stands for the distance from the point of observation P to the system of currents characterized by its moment $\boldsymbol{\mu}$. The point of this system of currents from which the distance \mathbf{R} is measured is of no significance because Eq. (4.137) itself has been obtained on the assumption that the distance from the point of observation to the system of currents is great in comparison with the dimensions of this system.

Thus, at these distances, the magnetic field of a closed system of currents is determined by its magnetic moment $\boldsymbol{\mu}$, just like the electric field of a neutral system of charges is determined by its electric moment p .

It is important that the value of the magnetic moment $\boldsymbol{\mu}$ of a system of currents complying with condition (4.135) does not depend on the choice of the conditional centre of the system O from which the distances \mathbf{R} are measured in Eq. (4.133). Indeed, if we displace the centre of the system from the point O over an arbitrary length \mathbf{a} to the point O' , then all the distances \mathbf{R} from the centre will change by $-\mathbf{a}$:

$$\mathbf{R} \rightarrow \mathbf{R} - \mathbf{a}$$

Therefore, the vector $\boldsymbol{\mu}$ determined by Eq. (4.133) will change by the value

$$\Delta\boldsymbol{\mu} = -\frac{1}{2c} \int [\mathbf{a}\mathbf{j}] dV = -\frac{1}{2c} \left[\mathbf{a} \int \mathbf{j} dV \right]$$

(because the constant vector \mathbf{a} can be put outside of the integral), which on the basis of Eq. (4.135) vanishes.

4. Let us now consider the resultant force \mathbf{F} acting in the external magnetic field \mathbf{H} on a closed system of currents complying both with condition (4.136) and with the equation (4.135) following from it.

According to Eq. (4.14), the component of this force \mathbf{F} along the z -axis is

$$F_z = \frac{1}{c} \int_V [\mathbf{j}\mathbf{H}]_z dV = \frac{1}{c} \int_V (j_x H_y - j_y H_x) dV \quad (4.138)$$

In a *homogeneous* external field, this force equals zero because the constant vector \mathbf{H} can be put outside of the integral, while the integral $\int \mathbf{j} dV$, according to Eq. (4.135), vanishes.

Let us assume that the field \mathbf{H} changes so slowly within the system of currents being considered that we may limit ourselves to the first terms of the expansion of \mathbf{H} by the powers of the distance from the conditional centre of the system of currents O along its entire length:

$$H_x(x, y, z) = H_x(0, 0, 0) + x \left(\frac{\partial H_x}{\partial x} \right)_0 + y \left(\frac{\partial H_x}{\partial y} \right)_0 + z \left(\frac{\partial H_x}{\partial z} \right)_0$$

and similarly for H_y and H_z . Here x , y , and z are the components of the distance \mathbf{R} from an arbitrary point of the field to the centre O , and $(\partial H_x/\partial x)_0$, $(\partial H_x/\partial y)_0$, etc. are the values of the corresponding derivatives at the point O . Thus, both $H_x(0, 0, 0)$ and $(\partial H_x/\partial x)_0$, $(\partial H_x/\partial y)_0$, etc. are constant and do not depend on x , y , and z . Introducing this expansion into Eq. (4.138) and using Eq. (4.135), we get

$$F_z = \frac{1}{c} \int_V \left\{ j_x \left(x \frac{\partial H_y}{\partial x} + y \frac{\partial H_y}{\partial y} + z \frac{\partial H_y}{\partial z} \right) - j_y \left(x \frac{\partial H_x}{\partial x} + y \frac{\partial H_x}{\partial y} + z \frac{\partial H_x}{\partial z} \right) \right\} dV$$

We shall prove that the following equations follow from condition (4.136) on the closed nature of currents:

$$\left. \begin{aligned} \int x j_x dV &= \int y j_y dV = \int z j_z dV = 0 \\ \int x j_y dV &= - \int y j_x dV = c\mu_z, \quad \int y j_z dV = - \int z j_y dV = c\mu_x \\ \int z j_x dV &= - \int x j_z dV = c\mu_y \end{aligned} \right\} \quad (4.139)$$

According to Eqs. (A.6) and (A.43₂)

$$x j_x = \frac{1}{2} \mathbf{j} \operatorname{grad} x^2 = \frac{1}{2} \operatorname{div} (\mathbf{j}x^2) - \frac{1}{2} x^2 \operatorname{div} \mathbf{j}$$

Since for steady currents $\text{div } \mathbf{j} = 0$, then upon using Gauss's theorem (A. 17) we get

$$\int x j_x dV = \frac{1}{2} \int \text{div} (\mathbf{j} x^2) dV = \frac{1}{2} \oint j_n x^2 dS$$

On the basis of condition (4.136), the last integral vanishes, Q.E.D. Further, in the identity

$$x j_y = \frac{1}{2} (x j_y - y j_x) + \frac{1}{2} (x j_y + y j_x)$$

the last term can be written in the form

$$\frac{1}{2} (x j_y + y j_x) = \frac{1}{2} \mathbf{j} \text{grad} (xy) = \frac{1}{2} \text{div} (\mathbf{j} xy) - \frac{1}{2} xy \text{div } \mathbf{j}$$

Since $\text{div } \mathbf{j} = 0$, then

$$x j_y = \frac{1}{2} (x j_y - y j_x) + \frac{1}{2} \text{div} (\mathbf{j} xy)$$

and, consequently, on the basis of Gauss's theorem

$$\int x j_y dV = \frac{1}{2} \int (x j_y - y j_x) dV + \frac{1}{2} \oint j_n xy dS$$

The last integral on the basis of condition (4.136) equals zero, while the first integral in the right-hand side, according to Eq. (4.133), equals $c\mu_z$, so that

$$\int x j_y dV = c\mu_z$$

The remaining equations (4.139) are obtained from the proved ones by the relevant transposition of the coordinate axes x , y , and z .

Using Eqs. (4.139), we get the following expression for F_z :

$$F_z = -\mu_z \frac{\partial H_y}{\partial y} + \mu_y \frac{\partial H_y}{\partial z} - \mu \frac{\partial H_x}{\partial x} + \mu_x \frac{\partial H_x}{\partial z}$$

Taking into account that from Eq. (4.36)

$$\text{div } \mathbf{H} = \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0$$

we obtain

$$F_z = \mu_x \frac{\partial H_x}{\partial z} + \mu_y \frac{\partial H_y}{\partial z} + \mu_z \frac{\partial H_z}{\partial z} = \frac{\partial}{\partial z} (\mu \mathbf{H})$$

because the magnetic moment of a system does not depend on the position of the point of the field, and therefore $\boldsymbol{\mu}$ may be put outside of the derivative. The expressions for F_x and F_y will be absolutely similar; writing them in the vector form, we finally get

$$\mathbf{F} = \nabla (\boldsymbol{\mu}\mathbf{H}) \quad (4.140)$$

which in its form completely coincides with Eq. (4.126).

Finally, the moment of the forces applied to the current element $\mathbf{j} dV$ equals

$$\frac{1}{c} [\mathbf{R} [\mathbf{j}\mathbf{H}]] dV = \frac{1}{c} \{ \mathbf{j} (\mathbf{R}\mathbf{H}) - \mathbf{H} (\mathbf{R}\mathbf{j}) \} dV$$

When integrating this relationship over the volume of the system of currents, we shall ignore the changes in \mathbf{H} over the length of the system, i.e. we shall consider \mathbf{H} to be a constant vector. Since according to Eq. (4.139)

$$\int (\mathbf{j}\mathbf{R}) dV = 0$$

then

$$\mathbf{N} = \frac{1}{c} \int \mathbf{j} (\mathbf{R}\mathbf{H}) dV$$

This integral differs from the second integral in Eq. (4.130) only in the substitution of \mathbf{H} for \mathbf{R}_0 . Consequently, a comparison of Eqs. (4.130) and (4.137) yields the following expression for the magnetic moment of the forces acting in a homogeneous field on a closed system of currents:

$$\mathbf{N} = \frac{1}{2c} \int [[\mathbf{R}\mathbf{j}] \mathbf{H}] dV = [\boldsymbol{\mu}\mathbf{H}] \quad (4.141)$$

which coincides with Eq. (4.127).

5. We shall now show that for a *line* current the results of this section completely coincide with those of Sec. 4.15.

Let us first consider Eq. (4.133) for the magnetic moment of currents. Passing over to line currents, we can transform it with the aid of Eq. (4.16), and as a result we get

$$\boldsymbol{\mu} = \frac{I}{2c} \oint_L [\mathbf{R} ds]$$

The integral in this expression has a simple geometrical meaning.

Inspection of Fig. 62 shows that the product $\frac{1}{2} [\mathbf{R} ds]$ is nothing but the vector value of the element $d\mathbf{S}$ of the surface \mathbf{S} of the cone

formed by the radii-vectors \mathbf{R} conducted from the conditional centre of the currents O to all the points of the contour L . Hence,

$$\frac{1}{2} \int_L [\mathbf{R} \, ds] = \int d\mathbf{S} = \mathbf{S} \text{ and } \boldsymbol{\mu} = \frac{I}{c} \mathbf{S}$$

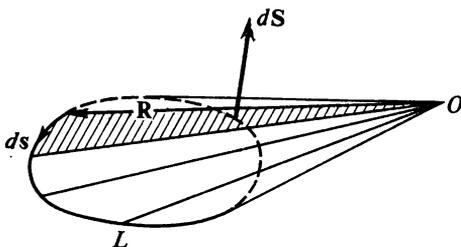


Fig. 62

where \mathbf{S} stands for the vector value of the surface resting on the current contour. This value, as indicated in Sec. 4.15, does not depend on the selection of the surface. Thus, the two definitions of the magnetic moment of currents $\boldsymbol{\mu}$ which we have given—Eqs. (4.122) and (4.133) — are indeed identical to each other for line currents, which are the only ones that Eq. (4.122) may be applied to.

Let us now consider the field of a system of currents at distances from the system that considerably exceed its dimensions. By Eqs. (4.29) and (4.137), the intensity of this field is

$$\mathbf{H} = \text{curl } \mathbf{A} = \text{curl } \frac{[\boldsymbol{\mu}\mathbf{R}]}{R^3} \tag{4.142}$$

whereas in Sec. 4.15 we obtained Eq. (4.124) for it:

$$\mathbf{H} = - \text{grad } \frac{(\boldsymbol{\mu}\mathbf{R})}{R^3} = \frac{3(\boldsymbol{\mu}\mathbf{R})\mathbf{R}}{R^5} - \frac{\boldsymbol{\mu}}{R^3} \tag{4.143}$$

We shall show that these expressions differ from each other only in their appearance. It follows from equations (4.142) and (A. 43₃) that

$$\mathbf{H} = \text{curl } \frac{[\boldsymbol{\mu}\mathbf{R}]}{R^3} = \frac{1}{R^3} \text{curl } [\boldsymbol{\mu}\mathbf{R}] - \left[[\boldsymbol{\mu}\mathbf{R}] \cdot \nabla \left(\frac{1}{R^3} \right) \right]$$

Since

$$\nabla \left(\frac{1}{R^3} \right) = - \frac{3\mathbf{R}}{R^5}$$

then

$$\begin{aligned} - \left[[\boldsymbol{\mu}\mathbf{R}] \cdot \nabla \left(\frac{1}{R^3} \right) \right] &= \frac{3}{R^5} [\boldsymbol{\mu}\mathbf{R}] \mathbf{R} = \\ &= \frac{3}{R^5} \{ \mathbf{R} \cdot (\boldsymbol{\mu}\mathbf{R}) - \boldsymbol{\mu}R^2 \} = \frac{3(\boldsymbol{\mu}\mathbf{R})\mathbf{R}}{R^5} - \frac{3\boldsymbol{\mu}}{R^3} \end{aligned}$$

Further,

$$\text{curl}_x [\mu \mathbf{R}] = \frac{\partial}{\partial y} (\mu_x y - \mu_y x) - \frac{\partial}{\partial z} (\mu_z x - \mu_x z) = 2\mu_x$$

because μ does not depend on the coordinates of the point of observation x , y , and z . Hence,

$$\text{curl} [\mu \mathbf{R}] = 2\mu$$

We finally get

$$\mathbf{H} = \text{curl} \frac{[\mu \mathbf{R}]}{R^3} = \frac{3(\mu \mathbf{R}) \mathbf{R}}{R^5} - \frac{\mu}{R^3}$$

which coincides with Eq. (4.143), Q.E.D.

Finally, as regards the forces acting on a current in an external field, then, as already indicated, the relevant formulas of this and the preceding sections do not differ from one another even in appearance.

4.17 Evolution of Notions of the Nature of Magnetism. Spin of Electrons

1. Let us now summarize the results we have obtained. We defined a magnetic field as a field of the forces of interaction of currents just like an electric field was defined as the field of the forces of interaction of electric charges. We further saw that if we exclude a vortex space from our consideration and introduce the appropriate conditional barriers into a field, then the field of currents can be reduced to the Coulomb field of fictitious magnetic charges joined up in pairs into dipoles and arranged in double layers or sheets.

It is known, however, that the historic development proceeded in the reverse direction. Magnetic phenomena were first discovered when studying the so-called "permanent" magnets. To explain these phenomena, the notion was advanced of magnetic charges interacting according to Coulomb's law, whose existence was considered to be just as real as that of electric charges. Experiments showed, however, that unlike electric charges, their magnetic counterparts of opposite signs cannot be separated from one another. This circumstance led to the conclusion (C. Coulomb, 1789) that every molecule of magnetism (i.e. bodies capable of becoming magnetized) always contains an equal quantity of magnetism of opposite signs and that magnetization consists either in the magnetic polarization of the molecules, i.e. in displacing the opposite charges of the molecules of the magnetic in opposite directions, or in turning of the magnetic axes of the molecules if the latter have a constant magnetic moment. The magnetic moment of molecules was considered to be due to the

non-symmetrical arrangement of the magnetic charges in them and was determined by an expression absolutely similar to that for the electric moment of a molecule [see Eq. (2.1)]:

$$\boldsymbol{\mu} = \sum_i m_i \mathbf{R}_i$$

where m_i are the separate magnetic charges or poles in the molecules and \mathbf{R}_i are their radii-vectors. From this viewpoint, a magnetic dipole is the simplest model of a molecule of a magnetic.

2. The magnetic properties of currents, namely, the ponderomotive interaction of currents on one hand and the interaction of currents with magnets on the other, were discovered only subsequently (H. Oersted, 1818, A. Ampere, 1820). Two kinds of sources of a magnetic field were thus found to exist—permanent magnets with magnetic dipoles and electric currents, i.e. moving electric charges, and both the magnets and the currents being acted upon by ponderomotive forces in an external magnetic field. The desire naturally appeared of eliminating this duality and reducing all the sources of a magnetic field to one category. Already Ampere, who had proved the equivalence of currents and magnetic sheets (i.e. a combination of magnetic dipoles) advanced the assumption that the seeming existence of magnetic dipoles in the molecules of magnetics may actually be due to the existence in them of current loops equivalent to the dipoles. In this case, all the sources of a magnetic field would be reduced to a single category, i.e. to currents. For almost an entire century, Ampere's hypothesis, later developed by W. Weber, remained a hypothesis and met a number of more or less reasonable objections. The hypothesis of Ampere's currents gained sure ground only after the modern notion of the atom appeared and became consolidated, i.e. the notion of the atom as a positive nucleus surrounded by a cloud of electrons rotating about it. Indeed, an electron rotating around a nucleus in the magnetic respect corresponds to a ring elementary current having a certain definite intensity I whose magnetic moment is determined by Eq. (4.122):

$$\boldsymbol{\mu} = \frac{1}{c} IS$$

where S is the area of the orbit of an electron. The resultant magnetic moment of a molecule equals the vector sum of the moments of the orbital motions of the separate electrons in it.

This "classical" electron theory has to its credit many major achievements: a rational explanation of diamagnetism, the prediction of gyromagnetic phenomena qualitatively confirmed experimentally, etc.

3. Some time near 1925, however, it was established that electrons (and protons) have more complicated properties than was thought earlier, namely, that every electron has not only a definite charge e

and a definite mass m , but also a quite definite and unchangeable in magnitude mechanical angular momentum or, simply, angular momentum K and magnetic moment μ . Here

$$\left. \begin{aligned} K &= \frac{1}{2} \frac{h}{2\pi} \\ \mu &= \frac{eh}{4\pi mc} \text{ cgs units} \end{aligned} \right\} \quad (4.144)$$

where h is Planck's constant*.

In other words, the situation is as if every electron (regardless of the nature of motion of its centre and regardless of whether it is revolving about the nucleus of an atom in a definite orbit or is freely moving outside of it) were continuously revolving about an axis passing through its centre with a constant angular velocity ω corresponding to the mechanical angular momentum K **.

From this viewpoint, the magnetic moment of an electron μ is created by the same circular motion of the elements of the charge of an electron about its axis, i.e. a combination of closed ring currents equivalent to this motion of the charges.

This illustrative notion of the so-called "proper rotation" of an electron does not correspond to actual facts. Even within the scope of the classical notion of an electron as of a minute sphere, this is clear from the fact that the numerical values of the quantities e , m , K , and μ cannot be made to agree with any permissible values of the radius and angular velocity of an electron. The magnetic moment of an electron has a purely quantum origin***, and only quantum mechanics has succeeded in explaining the magnetic properties of an electron quite satisfactorily.

The very convenient illustrative notion of the proper rotation of an electron, however, may naturally be used within definite limits because it does not result in contradictions with facts. We must not forget, however, that this notion is only of an auxiliary nature and does not correspond to actual facts. At present the complex of properties of an electron characterized by its angular momentum K and magnetic moment μ is denoted by the term "spin".

4. Since the experimental proofs of the existence of spin mainly relate to the field of atomic physics and quantum mechanics that

* Protons have the same mechanical angular momentum $\frac{1}{2} \frac{h}{2\pi}$, but their

magnetic moment is about 1/650-th of the magnetic moment of an electron. Therefore, in the majority of cases, the magnetic properties of protons and also of complex atomic nuclei may be disregarded.

** Thus, the analogy between an atom and the solar system becomes deeper: this "proper rotation" of an electron corresponds to the diurnal rotation of the Earth about its axis.

*** This manifests itself in that, according to Eq. (4.144), μ is proportional to the quantum constant h .

is outside of the scope of this book, we cannot stop to consider them here. There is also a considerable number of macroscopic phenomena whose explanation is possible only when the spin of electrons is taken into consideration. They include, for example, ferromagnetism in general and the question of the physical nature of the so-called Weiss molecular field in particular (Sec. 5.13), and the quantitative laws of gyromagnetic phenomena (Sec. 5.12).

For free electrons (for example, electrons forming a thermionic current, cathode rays, beta-rays), the corrections introduced by the presence of spin into the formulas of the classical theory of electrons are so negligible that they cannot virtually be detected experimentally. This is exactly why the spin of an electron could remain undiscovered for such a long time notwithstanding the thorough investigations of electronic phenomena that were begun at the end of the 19th century.

Let us consider, for instance, the forces acting on an electron in a magnetic field. Owing to the presence of spin, the Lorentz force [Eq. (4.20)]

$$\mathbf{F}_{\text{Lor}} = \frac{e}{c} [\mathbf{v}\mathbf{H}]$$

will be supplemented by the force acting in the field \mathbf{H} on a magnetic dipole whose moment $\boldsymbol{\mu}$ equals the spin magnetic moment:

$$\mathbf{F}_{\text{sp}} = \boldsymbol{\mu} \nabla \cdot \mathbf{H} = \mu \frac{\partial \mathbf{H}}{\partial \mu} \quad (4.145)$$

[cf. Eq. (4.128)]. To assess the order of magnitude of the quantities \mathbf{F}_{Lor} and \mathbf{F}_{sp} , we can replace the directional derivative $\partial \mathbf{H} / \partial \mu$ of the vector \mathbf{H} with respect to the vector $\boldsymbol{\mu}$ with the absolute value of the gradient of the numerical value of H , and assume in the Lorentz formula that \mathbf{v} is perpendicular to \mathbf{H} . Expressing μ with the aid of the second of Eqs. (4.144) through known physical constants, we get

$$\frac{F_{\text{sp}}}{F_{\text{Lor}}} = \frac{c\mu}{evH} |\nabla H| = \frac{h}{4\pi mv} \cdot \frac{1}{H} |\nabla H|. \quad (4.146)$$

The slowest electrons with which it is in general possible to conduct experiments and determine their path, the forces acting on them, etc. must have a velocity of the order of magnitude of at least $v = 6 \times 10^7$ cm/s (this is the velocity which an electron acquires when travelling through an accelerating potential difference of 1 volt). Thus, the maximum possible value of the factor $h/4\pi mv$, which we shall denote by Δl , is

$$\Delta l = \frac{h}{4\pi mv} = \frac{6.62 \times 10^{-27}}{4\pi \times 9.1 \times 10^{-28} \times 6 \times 10^7} \text{ cm}$$

i.e. about 10^{-8} cm. Equation (4.146) can be written as follows:

$$\frac{F_{\text{sp}}}{F_{\text{Lor}}} = \frac{\Delta l |\nabla H|}{H} \quad (4.147)$$

The numerator of the right-hand side equals the difference between the field intensities at the ends of the length $\Delta l \leq 10^{-8}$ cm, and, consequently, the ratio $F_{\text{sp}}/F_{\text{Lor}}$ equals the relative change in the field intensity over a distance of 10^{-8} cm. It is evident that in all possible experiments with free electrons in macroscopic magnetic fields this ratio will be vanishingly small and that the forces acting on the spin of an electron can thus manifest themselves only in intratomic phenomena.

It should be noted that this result is obtained in an absolutely general way without any assumptions on the velocity of an electron from the so-called uncertainty relation of quantum mechanics. 5. Thus, the magnetic properties of bodies are explained, first, by the rotation of electrons about the nuclei of the atoms forming a body, and, second, by the spin of the electrons.

It is significant that according to the quantum theory, the magnetic field induced by the magnetic spin moment of electrons is similar to the field of *electric currents* (and not to the field of magnetic dipoles consisting of magnetic charges) in the respect that it is a vortex field ($\text{div } \mathbf{H} = 0$ everywhere, whereas $\text{curl } \mathbf{H}$ does not equal zero everywhere). In other words, according to the quantum theory, the magnetic field induced by the spin of electrons can be reduced to the field of electric currents distributed in space in a definite way*.

In this sense, we can say that *all the magnetic properties of bodies are due to molecular electric currents*. The fundamentals of the electron theory of magnetism ensuing from this notion will be treated in the following chapter. The interpretation of the macroscopic theory of magnetism from the viewpoint of the old theories of magnetism will be considered in Sec. 5.14.

4.18 Absolute (Gaussian) and Other Systems of Units. The Electromagnetic Constant

1. In this book, we have used and will use the so-called Gaussian absolute system of units, which for brevity's sake we shall simply call the absolute system. It is based on defining the unit of electric charge according to the forces of interaction of point charges (Sec. 1.1.)

* Moreover, relativistic quantum mechanics directly leads to a general expression for the density of currents corresponding to a definite state of an electron the division of these currents into a part due to translational motion of the electron and a part due to its spin being in essence artificial.

The units of all the remaining electrical quantities, for example the electric field intensity, the field potential, current, and current density in this absolute system are derivatives of the basic unit of the quantity of electricity. Since this basic unit, in turn, is derived from the units of mass M , length L , and time T^* and with respect to these quantities has the dimension (Sec. 1.1)

$$[q] = M^{1/2}L^{3/2}T^{-1}$$

then the derived electrical units (field intensity, potential, etc.) also depend in a definite way on the choice of the units of mass, length, and time. We indicated the dimensions of these derived units at the relevant places of our text.

2. When encountering new physical quantities, we always tried to choose the units of these quantities so that in all the basic formulas establishing the relationship between these quantities all the numerical constants depending only on the choice of the units would become equal to unity (or 4π). We succeeded in doing this, however, only up to the moment when we passed over to studying the ponderomotive interaction of currents.

As we established in Sec. 4.2, the forces of interaction of current elements depend only on the magnitudes of the currents and on the length, direction and spacing of the current elements, i.e. on quantities whose units we had already established. It is quite natural that in the quantitative formulation of the law of current interaction [Eq. (4.6)] we had to introduce the constant c whose value depends on the previously chosen units. Since the values of these units have been chosen in a definite way, then this electromagnetic constant c also acquires a quite definite value that can be *measured* experimentally. We can use, for instance, Eqs. (4.10) for this purpose. They establish the relationship between the ponderomotive forces of interaction of currents and their magnitude and geometrical configuration because all these quantities can be determined independently of one another, while the relationship between them includes the constant c .

Theoretically, for example, we can use Eq. (4.88) determining the force of mutual attraction of two square current contours. After directly measuring in absolute units the quantities F , I_1 , I_2 , a , and d in this equation and introducing their numerical values into Eq. (4.88), we get the numerical value of the only unknown quantity—the constant c . In practice, however, these measurements involve considerable difficulties because it is very hard to construct such a system of conductors that on one hand their forces of interaction will be sufficiently great and convenient for measurement and on the other, the theoretical calculation of the value of their mutual inductance

* Because it is determined as the magnitude of a charge which acts with a force equal to unity on an equal charge at a unit distance from it.

L_{12} would not be too intricate mathematically. It must be borne in mind that the comparatively simple expression (4.78) for L_{12} derived for line currents stops being applicable when current contours are brought closer to each other. (For the optical methods used to measure the coefficient c see Sec. 7.7.)

The dimension of the electromagnetic constant c , according to Eq. (4.9), equals that of velocity: $[c] = [LT^{-1}]$ while its numerical value, as shown by measurements*, is very close to 3×10^{10} cm/s (with an accuracy up to 0.1%)

$$c = 3 \times 10^{10} \text{ cm/s}$$

3. The value of the constant c permits us to determine the value of the units of all the other quantities characterizing a constant field and also, as we shall see in the following, a varying field. For example, from the results of solving Problem 26 (p. 222) it follows that the intensity of the field of a ring current having the radius R_0 at its centre ($d = 0$) is

$$H = \frac{2\pi I}{cR_0} \quad (4.148)$$

Hence, we can say that the absolute unit of magnetic field intensity is 2π times smaller than the intensity of the field induced at the centre of a circle of unit radius by a line current having the magnitude $I = c$ flowing along the circumference of this circle. The dimension of H will be

$$[H] = \frac{[I]}{[cR]} = \frac{M^{1/2} L^{3/2} T^{-2}}{L^2 T^{-1}} = M^{1/2} L^{-1/2} T^{-1}$$

i.e. coincides with that of the intensity of an electric field (Sec. 1.2).

4. It should be noted that we can also arrive at the absolute (Gaussian) system of units proceeding from Coulomb's law for *fictional magnetic charges* instead of Coulomb's law for electric charges. Here it is essential first of all, naturally, to define the absolute unit of magnetism as the magnetic charge that repulses an equal charge at a unit distance from it with a force equal to unity. Next, the intensity of a magnetic field is defined as the magnitude of the force acting at a given point of a field on a unit positive magnetic charge, etc. Of course, this "magnetic" absolute system of units will be identical with the "electric" system which we have used in this book only if we retain the previous expression (4.144) establishing the relationship between an

* According to the recommended and approved values of the fundamental physical constants [*Uspekhi fiz. nauk*, 115: 623 (1975)], the following value has been adopted for the quantity c :

$$c = 299\,792\,458 (1.2) \text{ m} \cdot \text{s}^{-1}$$

electric current and the thickness of a magnetic sheet equivalent to the current

$$\tau = \frac{I}{c}$$

5. Unlike the system of units described above, which we shall call simply the *absolute* system and which is sometimes called the Gaussian or symmetrical system (see the table at the end of the section), there are also the so-called absolute *electrostatic* and absolute *electromagnetic* systems of units. The former is based on the definition of the unit of electricity proceeding from Coulomb's law, and the second on the definition of the unit of magnetism proceeding from a similar law for fictitious magnetic charges. In these systems, however, unlike the Gaussian one, the electromagnetic constant c is taken equal to unity. In accordance with the above considerations, however, it is impossible to completely eliminate from the formulas of the theory all the numerical constants whose value depends on the choice of the system of units. Consequently, in the electrostatic system of units, we have to assign the dimension $L^{-2}T^2$ to the permeability μ and assume that the "permeability of vacuum" μ_0 equals $\frac{1}{9}$

$$(3 \times 10^{10} \text{ cm/s})^{-2} = \frac{1}{9} \times 10^{-20} \text{ s}^2/\text{cm}^2$$

In the electromagnetic system, we have to ascribe the same dimension to the permittivity ε and assume that the "permittivity of vacuum" ε_0 equals the same value $\frac{1}{9} \times 10^{-20} \text{ s}^2/\text{cm}^2$.

We shall not stop to consider these systems because the Gaussian system of units corresponds to the modern notions of the nature of electromagnetic phenomena to a much greater extent. For example, from our consideration of the physical meaning of the quantity ε [Chap. 2, see, particularly, Eq. (2.21)] it follows that it is indeed a dimensionless quantity, and that for a vacuum ε becomes equal to unity; the constant c having the dimension of velocity in the Gaussian system, as we shall see on a later page, indeed equals the velocity of propagation of electromagnetic waves in a vacuum, etc.

Note. It does not follow from everything said above, however, that the other systems of electromagnetic units are not correct. Any system of units is conditional, and only the inherent lack of contradiction in a system, its convenience and the extent of conformity to modern physical notions can serve as the criterion in selecting a definite system of units.

6. We must indicate that *the absolute (Gaussian) units of electrical quantities coincide with the relevant electrostatic units, and the absolute units of magnetic quantities with the electromagnetic units* (Table 1).

We must further note that the *electromagnetic units of current and of the quantity of electricity* are *c* times greater than the corresponding *Gaussian units*, (and, therefore, the *electrostatic units* equal to them). Consequently, if *I* and *I'* are the values of the same current in absolute and in electromagnetic units, respectively, then

$$I' = \frac{I}{c}$$

TABLE 1. Electrical and Magnetic Quantities and Their Dimensions in the Absolute (Gaussian) System of Units

Physical quantity	Symbol	Dimension in absolute (Gaussian) system of units	
Quantity of electricity	<i>q</i>	$M^{1/2}L^{3/2}T^{-1}$	The absolute units coincide with the electrostatic ones
Electric field intensity	<i>E</i>	$M^{1/2}L^{-1/2}T^{-1}$	
Electric displacement	<i>D</i>	Ditto	
Polarization	<i>P</i>	Ditto	
Potential	ϕ	$M^{1/2}L^{1/2}T^{-1}$	
Capacitance	<i>C</i>	L	
Electromotive force and voltage	\mathcal{E}	$M^{1/2}L^{1/2}T^{-1}$	
Current	<i>I</i>	$M^{1/2}L^{3/2}T^{-2}$	
Current density	<i>j</i>	$M^{1/2}L^{-1/2}T^{-2}$	
Resistance of conductor	<i>R</i>	$L^{-1}T$	
Conductivity	κ	T^{-1}	
Magnetic field intensity	<i>H</i>	$M^{1/2}L^{-1/2}T^{-1}$	The absolute units coincide with the electromagnetic ones
Magnetic induction *	<i>B</i>	Ditto	
Magnetization *	<i>M</i>	Ditto	
Magnetic induction flux	Ψ	$M^{1/2}L^{3/2}T^{-1}$	
Self-inductance and mutual inductance	<i>L</i>	L	
Magnetic moment	μ	$M^{1/2}L^{5/2}T^{-1}$	
Quantity (fictitious) of magnetism	<i>m</i>	Ditto	

* See Chap. 5.

Since, on the other hand, in almost all the formulas of this chapter the constant *c* is encountered in the combination *I/c* (or *j/c*), then by expressing the current in electromagnetic units instead of in absolute ones we can eliminate this constant from the above formulas and thus simplify them. For instance, by introducing *I'* into Eq. (4.148), we get

$$H = \frac{2\pi I'}{R_0}$$

This permits us to say that the absolute unit of magnetic intensity *H* is 2π times greater than the intensity of the field induced at the

centre of a circle with a radius equal to unity by a current of one electromagnetic unit flowing along its circumference.

Further, Eq. (4.114) acquires the form

$$\tau = I'$$

Hence, the strength τ of a magnetic sheet equivalent to the current equals the intensity of this current I' expressed in electromagnetic units.

The consistent use of the electromagnetic units of the quantity of electricity and current, however, is convenient far from always. For example, when using electromagnetic units, the factor c^2 appears in Coulomb's law, namely,

$$F = c^2 \frac{q_1'q_2'}{R_2}$$

or the factor $1/\epsilon_0$ equal to it, where $\epsilon_0 = \frac{1}{9} \times 10^{-20} \text{ s}^2/\text{cm}^2$ is the "permittivity of a vacuum" (see above).

Electromagnetic units are nevertheless sometimes used when measuring not only magnetic, but also electrical quantities. Electrostatic units, in general, are not used for measuring magnetic quantities.

Knowing the ratio between the electromagnetic and electrostatic units of charge, we can easily find the relationship between the units of other quantities. For instance, the electric field intensity E equals unity if a unit charge placed in the field is acted upon by a force of one dyne. Consequently, the electromagnetic unit of intensity must be c times smaller than its electrostatic counterpart. These relationships for a number of fundamental electrical quantities are given in Table 2.

TABLE 2. Relationship Between Electromagnetic Units (cgsm or emu) and Electrostatic Units (cgse or esu)

Quantity of electricity q and current I	1 emu = c abs. units (esu)
Electric field intensity E	} 1 emu = $\frac{1}{c}$ abs. unit (esu)
Potential φ and e.m.f. \mathcal{E}	
Capacitance C	1 emu = c^2 abs. units (esu)
Resistance R	1 emu = $\frac{1}{c^2}$ abs. unit (esu)
Magnetic field intensity H	1 abs. unit (emu) = c esu
Inductance L	1 abs. unit (emu) = $\frac{1}{c^2}$ esu

Finally, Table 3 gives the names and values of the basic practical units. 7. The practical system of units is a correctly constructed one in the sense that like the Gaussian, electrostatic, and electromagnetic systems, all the practical units can be obtained as derived ones from a single fundamental unit. The unit (ampere, ohm, etc.) to be considered as the fundamental one is naturally of no importance. The exact determination of the values of the derived units according to

that of the fundamental one, however, involves appreciable experimental difficulties, and the results of the relevant measurements are constantly changing and yielding more accurate values with the improvement of the measuring equipment and techniques. Further, the fundamental units of the practical system (ampere, ohm, etc.) were, in turn, defined by indicating a definite numerical relationship of them to the absolute units based on the absolute unit of the quantity of electricity. And it is much more difficult to measure absolutely the quantity of electricity than, for instance, to determine absolutely the current or resistance.

For this reason, an international congress in 1908 established the so-called *international system of practical units* based on two fundamental units—the international ohm (int. ohm) and the international ampere (int. amp.). The values of these units were defined as follows: (1) the international ohm equals the resistance of a column of mercury 106.300 cm long at a temperature of 10 °C and a mass of the column equal to 14.4521 g; and (2) the international ampere equals the current which when passing through a solution of AgNO_3 in water liberates 0.001 118 0 g of silver a second. All the remaining international practical units (including the volt) are derived from these fundamental units.

TABLE 3. Practical Units

Quantity of electricity q	1 coulomb (C) = 10^{-1} emu = = 3×10^9 abs. units (esu)
Potential ϕ	1 volt (V) = 10^8 emu = $\frac{1}{300}$ abs. unit (esu)
E.m.f. and voltage \mathcal{E}	
Field intensity E	1 V/cm = 10^8 emu = $\frac{1}{300}$ abs. unit (esu)
Capacitance C	1 farad (F) = 10^{-9} emu = 9×10^{11} abs. units (esu)—centimetres
Current I	1 ampere (A) = 10^{-1} emu = 3×10^9 abs. units (esu)
Resistance R	1 ohm (Ω) = 10^9 emu = $\frac{1}{9} \times 10^{-11}$ abs. unit (esu)
Magnetic field intensity H	1 gauss (Gs) = 1 abs. unit (emu) = = 3×10^{10} esu
Self-inductance L	1 henry (H) = 10^9 abs. units (emu) = = $\frac{1}{9} \times 10^{-11}$ esu
Magnetic (induction) flux Ψ	1 maxwell (Mx) = 10^{-8} weber (Wb) = = 1 Gs·cm ² = 1 abs. unit (emu) = = 3×10^{10} esu
Energy U	1 joule (J) = 10^7 abs. cgs units (ergs)
Power P	1 watt (W) = 10^7 abs. cgs units (erg/s)

The international ampere and ohm were defined so that their values with the accuracy available at that time would equal, respectively, 10^{-1} and 10^9 electromagnetic units of current and resistance. At the same time, however, it was resolved that these units were established once and for all, and that their values would not be changed depending on their discrepancies from the absolute ohm (10^9 emu = 1 abs. ohm) and the absolute ampere (10^{-1} emu = 1 abs. amp.) that might be discovered.

Actually, however, every major country used its own standard e.m.f.'s (standard Weston cells) and resistances instead of the above fundamental units. Although the standards of different countries were brought into agreement with one another at their time and expressed in international units, by 1930 discrepancies reaching 0.01% accumulated between the standards of different countries. In addition, there was a discrepancy between the electrical and mechanical units; for example, the electrical and mechanical units of energy differed by 0.02%. Therefore, in 1935 the Comité International des Poids et Mesures (the International Committee of Weights and Measures) decided to replace the "international system of electrical units" with the absolute system again and introduce the latter into general use from January 1, 1940.

According to 1935 data*:

$$1 \text{ int. ohm} = 1.0005 \text{ abs. ohm}$$

$$1 \text{ int. amp.} = 0.9999 \text{ abs. amp.}$$

8. We shall mention, finally, the so-called rational system of units proposed by O. Heaviside, which following Lorentz's example are being used more and more often in theoretical investigations. It differs from the absolute Gaussian system not in the dimensions, but only in the numerical values of the fundamental units: a unit of electricity (and also of magnetism) in the rational system is taken equal to $1/\sqrt{4\pi}$ of the absolute unit of electricity (magnetism). Although a result of this change is the necessity of introducing the constant 4π into the denominator of Coulomb's law and some other formulas, this constant 4π is eliminated from a great number of basic formulas of the theory of the field, for example from the differential equations for the scalar [Eq. (1.65)] and vector [Eq. (4.33)] potentials, and from Eq. (2.18) relating the values of $\text{div } \mathbf{D}$ and ρ .

* Subsequently, the definition and values of the electrical units were repeatedly changed. The latest data (1973) can be found in *Uspekhi fiz. nauk*, **115**: 623 (1975). Particularly, the following relationships were established for the units of resistance and current:

$$1 \text{ ohm st.} = 0.999\,999\,47(19) \text{ SI unit}$$

$$1 \text{ amp. st.} = 1.000\,000\,7(26) \text{ SI unit}$$

where st. denotes the standards established by the Comité International des Poids et Mesures.

5

Magnetics (Magnetizable Media)

5.1 Magnetization of Magnetics. Molecular Currents and Conduction Currents

1. We know that the introduction of dielectrics into the field of free electric charges (see the definition of this term in Sec. 2.2) causes a change in the field due to *polarization* of the dielectric. Similarly, the introduction of magnetics (for instance, iron) into the magnetic field of currents causes a change in the field due to *magnetization* of the magnetic. We call all bodies capable of becoming magnetized *magnetics**, i.e. all bodies whose presence is capable of either changing or inducing a magnetic field. Whereas all dielectrics become depolarized simultaneously with the vanishing of the external electric field, however**, only the *majority* of magnetics that were magnetized under the action of an external magnetic field become completely demagnetized when the field vanishes (temporary or *induced* magnetization of para- and diamagnetics). At the same time, unlike dielectrics, there is a class of magnetics (the so-called *ferromagnetics*) that are capable of remaining magnetized even after the vanishing of the external field (the so-called *permanent* or *residual magnetization*), i.e. are capable not only of changing the magnetic field of currents by their presence, but also of independently inducing a magnetic field regardless of the presence of electric currents (the so-called *permanent magnets****).

* In essence, all material bodies have magnetic properties to some extent (true, very weak ones in the majority of cases).

** See Sec. 2.2, p. 116. The phenomenon of so-called residual polarization occurring in some dielectrics, especially when they are contaminated with impurities, has nothing in common in its physical nature with the residual magnetism of ferromagnetics. It is connected with the presence of leakage and polarization currents (imperfect insulators) causing the redistribution of the *free* electric charges. An exception are pyro- and ferroelectrics (see Sec. 2.10, p. 150) whose dielectric properties in a number of respects are indeed similar to the magnetic properties of ferromagnetics.

*** The ability of ferromagnetics to retain residual magnetization appreciably depends on their microscopic heterogeneity (residual elastic tensions, a polycrystalline structure, chemical impurities) and is almost absent, for example, in ferromagnetic monocrystals deprived of internal tensions. Hence, strictly speaking, the basic characteristic of ferromagnetics is not their residual magnetism

2. The field of magnetized magnetics, like any magnetic field, is induced by electric currents circulating in the magnetic*.

Let us first consider a magnetic that does not conduct electricity and is constructed of neutral molecules (gases, liquids) or of ions fixed in definite positions (an ionic crystal lattice or an amorphous solid dielectric). Although the *mean* density of the current in such a medium does equal zero and no electric charges are transferred in it over macroscopic distances, motion of electrons occurs inside separate molecules or ions corresponding to a definite distribution of the currents. These *currents* are called *molecular*. In unmagnetized magnetics, they are distributed absolutely chaotically, and their magnetic fields on an average are mutually compensated. A magnetized magnetic, on the other hand, is characterized by order in the molecular currents owing to which the resultant magnetic field of these currents differs from zero.

In magnetics that are conductors (metals, electrolytes, etc.), we must obviously distinguish *conduction currents* \mathbf{j}_{cond} corresponding to the motion of the charges carrying a macroscopic current (free electrons in metals, ions in electrolytes and ionized gases) and *molecular currents* \mathbf{j}_{mol} in the neutral molecules of electrolytes, in fixed ions forming the solid crystalline skeleton of metals, etc.:

$$\mathbf{j}_{\text{micro}} = \mathbf{j}_{\text{cond}} + \mathbf{j}_{\text{mol}} \quad (5.1)$$

where the subscript "micro" indicates the true microscopic density of the current in the medium as distinct from the average macroscopic density \mathbf{j} .

We shall adhere to this division of currents into two classes although it cannot always be conducted unambiguously** because this division greatly simplifies the derivation of the macroscopic equations of a field from the treated electron theory***. For our purposes, it is sufficient to assume that unlike conduction currents, molecular currents close within microscopically small spaces.

3. To construct a theory of magnetics, we must first of all find a convenient quantitative characteristic of the distribution of the molecular currents in the medium. The mean value of the density of

(hysteresis), but the non-linear nature of the dependence of their magnetization on the magnetic field intensity (which we shall treat in greater detail in Secs. 5.13 and 7.18) manifesting itself even in very weak fields and growing with an increase in the microscopic heterogeneity of a ferromagnetic.

* The spin magnetic moment of an electron can also be reduced to the action of the corresponding electric currents (see the end of this section).

** For example, the currents created by free electrons in metals cannot, generally speaking, be completely included in the class of conduction currents because, for instance, the magnetization of diamagnetic metals is mainly due to the ordered motion of the free electrons not connected with the transfer of a macroscopic current.

*** Naturally, the correctness of the macroscopic equations of a field can also be substantiated without any special assumptions of this kind (see the end of Sec. 7.1).

molecular currents \mathbf{j}_{mol} over a physically infinitely small volume cannot serve as such a characteristic. Indeed, the mean value of a current taken over the entire volume of a system of closed currents equals zero* although the magnetic moment and magnetic field of such a system must not necessarily equal zero. Particularly, the vector sum of the currents flowing in any molecule always equals zero.

In Secs. 4.15 and 4.16 we saw that a system of closed currents when its dimensions are sufficiently small is unambiguously characterized by its magnetic moment

$$\boldsymbol{\mu} = \frac{1}{2c} \int [\mathbf{R}\mathbf{j}] dV$$

The distribution of the molecular currents must also obviously be characterized by their magnetic moment. We know that the polarization vector \mathbf{P} is a measure of the polarization of a dielectric and equals the electric moment of a unit of its volume. Similarly, the *magnetization* vector \mathbf{M} is a measure of the magnetization of a magnetic and equals the magnetic moment of the molecular currents per unit of volume of the magnetic:

$$\mathbf{M} = \frac{1}{2c} \int [\mathbf{R}\mathbf{j}_{\text{mol}}] dV \quad (5.2)$$

where integration extends over a unit volume of the magnetic. As we have already noted in Sec. 4.16, the value of this integral on condition that the system of currents is closed does not depend on the choice of the origin of the radii-vectors \mathbf{R} .

If a magnetic consists of separate molecules (for example gaseous magnetics), its magnetization \mathbf{M} can also be determined as the vector sum of the magnetic moments of the molecules in a unit of its volume:

$$\mathbf{M} = \sum \boldsymbol{\mu} \quad (5.3)$$

where $\boldsymbol{\mu}$ stands for the moment of a separate molecule of the magnetic. It is easy to see that for magnetics consisting of separate molecules Eq. (5.3) is equivalent to Eq. (5.2).

Finally, if the magnetization of a magnetic is not constant over its volume, the magnetization vector \mathbf{M} can be determined as the mean (over an infinitely small volume ΔV) density of the magnetic moment of the molecular currents:

$$\mathbf{M} = \frac{1}{2c} [\mathbf{R}\mathbf{j}_{\text{mol}}] = \frac{1}{2c \Delta V} \int_{\Delta V} [\mathbf{R}\mathbf{j}_{\text{mol}}] dV \quad (5.4)$$

By analogy with the electric polarization \mathbf{P} , the magnetization \mathbf{M} can also be called the *magnetic polarization*.

* Because as proved in Sec. 4.16, from condition (4.136) we get Eq. (4.135).

4. The construction of the theory of magnetism on the basis of a consideration of steady closed molecular currents may call forth doubts of two kinds.

First, from the viewpoint of elementary notions of the structure of an atom, the motion of electrons in atoms and molecules is not completely equivalent to *steady* currents because the field of electrons is not constant in time but changes periodically in accordance with the period of rotation of an electron in its orbit (about the nucleus of an atom or along a complicated orbit in a molecule, etc.).

From the viewpoint of Bohr's theory of the atom, this difficulty is eliminated by the fact that the period of revolution of electrons in their orbits are exceedingly small and are comparable with the period of light waves (10^{-14} to 10^{-15} s), so that in macroscopic observations we perceive only the mean value of this field in time. Therefore in constructing a macroscopic theory, we have the right to replace an electron moving inside an atom with a steady closed current (a "molecular current") whose constant field is identical with the mean value of the field of an electron during one period.

Modern quantum mechanics, however, has completely eliminated this difficulty by showing that the illustrative notion of an electron moving in an atom in definite orbits is only a first very rough approximation to what actually occurs, and that the magnetic field of atoms in a stationary state is constant in time and can be reduced to the field of steady closed currents distributed inside an atom or molecule with a definite density \mathbf{j} .

Second, the circumstance that the magnetic properties of atoms and molecules are due not only to the motion of the electrons in them, but also to the *spin* of the electrons, is also doubtful. Indeed, the magnetic spin moment of electrons is often considered to be similar to a magnetic dipole. As we have already indicated in Sec. 4.17, however, according to quantum mechanics, the magnetic field induced by the spin magnetic moment of an electron can also be reduced to the field of electric currents distributed in a definite manner in space.

At any rate, the magnetic field induced by spin, like any field of currents, is a vortex field and must be described by the vector potential \mathbf{A} instead of by the scalar potential ψ (see Sec. 5.12).

Thus, the statement that the magnetic properties of magnetics are due to molecular currents is quite justified. For some purposes, however, it is very convenient to consider the magnetization \mathbf{M} of magnetics as consisting, first, of the magnetic moments of the currents corresponding to the translational (orbital) motion of the electrons, and, second, of the dipole spin magnetic moments of the electrons. Such a division changes nothing in the reasoning in the following sections devoted to the derivation of the general equations of a magnetic field in magnetics, but will be helpful when considering gyromagnetic effects (Sec. 5.12) and the mechanism of magnetization of ferromagnetics (Sec. 5.13).

5.2 Vector Potential of a Magnetic Field in the Presence of Magnetics. Mean Density of Space and Surface Molecular Currents

1. After resolving in accordance with Eq. (5.1) the total density $\mathbf{j}_{\text{micro}}$ in an arbitrary medium into the density of the conduction currents \mathbf{j}_{cond} and that of the molecular currents \mathbf{j}_{mol} , we get the following expression for the vector potential of a magnetic field:

$$\mathbf{A} = \frac{1}{c} \int \frac{\mathbf{j}_{\text{micro}} dV}{R} = \frac{1}{c} \int \frac{\mathbf{j}_{\text{cond}} dV}{R} + \frac{1}{c} \int \frac{\mathbf{j}_{\text{mol}} dV}{R}$$

Introducing the symbols \mathbf{A}_0 and \mathbf{A}' for the vector potentials of the conduction and molecular currents, respectively, i.e.

$$\mathbf{A}_0 = \frac{1}{c} \int \frac{\mathbf{j}_{\text{cond}} dV}{R} \text{ and } \mathbf{A}' = \frac{1}{c} \int \frac{\mathbf{j}_{\text{mol}} dV}{R} \quad (5.5)$$

we can write

$$\mathbf{A} = \mathbf{A}_0 + \mathbf{A}' \quad (5.6)$$

These expressions include the true microscopic current densities, whereas in the macroscopic theory we must operate with the mean values of microscopic quantities and must therefore transform Eqs. (5.5) correspondingly.

The mean value of \mathbf{j}_{cond} over an infinitely small volume is obviously the density \mathbf{j} of the currents which the macroscopic theory only deals with without explicitly introducing the molecular currents into consideration:

$$\mathbf{j} = \bar{\mathbf{j}}_{\text{cond}} \quad (5.7)$$

Thus, in the macroscopic theory, we can simply substitute \mathbf{j} for \mathbf{j}_{cond} in the expression for \mathbf{A}_0 :

$$\mathbf{A}_0 = \frac{1}{c} \int \frac{\mathbf{j} dV}{R} \quad (5.8)$$

Hence, to determine the mean value of the vector potential of the molecular currents \mathbf{A}' , we must express the mean value of the density of the molecular currents $\bar{\mathbf{j}}_{\text{mol}}$ through quantities which the macroscopic theory operates with, namely through the magnetization. It is simpler, however, to calculate directly the mean value of the vector \mathbf{A}' as follows.

The vector potential of a system of closed currents, provided that its dimensions are sufficiently small, according to Eq. (4.137) equals $[\boldsymbol{\mu}\mathbf{R}]/R^3$, where $\boldsymbol{\mu}$ is the magnetic moment of the system. On the other hand, the magnetic moment of an element of volume dV

of a magnetic characterizing the molecular currents circulating in it, according to Eqs. (6.2) and (6.4), equals $\mathbf{M} dV^*$. Therefore the vector potential of the field induced by an element of volume dV of a magnetic equals

$$\frac{[\mathbf{MR}]}{R^3} dV$$

where \mathbf{R} is a radius-vector conducted from the element of volume dV to the "point of observation" at which the value of the vector potential is being determined.

Finally, the vector potential \mathbf{A}' of the totality of molecular currents circulating in all the elements of a magnetic is determined by the integral

$$\mathbf{A}' = \int \frac{[\mathbf{MR}]}{R^3} dV \tag{5.9}$$

which can obviously be extended over the entire infinite space (because outside of the magnetics $\mathbf{M} = 0$). Thus, the vector potential \mathbf{A}' of the field of molecular currents is completely determined by the magnetization of the medium \mathbf{M} .

2. It will be good to transform Eq. (5.9) somewhat. According to Eqs. (A.43₃) and (A.10),

$$\begin{aligned} \text{curl}_q \left(\frac{1}{R} \mathbf{M} \right) &= \left[\nabla_q \frac{1}{R} \cdot \mathbf{M} \right] + \frac{1}{R} \text{curl } \mathbf{M} = \\ &= \frac{[\mathbf{RM}]}{R^3} + \frac{1}{R} \text{curl } \mathbf{M}^{**} \end{aligned}$$

and, consequently, Eq. (5.9) can be written as follows:

$$\mathbf{A}' = \int \frac{\text{curl } \mathbf{M}}{R} dV - \int \text{curl}_q \left(\frac{1}{R} \mathbf{M} \right) dV$$

The last integral can be transformed with the aid of vector analysis relationship (A.56) into an integral over the surface S enclosing the volume of integration V^{***} :

$$\int_V \text{curl}_q \left(\frac{1}{R} \mathbf{M} \right) dV = \oint_S \frac{[\mathbf{nM}]}{R} dS$$

* It must be noted that the magnetization vector \mathbf{M} is a *macroscopic* quantity because according to Eq. (5.4) it equals the *mean* density of the magnetic moment in an infinitely small volume.

** The subscript q in the expression $\text{curl}_q \mathbf{M}$ may be omitted without any danger of misunderstandings because the vector \mathbf{M} is a function of only the source point of the radius-vector \mathbf{R} .

*** This transformation may be directly applied to our integral because upon the formation of spatial derivatives we differentiate R and \mathbf{M} with respect to the coordinates of the *source point* of the vector \mathbf{R} coinciding with the element of the volume of integration dV (see the footnote to page 114 in Sec. 2.2).

If the field has no surfaces of discontinuity of the magnetization vector \mathbf{M} , then the last integral can be taken over an infinitely remote surface enveloping the *total field*, and vanishes (if the magnetization \mathbf{M} decreases at infinity at a higher rate than $1/R$).

Otherwise, as usual, the surface integral will also have to be extended over the surface S'_1 separating the surface S_1 of discontinuity of the vector \mathbf{M} from the volume of integration V .

Contracting the surface S'_1 until it coincides with the surface of discontinuity S_1 and repeating with insignificant changes our reasoning of Sec. 1.12, we see that

$$\begin{aligned} \lim \oint_{S'_1} \frac{[\mathbf{nM}]}{R} dS &= \int_{S_1} \frac{[\mathbf{n}_1\mathbf{M}_1] + [\mathbf{n}_2\mathbf{M}_2]}{R} dS = \\ &= - \int_{S_1} \frac{[\mathbf{N} \cdot \mathbf{M}_2 - \mathbf{M}_1]}{R} dS \end{aligned}$$

where \mathbf{M}_1 and \mathbf{M}_2 are the values of \mathbf{M} at both sides of the surface of discontinuity, and \mathbf{N} is a normal to this surface directed from 1 to 2. Denoting this normal by \mathbf{n} , we get

$$\mathbf{A}' = \int \frac{\text{curl } \mathbf{M}}{R} dV + \int \frac{[\mathbf{n} \mathbf{M}_2 - \mathbf{M}_1]}{R} dS \quad (5.10)$$

Thus, the total vector potential $\mathbf{A} = \mathbf{A}_0 + \mathbf{A}'$ of a magnetic field in an arbitrary medium is expressed in the macroscopic theory through the macroscopic density of the currents \mathbf{j} and through the vector \mathbf{M} characterizing the magnetization of the medium:

$$\begin{aligned} \mathbf{A} &= \frac{1}{c} \int \frac{\mathbf{j}}{R} dV + \int \frac{\text{curl } \mathbf{M}}{R} dV + \\ &+ \int \frac{[\mathbf{n} \cdot \mathbf{M}_2 - \mathbf{M}_1]}{R} dS \end{aligned} \quad (5.11)$$

3. Let us compare the macroscopic expression (5.10) for the vector potential \mathbf{A}' of molecular currents with the one obtained from the microscopic expression (5.5) for \mathbf{A}' by directly calculating its mean value, i.e. by replacing the microscopic density \mathbf{j}_{mol} in Eq. (5.5) with its mean value $\bar{\mathbf{j}}_{\text{mol}}$ over an infinitely small volume:

$$\mathbf{A}' = \frac{1}{c} \int \frac{\bar{\mathbf{j}}_{\text{mol}} dV}{R} \quad (5.12)$$

This comparison shows, first, that the mean density of space molecular currents $\bar{\mathbf{j}}_{\text{mol}}$ is related to the magnetization of the medium as follows:

$$\bar{\mathbf{j}}_{\text{mol}} = c \text{curl } \mathbf{M} \quad (5.13)$$

Second, it follows from this comparison that the assumption of the existence of discontinuity surfaces of the magnetization vector is equivalent to the assumption of the existence together with the space currents of *surface molecular currents* whose mean density is proportional to the surface curl of \mathbf{M} :

$$\bar{\mathbf{j}}_{\text{mol}} = c \text{Curl } \mathbf{M} = c[\mathbf{n} \cdot \mathbf{M}_2 - \mathbf{M}_1] \quad (5.14)$$

Indeed, upon this assumption, Eq. (5.12) must be supplemented with a term taking the surface currents into account:

$$\mathbf{A}' = \frac{1}{c} \int \frac{\bar{\mathbf{j}}_{\text{mol}}}{R} dV + \frac{1}{c} \int \frac{\bar{\mathbf{j}}_{\text{mol}}}{R} dS \quad (5.15)$$

As a result, Eq. (5.15) on the basis of Eqs. (5.13) and (5.14) becomes equivalent to Eq. (5.10).

Naturally, the very assumption of the possibility of the existence of discontinuity surfaces of physical quantities and surface currents is characteristic of the macroscopic interpretation of a field and is absolutely alien to the microscopic theory.

Equation (5.13) for the mean density of molecular currents, as should be expected, complies with the condition that the currents are closed because according to Eq. (A.42₂)

$$\text{div } \bar{\mathbf{j}}_{\text{mol}} = c \text{div curl } \mathbf{M} = 0$$

Further, the expression for the mean surface density of the molecular currents coincides with the one that can be obtained from the expression for the volume density by a limit transition of the type of Eq. (4.56).

It must be noted that in uniformly magnetized media ($\mathbf{M} = \text{const}$) the mean density of the molecular currents according to Eq. (5.13) equals zero*. Indeed, if adjacent elements of volume of a medium are magnetized absolutely identically, then currents of one definite direction cannot predominate in it anywhere. On the boundary between the magnetized magnetics and a vacuum, however, in accordance with Eq. (5.14) there are surface currents with the density $\mathbf{i} = \pm c [\mathbf{nM}]$, because $\mathbf{M} = 0$ in a vacuum.

We obtained Eqs. (5.13) and (5.14) establishing the relationship between the distribution of the molecular currents and the spatial derivatives of the magnetization vector (and also the jump in its tangential components on discontinuity surfaces) in a quite roundabout way. It would be desirable to obtain them directly from the basic equation (5.2) determining the magnetization vector \mathbf{M} and expressing the value of \mathbf{M} through $\bar{\mathbf{j}}_{\text{mol}}$. We shall conduct the relevant calculations in Sec. 5.8 with some simplifying assumptions.

* Similarly to how the mean density of bound charges in a uniformly polarized dielectric equals zero.

4. Let us consider as an example a cylindrical magnet uniformly magnetized over its entire volume parallel to its axis. The mean density of the space molecular currents will everywhere equal zero because when $\mathbf{M} = \text{const}$ we have $\text{curl } \mathbf{M} = 0$. There will also be no molecular currents on the bases of the cylinder because a normal to these bases is parallel to \mathbf{M} . A normal to the side surface of the cylinder is perpendicular to \mathbf{M} , and therefore the density of the surface molecular currents on the side surface of a cylinder will differ from zero and will numerically equal

$$i_{\text{mol}} = cM \quad (5.16)$$

[in Eq. (5.14) we assume that $\mathbf{M}_1 = \mathbf{M}$ and $\mathbf{M}_2 = 0$ because outside of the magnet $\mathbf{M} = 0$].

These closed ring surface currents form a right-handed system with the direction of magnetization \mathbf{M} .

Thus, from the viewpoint of the electron theory, a magnet is equivalent to a cylindrical solenoidal current (see Sec. 4.8). It follows from a comparison of Eqs. (5.16) and (4.63) that the current I in a solenoid equivalent to a given magnet can be determined from the equation

$$i = nI = cM \quad (5.17)$$

where n is the number of solenoid turns per unit length. The origin of the surface currents at the boundary between a magnetic and a vacuum can be explained by very simple reasoning. Figure 63 is a very schematic cross-sectional view of a magnet. The totality of molecular currents inside the magnet can be schematically depicted

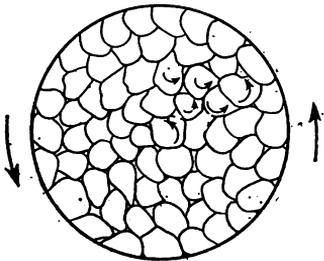


Fig. 63

as a totality of identical currents flowing around each cell (molecule) of the magnet in the same direction, for instance counterclockwise. The currents of adjacent molecules compensate one another inside the magnet, whereas on the surface they combine into a ring current flowing around the magnet in a circle. (The difference between the field inside a magnet and that inside an equivalent solenoid will be treated in Sec. 5.15.)

For reasoning more precisely in a quantitative respect, let us consider a thin layer of a magnet confined between two planes per-

pendicular to its axis. If the height of this layer is l , its cross-sectional area is S and its volume is $V = lS$, and the sum of the magnetic moments of the molecules in it is $\sum \mu$, then

$$M = \frac{1}{V} \sum \mu$$

Using Eq. (4.122) to express the magnetic moment of each molecule through the intensity and the area of the corresponding molecular current, we get

$$M = \frac{1}{cV} I \sum S$$

where we have assumed for simplicity that all the molecular currents are line ones and are identical. The magnetic moment of the molecules will not change if we change the current I and the area of the molecular currents so that their product remains constant. Let us choose these quantities in accordance with Fig. 63 so that adjacent molecular currents are directly next to one another. Hence, $\sum S$ will numerically equal the cross-sectional area of the magnet S , and

$$M = \frac{IS}{cV} = \frac{IS}{cSl} = \frac{1}{c} \frac{I}{l}$$

The ratio of the current I flowing along the surface of the layer being considered to the height of this layer l by definition equals the surface density of the current i . Thus, the last equation coincides with Eqs. (5.16) and (5.17).

5.3 Differential Equations of the Macroscopic Magnetic Field in Magnetics. Magnetic Field Intensity in Magnetics and Magnetic Induction Vector

1. Our task in this section will be to derive an equation for the mean *macroscopic* values of the quantities \mathbf{H} and \mathbf{j} characterizing a field by averaging the equations for a true *microscopic* field. We shall proceed from the assumption that the basic equations of the magnetic field of steady currents (4.36) and (4.38) strictly hold for a true microscopic field, i.e.

$$\operatorname{div} \mathbf{H}_{\text{micro}} = 0 \quad \text{and} \quad \operatorname{curl} \mathbf{H}_{\text{micro}} = \frac{4\pi \mathbf{j}_{\text{micro}}}{c}$$

if by $\mathbf{j}_{\text{micro}}$ we understand exactly the “microscopic” value of the current density at a given point of a field. Our task will consist in establishing the equations determining the mean macroscopic value of the vector $\mathbf{H}_{\text{micro}}$ in an infinitely small volume (see Sec. 2.6), which we shall denote by $\bar{\mathbf{H}}_{\text{micro}}$. Since according to Eq. (2.45) the mean

value of derivatives with respect to coordinates equals the derivatives of the mean value of the quantity being differentiated, then it follows from the microscopic equations of a field that

$$\operatorname{div} \bar{\mathbf{H}}_{\text{micro}} = 0 \quad (5.18)$$

$$\operatorname{curl} \bar{\mathbf{H}}_{\text{micro}} = \frac{4\pi \bar{\mathbf{j}}_{\text{micro}}}{c} \quad (5.19)$$

According to Eq. (5.1), the density of the currents in an arbitrary medium consists of that of conduction currents and molecular currents. The mean value $\bar{\mathbf{j}}_{\text{cond}}$, according to Eq. (5.7), is the usual density \mathbf{j} of the macroscopic current in the conductors, whereas the mean value $\bar{\mathbf{j}}_{\text{mol}}$, by Eq. (5.13), is expressed through the curl of magnetization. Thus,

$$\bar{\mathbf{j}}_{\text{micro}} = \bar{\mathbf{j}}_{\text{cond}} + \bar{\mathbf{j}}_{\text{mol}} = \bar{\mathbf{j}} + c \operatorname{curl} \mathbf{M} \quad (5.20)$$

Introducing this expression into Eq. (5.19), we obtain

$$\operatorname{curl} \bar{\mathbf{H}}_{\text{micro}} = \frac{4\pi}{c} \bar{\mathbf{j}} + 4\pi \operatorname{curl} \mathbf{M} \quad (5.21)$$

Equations (5.18) and (5.21) are the fundamental differential equations of a magnetic field in an arbitrary magnetic medium. 2. The intensity of a macroscopic electric field, by definition, equals the mean intensity $\bar{\mathbf{E}}_{\text{micro}}$ of a microscopic field (see Sec. 2.7). It would be absolutely natural to determine the intensity of a macroscopic magnetic field in a similar way.

Historically, however, a different definition took root that is very natural from the viewpoint of the notion of the existence of magnetic charges in molecules (see Sec. 5.14), namely, the intensity of the macroscopic field in magnetics, which in the following we shall simply designate by the letter \mathbf{H} , is determined by the following relationship:

$$\mathbf{H} = \bar{\mathbf{H}}_{\text{micro}} - 4\pi \mathbf{M} \quad (5.22)$$

The mean value of the intensity of a microscopic field is called the vector of *magnetic induction* and is designated by the letter \mathbf{B} :

$$\mathbf{B} = \bar{\mathbf{H}}_{\text{micro}} \quad (5.23)$$

Equation (5.21) can be written as follows:

$$\operatorname{curl} (\bar{\mathbf{H}}_{\text{micro}} - 4\pi \mathbf{M}) = \frac{4\pi}{c} \bar{\mathbf{j}}$$

so that the use of the new symbols transforms it to

$$\operatorname{curl} \mathbf{H} = \frac{4\pi}{c} \bar{\mathbf{j}} \quad (5.24)$$

while Eqs. (5.18) and (5.22) become

$$\operatorname{div} \mathbf{B} = 0 \quad (5.25)$$

$$\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M} \quad (5.26)$$

Equations (5.24), (5.25), and (5.26) are a system of the fundamental differential equations of a field that must be supplemented only by equations establishing the relationship between the magnetization \mathbf{M} and field intensity \mathbf{H} . We shall consider the relationship between these quantities in the following section. When magnetized media are absent, we have $\mathbf{M} = 0$; \mathbf{H} and \mathbf{B} coincide with each other, and Eqs. (5.24) and (5.25) coincide with the earlier derived equations for a magnetic field in a vacuum (4.36) and (4.38).

Throughout the following, unless otherwise indicated, by the magnetic field intensity \mathbf{H} we shall understand a vector determined by relationship (5.22) and satisfying Eqs. (5.24) and (5.26).

Upon a formal comparison of the equations for an electric and a magnetic fields

$$\operatorname{div} \mathbf{D} = 4\pi\rho, \quad \operatorname{curl} \mathbf{E} = 0, \quad \mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$$

$$\operatorname{div} \mathbf{B} = 0, \quad \operatorname{curl} \mathbf{H} = \frac{4\pi}{c} \mathbf{j}, \quad \mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$$

the impression appears that the quantities \mathbf{E} and \mathbf{H} , on one hand, and \mathbf{D} and \mathbf{B} , on the other, are similar, whereas in essence, as we have just indicated, the analogue of the intensity of a macroscopic electric field \mathbf{E} is the magnetic induction \mathbf{B} (equal to the mean intensity of a microscopic magnetic field), while the analogue of the electric displacement \mathbf{D} is the intensity of a macroscopic magnetic field \mathbf{H}^* . This tells, for example, in that, as we shall see in Sec. 5.6, the forces acting on electric currents are determined by the magnetic induction \mathbf{B} , whereas the forces acting on electric charges are determined by the electric field intensity \mathbf{E} .

3. We shall note in concluding that in the customary symbols the equation $\mathbf{H}_{\text{micro}} = \operatorname{curl} \mathbf{A}$ [cf. Eq. (4.29)] which makes it possible to reduce the definition of the field intensity to calculation of the vector potential, is written as follows:

$$\mathbf{B} = \operatorname{curl} \mathbf{A} \quad (5.27)$$

where by \mathbf{A} one must naturally understand the mean macroscopic value of the vector potential. Equation (5.25) can be considered as a direct corollary of Eq. (5.27). The macroscopic field intensity vector

* This circumstance manifests itself, in particular, in that upon the four-dimensional formulation of the equations of an electromagnetic field in the theory of relativity, it becomes necessary to combine, on one hand, the vectors \mathbf{E} and \mathbf{B} and, on the other hand, \mathbf{H} and \mathbf{D} in pairs in two four-dimensional tensors of the second rank.

\mathbf{H} , however, is not a solenoidal one and therefore cannot be expressed by the curl of an auxiliary vector potential.

Finally, the differential equation for the macroscopic value of the vector potential

$$\nabla^2 \mathbf{A} = -\frac{4\pi}{c} \bar{\mathbf{j}}_{\text{micro}} = -\frac{4\pi}{c} (\mathbf{j} + c \text{curl } \mathbf{M}) \quad (5.28)$$

can be obtained either by averaging Eq. (4.33) or directly from Eq. (5.11) in the same way as in Sec. 4.5 Eq. (4.33) was obtained from Eq. (4.28).

4. As regards the *boundary conditions* for a magnetic field, they directly follow from the differential equations of the field by a limit transition from the case of thin layers of a finite thickness in which the space currents \mathbf{j} and magnetization \mathbf{M} remain finite and continuous, to the limiting case of infinitely thin discontinuity surfaces.

Thus, according to Eq. (1.30), we get the following equation for the normal components of the magnetic induction vector \mathbf{B} from Eq. (5.25):

$$\text{Div } \mathbf{B} = B_{2n} - B_{1n} = 0 \quad (5.29)$$

which is a generalization of Eq. (4.46) for the case when magnetics are present. As regards the field intensity \mathbf{H} , the differential equation for this vector retains the same form for magnetic media [Eq. (5.24)] as for a vacuum [Eq. (4.38)] and, consequently, leads to the same boundary condition for its tangential components [cf. Eqs. (4.51) and (4.55)]:

$$\text{Curl } \mathbf{H} = [\mathbf{n} \cdot \mathbf{H}_2 - \mathbf{H}_1] = \frac{4\pi}{c} \mathbf{i} \quad (5.30)$$

In the absence of surface currents ($\mathbf{i} = 0$), this equation can also be written in the following form [cf. Eq. (4.53)]:

$$H_{2t} = H_{1t} \quad (5.31)$$

5.4 Dependence of Magnetization on Magnetic Field Intensity. Para-, Dia-, and Ferromagnetics

1. For the system of equations for a field derived in the preceding section to become complete, it must be supplemented with a definite relationship between the magnetization of the medium \mathbf{M} and the magnetic field intensity \mathbf{H} . The polarization of dielectrics \mathbf{P} is proportional to the intensity \mathbf{E} of the electric field in them [Eq. (2.15)]. Magnetics, however, can be divided into three classes according to the nature of the dependence of their magnetization \mathbf{M} on the intensity of the magnetic field \mathbf{H} .

In the so-called paramagnetics and diamagnetics, \mathbf{M} is proportional to \mathbf{H} *:

$$\mathbf{M} = \chi \mathbf{H} \quad (5.32)$$

The value of the proportionality factor χ called the (bulk) *magnetic susceptibility* and quite similar to the polarizability α of a dielectric** depends on the physicochemical properties of a given magnetic. The susceptibility χ of *paramagnetic bodies* has (like the polarizability of dielectrics) a *positive* value, i.e. the direction of magnetization \mathbf{M} coincides with the direction of the field \mathbf{H} . *Diamagnetics* differ in that their susceptibility χ is *negative*, i.e. the direction of magnetization of diamagnetics is *opposite* to the direction of the field \mathbf{H} magnetizing them. The explanation of this property of diamagnetics seeming to be a paradox will be given on a later page.

Finally, the magnetization \mathbf{M} of the third class of magnetism, called *ferromagnetics* in accordance with the Latin name of their most important representative—iron (ferrum), not only fails to be proportional to the field intensity \mathbf{H} , but in general is not related to it by a more or less simple functional relationship. For example, in ferromagnetics, the so-called *hysteresis* is observed, i.e. the dependence of the magnetization on the *prehistory* of the given specimen of a ferromagnetic substance. This means that the value of the magnetization \mathbf{M} of a ferromagnetic depends not only on the intensity \mathbf{H} of the magnetic field in it, but also on whether or not the given specimen of a ferromagnetic was previously in a magnetic field, on the numerical value and direction of the intensity of this field, etc. *Remanent* or *permanent* magnetism of ferromagnetics is closely related to hysteresis. As mentioned above, it consists in that after the vanishing of the external magnetizing field ferromagnetics can retain magnetization and as a result continue to induce their “own” magnetic field (permanent magnets).

Thus, although we can formally define the susceptibility χ of ferromagnetics as the ratio of the magnetization \mathbf{M} to the field intensity \mathbf{H} [Eq. (5.32)], in ferromagnetics, however (unlike diamagnetics and paramagnetics), this coefficient is not a material constant depending only on the chemical composition of a body, its temperature and other physical conditions. A tremendous number of experimental investigations have been devoted to determining the very intricate dependence of χ on the field intensity \mathbf{H} , on the prehistory of the given

* For *anisotropic* (crystalline) dia- and paramagnetics, which we shall not consider in this book, the relationship between the components of the magnetization vector \mathbf{M} and those of the vector \mathbf{H} remains linear, but the vectors \mathbf{M} and \mathbf{H} , generally speaking, do not coincide in their direction, and Eq. (5.32) has to be replaced with a more complicated relationship similar to Eq. (2.16).

** It will be shown at the end of this section that, strictly speaking, the quantity $\chi/(1 + 4\pi\chi) = \chi/\mu$ instead of the susceptibility χ corresponds to the polarizability of dielectrics α [see Eq. (5.36)].

specimen of a ferromagnetic substance, etc. We shall indicate as an example that the susceptibility χ of soft iron grows from 5-10 units to several hundred units (χ is a dimensionless number) with an increasing field, and then drops again. The value of this coefficient depends very appreciably on the method of fabrication and heat treatment of the specimen of the metal, and also on the presence of negligible chemical impurities in it.

2. Apart from all other circumstances, already the single fact of the non-linear dependence of the magnetization \mathbf{M} on the field intensity introduces exceedingly great complications even into the formal phenomenological theory of the field in ferromagnetics because non-linearity of the field equations results in violation of the principle of superposition of fields*.

Owing to the intricate nature of the magnetic properties of ferromagnetics, we shall limit ourselves in the next few sections to a treatment of dia- and paramagnetics and postpone the consideration of ferromagnetics to the end of this chapter. Therefore, up to Sec. 5.12 inclusively, unless explicitly otherwise indicated, we shall understand magnetics to mean only para- and diamagnetics. In a number of matters (for example in the matter of the ponderomotive forces acting on magnetics—Sec. 5.7), for simplification of our reasoning we shall be able to take advantage of the circumstance that only ferromagnetics have an appreciable susceptibility (the value of χ for them may reach hundreds of units in them), whereas the susceptibility of para- and especially of diamagnetics is exceedingly small**.

3. Introducing Eq. (5.32) into Eq. (5.26), we get

$$\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M} = (1 + 4\pi\chi)\mathbf{H} \quad (5.33)$$

By analogy with the permittivity ϵ , let us introduce the *permeability* of a medium μ which is determined by the equation

$$\mu = 1 + 4\pi\chi \quad (5.34)$$

Equation (5.33) thus becomes

$$\mathbf{B} = \mu\mathbf{H} \quad (5.35)$$

It follows from what has been said above about the susceptibility χ that in diamagnetics $\mu < 1$, in a vacuum $\mu = 1$, and, finally, in paramagnetics $\mu > 1$.

* The intensity of the field of several field sources equals the sum of the intensities of the fields induced by each of these sources separately (the principle of superposition of fields) only if the equations of the fields are linear.

** The ratio of the susceptibility χ to the density of a body ρ (the so-called susceptibility of a unit mass) approximately ranges from 10^{-4} to 10^{-5} for paramagnetics, and from 10^{-6} to 10^{-7} cgs units for diamagnetics. For bismuth, which has the highest diamagnetic properties of all substances, χ/ρ approximately equals 2×10^{-5} .

For the likening of superconductors to an ideal diamagnetic, i.e. to a body with a permeability $\mu = 0$ and a susceptibility $\chi = -1/4\pi$, see the footnote on p. 217.

It should be noted that from Eqs. (5.32) and (5.35) we get

$$\mathbf{M} = \frac{\chi}{\mu} \mathbf{B} \tag{5.36}$$

Since it is exactly \mathbf{B} , and not \mathbf{H} that is the mean intensity of a microscopic field [Eq. (5.23)], then the coefficient χ/μ has a simpler physical meaning than the coefficient χ (particularly, it is precisely χ/μ and not χ that corresponds to the polarizability of dielectrics α in an electric field).

5.5 Complete System of Equations for the Field of Steady Currents. Homogeneous Magnetic Medium

1. The system of equations obtained in Sec. 5.4:

$$\left. \begin{aligned} \text{curl } \mathbf{H} &= \frac{4\pi}{c} \mathbf{j}, \text{ div } \mathbf{B} = 0 \\ \mathbf{B} &= \mu \mathbf{H} \\ \text{Div } \mathbf{B} &= 0, \text{ Curl } \mathbf{H} = \frac{4\pi}{c} \mathbf{i} \end{aligned} \right\} \tag{C}$$

is a *complete* system of equations of a constant magnetic field in an arbitrary (but not ferromagnetic) medium. This means that a magnetic field is determined *unambiguously* by the system (C) if only we know the distribution of the space and surface electric currents \mathbf{j} and \mathbf{i} and the value of the permeability μ [or, which according to Eq. (5.34) is the same, the value of the susceptibility χ] at each point of the medium, and if at infinity the condition (4.59) is complied with: HR^2 remains finite when $R \rightarrow \infty$.

The proof of the completeness of the system (C) is similar to those of the completeness of the systems of equations (A) and (B) set out in Secs. 2.3 and 4.8, and we invite our reader to verify this.

It follows from the unambiguity of the system (C), particularly [cf. the similar derivation from the system (A') in Sec. 2.3], that in the absence of conduction currents (and in the absence of ferromagnetics) the constant magnetic field identically equals zero. Hence, the presence of (non-ferromagnetic) magnetics only modifies the field of the currents; in the absence of the latter the magnetization of magnetics cannot retain a value that is constant in time and differs from zero—it drops to zero, and the magnetic field vanishes.

When $\mu = 1$, the system (C), as should be expected, completely coincides with the equations for a magnetic field in a vacuum [system (B), Sec. 4.8].

2. Let us consider a homogeneous magnetic medium (μ and χ are constant) in which the system of equations (C) acquires the form

$$\left. \begin{aligned} \text{curl } \mathbf{H} &= \frac{4\pi}{c} \mathbf{j} \\ \text{div } \mathbf{H} &= 0 \end{aligned} \right\} \quad (5.37)$$

because when μ is constant

$$\text{div } \mathbf{H} = \text{div } \frac{\mathbf{B}}{\mu} = \frac{1}{\mu} \text{div } \mathbf{B} = 0$$

(We have not written out the boundary conditions unambiguously following from the differential equations for brevity's sake.)

This system of equations for the vector \mathbf{H} completely coincides with the system (B) of equations for the magnetic field of currents in the absence of magnetics (Sec. 4.8): the constant μ falls out from it. Thus, *when the entire field is filled with a homogeneous magnetic the intensity of the magnetic field of currents does not change*, remaining the same as in the absence of the magnetics (i.e. as when $\mu = 1$); the magnetic induction \mathbf{B} , however, grows μ times. This is how the correspondence between the *intensity* of a magnetic field \mathbf{H} and the *displacement* of an electric field \mathbf{D} noted in Sec. 5.3 manifests itself: the electric displacement \mathbf{D} also does not change if with a given distribution of the (free) charges the entire field is filled with a homogeneous dielectric; the intensity of the electric field, however, grows $1/\epsilon$ times* (see Sec. 2.4).

3. The vector potential of a magnetic field in the general case is determined by Eqs. (5.27), (4.35), (5.28), and (5.11), which we shall compare here once more:

$$\left. \begin{aligned} \mathbf{B} &= \text{curl } \mathbf{A}, \quad \text{div } \mathbf{A} = 0, \quad \nabla^2 \mathbf{A} = -\frac{4\pi}{c} (\mathbf{j} + c \text{curl } \mathbf{M}) \\ \mathbf{A} &= \frac{1}{c} \int \frac{(\mathbf{j} + c \text{curl } \mathbf{M})}{R} dV + \\ &+ \frac{1}{c} \int \frac{[\mathbf{n} \cdot \mathbf{M}_2 - \mathbf{M}_1]}{R} dS \end{aligned} \right\} \quad (5.38)$$

In the last two of these equations, we can express \mathbf{M} through \mathbf{H} with the aid of Eq. (5.32). For a non-homogeneous medium, however ($\mu \neq \text{const}$), this leads to quite complicated expressions. In addition, the continuity of the vectors \mathbf{H} and \mathbf{B} , and at the same time the continuity of the derivatives of the vector \mathbf{A} , generally speaking, is violated at the interface of media having different permeabilities μ .

* When comparing \mathbf{H} with \mathbf{D} and \mathbf{B} with \mathbf{E} we must evidently compare μ with $1/\epsilon$ instead of with ϵ .

For this reason, we shall not stop to consider the general case here and shall limit ourselves only to a treatment of the vector potential for the *field of currents in a homogeneous medium* (μ and χ are constant). In this case, the vector \mathbf{M} is continuous, and the surface integral in the last of the equations (5.38) vanishes. Further, taking into account Eqs. (5.32), (5.24), and (5.34), we get

$$\mathbf{j} + c \operatorname{curl} \mathbf{M} = \mathbf{j} + c\chi \operatorname{curl} \mathbf{H} = \mathbf{j} + 4\pi\chi\mathbf{j} = \mu\mathbf{j}$$

and, consequently,

$$\mathbf{A} = \frac{\mu}{c} \int \mathbf{j} \frac{dV}{R}, \quad \nabla^2 \mathbf{A} = -\frac{4\mu\pi}{c} \mathbf{j} \quad (5.39)$$

Thus, the vector potential of the currents in a homogeneous magnetic medium is μ times greater than in a vacuum. This completely corresponds to the field intensity of currents in a homogeneous medium being independent of the permeability of the medium mentioned above. Indeed, on the basis of Eqs. (5.35) and (5.27), we get

$$\mathbf{H} = \frac{1}{\mu} \mathbf{B} = \frac{1}{\mu} \operatorname{curl} \mathbf{A}$$

In the field of line currents (i.e. in essence at distances from the currents that are great in comparison with their cross section), the first of Eqs. (5.39) acquires the form [cf. Eq. (4.30)] ($\mu = \text{const}$)

$$\mathbf{A} = \frac{\mu I}{c} \oint \frac{ds}{R} \quad (5.40)$$

In general, Eqs. (5.39) and (5.40) strictly hold only when μ is constant not only in the entire space surrounding the conductors along which the current flows, but also if these conductors themselves have the same permeability μ .

Problem 32. Show that for a current flowing along an infinite straight cylindrical conductor the first of equations (5.39) strictly holds even if the permeability of the conductor μ' differs from that of the surrounding medium μ . The conductor may be confined in a cylindrical shell having an arbitrary permeability μ'' (insulation).

5.6 Mechanical Forces Acting on Currents in a Magnetic Field. Interaction of Currents

1. The density of the forces acting on currents in a magnetic field in the absence of magnetics is determined by Eq. (4.15):

$$\mathbf{f} = \frac{1}{c} [\mathbf{j}\mathbf{H}]$$

Owing to the atomistic structure of conductors, the true microscopic field $\mathbf{H}_{\text{micro}}$ changes in them quite considerably even over atomic distances. When using Eq. (4.15) in the microscopic theory, however, we must evidently understand \mathbf{H} in it to be the *mean* value of the microscopic field. We saw that in magnetic media this mean value $\overline{\mathbf{H}}_{\text{micro}}$ is customarily denoted by the symbol \mathbf{B} and is called the *induction* of the magnetic field. Hence, if we take into account that conductors through which a current flows are capable of becoming magnetized, then Eq. (4.15) must be written as follows*:

$$\mathbf{f} = \frac{1}{c} [\mathbf{j} \overline{\mathbf{H}}_{\text{micro}}] = \frac{1}{c} [\mathbf{j} \mathbf{B}] \quad (5.41)$$

When $\mu = 1$, the induction \mathbf{B} equals \mathbf{H} so that Eq. (5.41) coincides with the previous equation (4.15).

Thus, the forces acting on a current in a magnetic field are proportional to the *induction* and not to the intensity of this field.

Turning from currents having a finite cross section to line ones, we can easily see that an element ds of length of a line current is acted upon by the force [cf. Eq. (4.1)]

$$\mathbf{F} = \frac{1}{c} I [ds \mathbf{B}] \quad (5.42)$$

2. Let us now consider the forces acting on an arbitrary closed current as a whole. In determining these forces, we can repeat all our reasoning in Secs. 4.9 to 4.11 where no account was taken of magnetization, the only change being that in accordance with the transition from Eq. (4.15) to Eq. (5.41) we shall have to substitute \mathbf{B} for \mathbf{H} in all the formulas of these sections. This relates, particularly, to the determination of the magnetic flux Φ through a current contour L ; in deriving Eq. (4.66) for Φ we used Eq. (4.29):

$$\mathbf{H} = \text{curl } \mathbf{A}$$

which for magnetic media must be replaced by Eq. (5.27):

$$\mathbf{B} = \text{curl } \mathbf{A}$$

Accordingly, instead of Eq. (4.66), we get a similar expression for the *flux of magnetic induction* through the surface S resting on the contour L ; we shall designate this flux by Ψ to distinguish it from Φ :

$$\Psi = \int_S B_n dS = \int_S \text{curl}_n \mathbf{A} dS = \oint_L A_s ds \quad (5.43)$$

* The derivation of Eq. (5.41) given here has the shortcoming that we take no account in it of the possible difference between the mean value of the product $[\overline{\mathbf{j}_{\text{cond}} \mathbf{H}_{\text{micro}}}]$ and the product of the mean values of the multipliers in $[\mathbf{j} \mathbf{B}]$. A stricter proof of Eq. (5.41) will be given in Sec. 6.8.

This formula shows that the flux of magnetic induction through an arbitrary surface S depends only on the position and shape of the contour L of this surface and has the same value for all the surfaces resting on the same contour. When $\mu = 1$, we get $\mathbf{B} = \mathbf{H}$ and $\Phi = \Psi$, and Eq. (5.43) coincides with the previous equation (4.66).

The potential function of currents in a magnetic field when the magnetic properties of the medium are taken into account is expressed by the formula

$$U = - \frac{1}{c} I\Psi \tag{5.44}$$

which is obtained from the previous formula (4.68) by substituting Ψ for Φ . This function can be used to determine both the generalized ponderomotive forces of a magnetic field Θ_i and the work of these forces δW [see Eqs. (4.69) and (4.70)]:

$$\Theta_i = - \frac{\partial U}{\partial q_i} \text{ and } \delta W = -(\delta U) \tag{5.45}$$

3. All the formulas given in this section can obviously be applied to any non-homogeneous magnetic medium. Let us now consider the ponderomotive interaction of two line currents, assuming for simplicity that the entire field is filled with a medium that is homogeneous in the magnetic respect ($\mu = \text{const}$). In this case, the vector potential of the field \mathbf{A} is expressed by Eq. (5.40). Repeating the calculations of Secs. 4.10 and 4.11 and taking into consideration that \mathbf{A} is proportional to μ , we arrive at the following formulas ($i, k = 1, 2$):

$$\Psi_{ik} = \frac{1}{c} L_{ik}I_k \tag{5.46}$$

$$\left. \begin{aligned} L_{21} = L_{12} &= \mu \oint_{L_1} \oint_{L_2} \frac{ds_1 ds_2}{R} \text{ [cf. Eq. (4.78)]} \\ L_{11} &= \frac{\mu}{I_1^2} \iint_{V_1} \frac{j_1 j_1' dV dV'}{R} \text{ [cf. Eq. (4.92)]} \end{aligned} \right\} \tag{5.47}$$

The potential function of the currents U as previously will be determined by Eq. (4.98):

$$\begin{aligned} U &= U_{11} + U_{12} + U_{22} = \\ &= - \frac{1}{c^2} \left(\frac{1}{2} L_{11}I_1^2 + L_{12}I_1I_2 + \frac{1}{2} L_{22}I_2^2 \right) \end{aligned} \tag{5.48}$$

Thus, in a homogeneous magnetic medium, the mutual inductance and self-inductance of currents L_{12} and L_{11} , and, consequently, the potential function U and the ponderomotive forces of interaction of

the currents Θ_i are directly proportional to the permeability of the medium μ .

4. When a medium is not homogeneous in a magnetic respect ($\mu \neq \text{const}$), the vector potential of the currents \mathbf{A} cannot be expressed by a simple formula such as Eq. (5.40), and, therefore, Eqs. (5.47) can no longer be applied. If there are no ferromagnetics in a field, however, then the vector potential of the arbitrary current I will evidently be proportional to this current I as previously*. Consequently, the flux of magnetic induction $\Psi_{21} = \int \mathbf{A}_2 \cdot d\mathbf{s}_1$ sent by the current I_2 through the contour of the current I_1 [cf. Eq. (4.75)] can as previously be expressed in the form of the product:

$$\Psi_{21} = \frac{1}{c} I_2 L_{21}$$

The mutual inductance will depend only on the geometrical configuration of the currents, their distribution over the cross sections of the conductors, etc., but not on the current in them.

Thus, unlike Eq. (5.47), Eq. (5.46) and also, as we can easily see, Eq. (5.48) remain true in an arbitrary magnetic medium *in the absence of ferromagnetics*, since when deriving them we assumed that the vector potential is proportional to the current. It should be noted that when deriving Eq. (4.100), i.e.

$$U = - \frac{1}{2c} \int \mathbf{A} \mathbf{j} \, dV$$

this assumption was not made.

5. In concluding, we shall note the following. The values of the vectors \mathbf{H} and \mathbf{B} characterizing a magnetic field can be determined experimentally by measuring the ponderomotive forces acting in this field on current-carrying conductors and permanent magnets (for instance on a magnetic pointer which in certain conditions can be considered as a magnetic dipole).

For this purpose, it is most convenient to use either Eq. (5.41) relating the force \mathbf{f} acting on an element of current to the induction \mathbf{B} or Eq. (4.127) relating the moment of the couple acting on a magnetic dipole having the moment $\boldsymbol{\mu}$ to the field intensity \mathbf{H} ** . Care must be taken during such measurements to see that the introduction of a measuring instrument (a current or pointer) does not cause an appreciable change in the field being measured.

* In ferromagnetic media, proportionality is violated between the vector \mathbf{B} determined by the curl of the vector potential \mathbf{A} and the intensity of the field \mathbf{H} whose curl is determined by the density of the currents \mathbf{j} .

** We shall see in Sec. 5.13 that the forces acting on elongated permanent bar magnets are determined not by the induction \mathbf{B} , but by the intensity \mathbf{H} of the external field.

Measuring instruments can be introduced into a field only if the portion of it being studied is filled with a gaseous or liquid medium. If the medium is solid, the appropriate openings must be made in it to make measurements possible. The field \mathbf{H}' inside of these openings will naturally differ from the field \mathbf{H} and \mathbf{B} in adjacent points of the solid medium.

It is not difficult, however, to establish a relationship between \mathbf{H}' on one hand and \mathbf{H} and \mathbf{B} on the other (See Problem 33).

Problem 33. Show on the basis of Eqs. (5.29) and (5.31) that the intensity of the field \mathbf{H}' in the middle part of a long and narrow slot made in a solid magnetic equals the *intensity* of the field \mathbf{H} in the points of the magnetic adjoining the slot if the latter is *parallel* to the vector \mathbf{H} , and that \mathbf{H}' equals the *induction* \mathbf{B} at adjacent points of the magnetic if this slot is *perpendicular* to the vector \mathbf{H} . Compare with Problem 17 (Sec. 2.3).

5.7 Ponderomotive Forces Acting on Magnetics in a Magnetic Field

1. The mechanical forces acting on magnetics in a magnetic field should consist of the forces acting on the molecular currents. According to Eqs. (4.140) and (4.128), the force acting on a system of current loops characterized by the magnetic moment $\boldsymbol{\mu}$ is

$$\mathbf{F} = \nabla(\boldsymbol{\mu}\mathbf{H}) = \boldsymbol{\mu} \nabla \cdot \mathbf{H} + [\boldsymbol{\mu} \text{ curl } \mathbf{H}] \quad (5.49)$$

For definiteness, we shall assume that a magnetic consists of separate molecules. (Otherwise it will be necessary to only slightly alter our reasoning, the result will remain unchanged.) Applying Eq. (5.49) to the separate molecules of a magnetic, we must evidently understand $\boldsymbol{\mu}$ to stand for the magnetic moment of a molecule, and \mathbf{H} to stand for the intensity of the true microscopic field $\mathbf{H}_{\text{micro}}$ at the place where a molecule is. The magnetization of para- and diamagnetics in all the fields available to us is so weak (see footnote** on p. 316) that we have the right to disregard in them the difference between the mean value of the *field acting on a molecule* (magnetic dipole) and the mean value of \mathbf{B} of the field $\mathbf{H}_{\text{micro}}$ over all the points of an infinitely small volume (cf. Sec. 2.9). Therefore, the *mean* value of the force acting on a separate molecule of a magnetic equivalent to an elementary current will be determined by the *mean* intensity of the microscopic field $\bar{\mathbf{H}}_{\text{micro}}$, which by Eq. (5.23) equals the magnetic induction vector \mathbf{B} :

$$\mathbf{F} = \boldsymbol{\mu} \nabla \cdot \mathbf{B} + [\boldsymbol{\mu} \text{ curl } \mathbf{B}] \quad (5.50)$$

where we have omitted the symbol of the mean value over \mathbf{F} .

The density \mathbf{f} of the ponderomotive forces acting on a magnetic, i.e. the force acting on a unit volume of a magnetic, will equal the

sum of the forces acting on the separate molecules in the unit volume:

$$\mathbf{f} = \sum \mathbf{F} = \sum \boldsymbol{\mu} \nabla \cdot \mathbf{B} + \sum [\boldsymbol{\mu} \text{curl } \mathbf{B}] = (\sum \boldsymbol{\mu}) \nabla \cdot \mathbf{B} + [\sum \boldsymbol{\mu} \text{curl } \mathbf{B}]$$

Using Eq. (5.3), we finally get

$$\mathbf{f} = \mathbf{M} \nabla \cdot \mathbf{B} + [\mathbf{M} \text{curl } \mathbf{B}]^* \quad (5.51)$$

Finally, if conduction currents \mathbf{j} also flow through the volume of the magnetic being considered in addition to the molecular currents, then \mathbf{f} is determined by the sum of Eqs. (5.41) and (5.51):

$$\mathbf{f} = \frac{1}{c} [\mathbf{jB}] + \mathbf{M} \nabla \cdot \mathbf{B} + [\mathbf{M} \text{curl } \mathbf{B}] \quad (5.52)$$

2. In equations (5.51) and (5.52), we can express the magnetization \mathbf{M} through the induction \mathbf{B} . According to Eqs. (5.36) and (5.34),

$$\mathbf{M} = \frac{\chi}{\mu} \mathbf{B} = \frac{\mu - 1}{4\pi\mu} \mathbf{B} \quad (5.53)$$

(here μ is the permeability).

Inserting this expression into Eq. (5.51) and using Eq. (A.47), we get

$$\mathbf{f} = \frac{\mu - 1}{4\pi\mu} \{\mathbf{B} \nabla \cdot \mathbf{B} + [\mathbf{B} \text{curl } \mathbf{B}]\} = \frac{\mu - 1}{8\pi\mu} \nabla B^2 \quad (5.54)$$

in the absence of conduction currents in the element of the medium being considered, whereas when $\mathbf{j} \neq 0$ we have

$$\mathbf{f} = \frac{1}{c} [\mathbf{jB}] + \frac{\mu - 1}{8\pi\mu} \nabla B^2 \quad (5.55)$$

Expression (5.54) is quite similar (if we do not count the factor μ in the denominator) to Eq. (2.95) determining the density of the forces acting on dielectrics $\left(\mathbf{f} = \frac{\epsilon - 1}{8\pi} \nabla E^2\right)$. The quantity $\epsilon - 1$ is always positive, however, whereas $\mu - 1 = 4\pi\chi$ is positive only in paramagnetics, while in diamagnetics $\mu - 1 < 0$. Consequently, the force (5.55) carries paramagnetic substances into the regions where the field induction \mathbf{B} is maximum and, conversely, tends to remove diamagnetic substances from these regions (cf. Sec. 2.13). Since the intensity of the field of a conventional bar magnet grows when we approach its pole, then paramagnetics (for example copper) should be attracted by the magnet, and diamagnetics (for example bismuth) repelled by it.

* It should be noted that when deriving this formula we disregarded the difference between the mean value of vector expressions such as $\overline{\boldsymbol{\mu} \nabla \cdot \mathbf{H}_{\text{micro}}}$ and the corresponding expressions formed from the mean values of the vectors $\overline{\boldsymbol{\mu}}$ and $\overline{\mathbf{H}_{\text{micro}}}$.

Owing to the very low susceptibility of diamagnetic substances (see p. 316), the forces acting on diamagnetics are, generally speaking, very low. Nevertheless the majority of the experimental methods of determining the values of the susceptibility χ and the permeability μ for para- and diamagnetics are based on the measurement of ponderomotive forces (5.54) acting on these bodies in a magnetic field.

It must be noted that many different expressions for \mathbf{f} that differ from Eq. (5.54) are encountered in publications on the subject. Since the permeability μ differs very little from unity in dia- and paramagnetics, however, all these expressions are approximately equivalent to one another. In Sec. 6.8, we shall give a strict derivation of the value of the ponderomotive forces in magnetics from the expression for the energy of a magnetic field; this derivation will also contain the conditions for applying the formulas of the present section.

5.8 Supplement to the Derivation of the Macroscopic Equations for a Magnetic Field in Magnetics.*

1. In Sec. 5.3, we derived differential equations of a macroscopic field by averaging the corresponding microscopic equations and used Eq. (5.13):

$$\bar{\mathbf{j}}_{\text{mol}} = c \text{curl } \mathbf{M}$$

which we obtained in Sec. 5.2 in a quite roundabout way. Owing to the fundamental importance of Eq. (5.13), we shall devote the present section to the direct derivation of this equation from the fundamental principles of the electron theory of magnetics.

2. Let us consider an infinitely small volume of a magnetic V confined by the surface S . By definition [see Eq. (2.44)]

$$\bar{\mathbf{j}}_{\text{mo}} V = \int_V \mathbf{j}_{\text{mol}} dV \tag{5.56}$$

where $\bar{\mathbf{j}}_{\text{mol}}$ stands for the mean macroscopic value of the density of molecular currents.

In the magnetic respect, a molecular current, like any elementary current, is completely characterized by giving its magnetic moment μ . Therefore, to simplify our calculations, we can assume that molecular currents are line ones and that the contour of each molecular current is a circle. If the radius of a ring current is a and its intensity is I , then by Eq. (4.122), the magnetic moment of the current is

$$\mu = \frac{IS}{c} = \frac{I\pi a^2}{c}$$

* This section may be omitted when reading the book the first time.

Neither a nor I will explicitly enter the final result of our calculations so that our special assumption will in essence not limit the general nature of our reasoning. In addition, the lawfulness of replacing the molecular currents with line ones can be strictly substantiated by resolving each molecular current into a totality of infinitely thin current filaments each of which is a line current.

Considering all the molecular currents to be line ones, we can write, according to Eq. (4.11),

$$\mathbf{j}_{\text{mol}} dV = I ds$$

Consequently, Eq. (5.56) becomes

$$\bar{\mathbf{j}}_{\text{mol}} V = \sum_V I ds = I \sum_V ds$$

where summation should be extended over all the elements of the molecular currents within the volume V (for simplicity we assume that all these currents are the same).

3. If an elementary current is *entirely* inside the volume V , then the vector sum of all its elements will equal zero (because the current is closed). Hence, the sum $\sum_V ds$ will be the sum of the elements of the

molecular currents that are *cut* by the surface S confining the volume V and are thus only partly inside of V . Further, if the surface S cuts a ring current along chord AB (Fig. 64), then the vector sum of the elements ds of this current inside the surface S will obviously equal the closing chord AB . Consequently, if this chord AB in its magnitude and direction equals \mathbf{s} , then $\sum ds = \mathbf{s}$. The magnitude and direction of the vector \mathbf{s} will depend on the direction of the plane of the current, characterized by the direction of the vector $\boldsymbol{\mu}$ perpendicular to it, and on the distance from the centre of the current to the surface S .

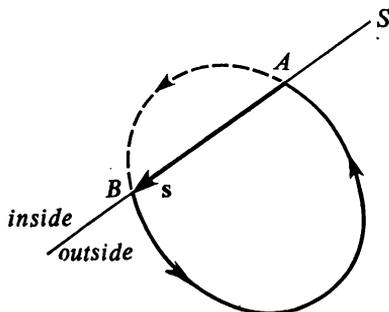


Fig. 64

Let us assume that $\boldsymbol{\mu}$ forms the angle α with an outward normal \mathbf{n} to the surface S and that the centre O of the current is at the distance $OO' = x$ from the surface S (Fig. 65). We shall consider x to be negative or positive depending on whether O is inside or outside

of S . We draw through OO' a plane perpendicular to AB and denote its point of intersection with AB by P . We shall denote by φ the dihedral angle OPO' between the surface S and the plane of the current. This angle does not exceed $\pi/2$ and equals either the angle

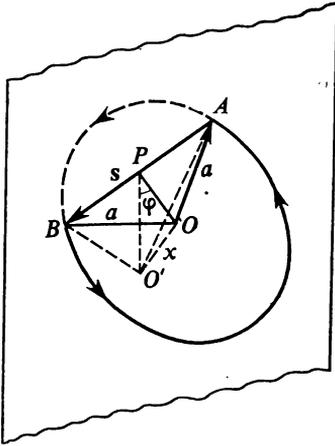


Fig. 65

α between the normals \mathbf{n} and $\boldsymbol{\mu}$ to the surface S and the plane of the current, or $\pi - \alpha$ (if $\alpha > \pi/2$). Since OO' is perpendicular to the surface, then

$$OP = \frac{|x|}{\sin \varphi}$$

Further, since OA and OB equal the radius of the current a , then

$$s = 2\sqrt{OA^2 - OP^2} = 2\sqrt{a^2 - \frac{x^2}{\sin^2 \varphi}} = \frac{2\sqrt{a^2 \sin^2 \varphi - x^2}}{\sin \varphi}$$

4. Let us now consider the complex of molecular currents, the direction of whose magnetic moment $\boldsymbol{\mu}$ is within the infinitely small solid angle $d\omega$. We shall denote the number of these currents in a unit volume of a magnetic by $N(\omega) d\omega$. The number of such currents in a layer with the thickness dx and the base dS at a distance x from the element dS of the boundary surface S will be

$$N(\omega) d\omega dS dx$$

If

$$-a \sin \varphi < x < +a \sin \varphi$$

then all these currents will be intersected by the boundary surface, the intersections being along equal and parallel chords s . Therefore

the absolute value of the sum of the terms of the sum $I \sum ds$ relating to these currents will equal

$$|I \sum ds| = I s N(\omega) d\omega dS dx = 2I \frac{\sqrt{a^2 \sin^2 \varphi - x^2}}{\sin \varphi} N(\omega) d\omega dS dx$$

Integrating this expression with respect to x from $-a \sin \varphi$ to $+a \sin \varphi$, we get the value of the corresponding sum for all the currents intersected by an element of the boundary surface dS , the direction of whose magnetic moment is within the angle $d\omega^*$:

$$\begin{aligned} |I \sum ds| &= 2IN(\omega) d\omega dS \int_{-a \sin \varphi}^{+a \sin \varphi} \frac{\sqrt{a^2 \sin^2 \varphi - x^2}}{\sin \varphi} dx = \\ &= N(\omega) d\omega dS \cdot \pi a^2 \sin \varphi = N(\omega) d\omega dS \cdot c\mu \sin \varphi \end{aligned} \quad (5.57)$$

To determine the direction of the vector $I \sum ds$ relating to the currents of the indicated direction, we shall note that the chord s of each current is both in the plane of the element dS and in the plane of the current itself and, consequently, is perpendicular to both \mathbf{n} and $\boldsymbol{\mu}$. Hence, s , and also $I \sum ds$, must be parallel to $\pm[\mathbf{n}\boldsymbol{\mu}]$. Inspection of Figs. 64 and 65 shows that the positive sign must be chosen here. Since, on the other hand,

$$|[\mathbf{n}\boldsymbol{\mu}]| = \mu |\sin \alpha| = \mu \sin \varphi$$

(because the angle $\varphi \leq \pi/2$ equals either α or $\pi - \alpha$), then Eq. (5.57) can be written in the vector form as follows:

$$I \sum ds = N(\omega) \xi d\omega \cdot dS c [\mathbf{n}\boldsymbol{\mu}] = c[dS\boldsymbol{\mu}] N\omega d\omega \quad (5.58)$$

where, as usual, we consider the vector dS to be directed along the outward normal \mathbf{n} .

To obtain the complete sum of all the elements ds of length of the currents cut off by the element of the boundary surface dS , it is obviously sufficient to integrate Eq. (5.58) with respect to ω . This yields

$$I \sum ds = c \int [dS \boldsymbol{\mu}] N(\omega) d\omega = c \left[dS \int \boldsymbol{\mu} N(\omega) d\omega \right] \quad (5.59)$$

The integral $\int \boldsymbol{\mu} N(\omega) d(\omega)$ in this expression is the vector sum of the moments of all the molecules in a unit volume of a magnetic and, consequently, according to Eq. (5.3) equals the magnetization of the magnetic \mathbf{M} . Thus,

$$I \sum ds = c [dS \cdot \mathbf{M}] \quad (5.60)$$

* We use the formula

$$\int \sqrt{b^2 - x^2} dx = \frac{1}{2} \left(b^2 \arcsin \frac{x}{b} + x \sqrt{b^2 - x^2} \right)$$

Finally, the complete sum of all the elements ds of length of the molecular currents cut off by all the elements of the closed surface S will be

$$\bar{\mathbf{j}}_{\text{mol}}V = I \sum_V ds = c \oint_S [d\mathbf{S} \cdot \mathbf{M}] \quad (5.61)$$

The last integral on the basis of Eq. (A.56) can be transformed into an integral over the volume V so that we finally get

$$\frac{1}{c} \bar{\mathbf{j}}_{\text{mol}} = \frac{1}{V} \oint_S [d\mathbf{S} \cdot \mathbf{M}] = \frac{1}{V} \int_V \text{curl } \mathbf{M}'_A dV \quad (5.62)$$

The right-hand side of this equation, according to Eq. (2.44), is the mean value of the curl of \mathbf{M} in an infinitely small volume V . Since the vector \mathbf{M} itself is a microscopic quantity and according to Eq. (5.4) equals the mean (over an infinitely small volume) density of the magnetic moment of a magnetic, then the mean value of curl \mathbf{M} may be replaced simply by curl \mathbf{M} . Thus, we get

$$\frac{1}{c} \bar{\mathbf{j}}_{\text{mol}} = \text{curl } \mathbf{M}$$

which coincides with Eq. (5.13), Q.E.D.

5. It follows from this equation, particularly, that the mean density of molecular currents in a uniformly magnetized medium equals zero. It is easy to see that this is true directly, by considering, for instance, the volume V of a cubical shape. Owing to the constancy of the vector \mathbf{M} , the opposite faces of the cube will be intersected by an identical number of molecular currents of the given direction $\boldsymbol{\mu}$. This will happen in such a way that the sections of these currents cut off by one face on an average will be exactly supplemented to a complete closed contour by the sections cut off by the opposite face. Therefore, the vector sum of the cut off sections will equal zero.

5.9 Mechanism of Magnetization of Magnetics. Larmor's Theorem

1. Up to now, we made no special assumptions on the mechanism of magnetization of magnetics, and, apart from absolutely general laws of the theory of magnetism, based our treatment only on the fact that the magnetization of dia- and paramagnetics is proportional to the intensity of the magnetic field in them. Now we shall consider a general outline of the mechanism itself of the magnetization of various bodies. This will permit us to find out why magnetization in dia- and paramagnetics has different signs, to establish the relationship between the susceptibility and the atomistic structure of a magnetic, etc.

Any theory of atomic phenomena and processes must be based on quantum mechanics. For a number of reasons, the specific quantum laws play a much more essential part in phenomena of magnetization than, for instance, in phenomena of the polarization of dielectrics. First of all, the consistent electron theory within the scope of classical physics inevitably leads to the conclusion that the magnetization of any body must always equal zero! (See Sec. 5.12.) Further, even if we make the assumption, alien to classical physics, on the discreteness of the possible states of motion of the electrons in atoms and molecules (see Sec. 5.12), this will nevertheless be insufficient for explaining ferromagnetic phenomena.

A treatment of the quantum theory, however, is beyond the scope of this book. Therefore, in considering the mechanism of magnetization, we shall mainly have to proceed from the semiclassical and semiquantum notions of *Bohr's theory* of the atom that will allow us to *qualitatively* orient ourselves in a number of phenomena we are interested in.

2. Let us consider the very simple case of magnetization of monatomic gases and meanwhile ignore the spin of the electrons. In the absence of an external magnetic field, the electrons in each atom are in a definite state of motion. When the magnetic is introduced into the magnetic field \mathbf{H} , the motion of the electrons will change because the Lorentz force [Eq. (4.20)]

$$\mathbf{F}_L = \frac{e}{c} [\mathbf{v}\mathbf{H}]$$

will begin to act on them.

According to *Larmor's theorem*, which we shall prove below, this change in the motion of the electrons in a first approximation consists in the superposition onto the undisturbed motion of the electrons of the additional rotation (*precession*) of all the electrons about the direction of the magnetic field \mathbf{H} with the angular velocity

$$\mathbf{o} = - \frac{e\mathbf{H}}{2mc} \quad (5.63)$$

where e = charge of an electron ($e < 0$)

m = its mass.

In other words, if we introduce an auxiliary coordinate system S' with its centre at the nucleus of the atom and rotating with the angular velocity \mathbf{o} about an axis passing through the nucleus that coincides in direction with \mathbf{H} , then with respect to this system S' the motion of the electrons in the presence of the field will in the first approximation be the same as it was in the absence of the field \mathbf{H} with respect to the inertial (stationary) coordinate system S .

Indeed, owing to the symmetry of the Coulomb field of the nucleus, the interaction of the electrons with the nucleus will not be changed

by their additional precession. The interaction of the electrons with one another will also remain unchanged because the general precession of the electrons will not change their relative disposition. Since the system S' rotates, however, the forces that maintained the motion of the electrons in the inertial coordinate system will no longer be sufficient to maintain this motion in the system S' ; it will also be necessary to balance the inertial forces, namely, the centrifugal and the Coriolis forces.

The centrifugal force is proportional to the product of the distance from the axis of rotation to an electron and the square of the angular velocity ω^2 , i.e. in accordance with Eq. (5.63) it is proportional to the square of the field intensity H^{2*} . Therefore in a first approximation, when only quantities proportional to the first power of the field H are taken into consideration, we may disregard the centrifugal forces.

The Coriolis force applied to the i -th electron ($i = 1, 2, \dots, n$ where n is the number of electrons in an atom) is proportional to the first power of \mathbf{H} and equals

$$\mathbf{F}_{C,i} = 2m [\mathbf{v}'_i \mathbf{\omega}]$$

where \mathbf{v}'_i is the ("relative") velocity of the i -th electron in the rotating system S' related to its ("absolute") velocity \mathbf{v}_i in the inertial system S by the expression

$$\mathbf{v}'_i = \mathbf{v}_i - [\mathbf{\omega} \mathbf{r}_i]$$

where \mathbf{r}_i is the distance from the i -th electron to the axis of rotation. We may substitute \mathbf{v}_i for \mathbf{v}'_i in the expression for $\mathbf{F}_{C,i}$ with an accuracy to terms of the second order of magnitude with respect to \mathbf{H} :

$$\mathbf{F}_{C,i} = 2m [\mathbf{v}_i \mathbf{\omega}]$$

Introducing into this equation the value of $\mathbf{\omega}$ from Eq. (5.63), we get

$$\mathbf{F}_{C,i} = - \frac{e}{c} [\mathbf{v}_i \mathbf{H}]$$

which equals the Lorentz force (4.20) acting on the i -th electron, with the opposite sign. Thus, the Lorentz force (4.20) is indeed balanced by the Coriolis force (with an accuracy up to terms of the second order of magnitude relative to \mathbf{H}).

Thus, we have proved that in the presence of a field the above-mentioned motion of the electrons in an atom is possible. However, this motion is modified by their precession as a whole with an angular

* Because in an atom the distance from an electron to the axis of rotation passing through the nucleus of the atom does not depend on H in a first approximation. Conversely, when a free electron moves in a magnetic field the radius R of its orbit, according to Eq. (4.24), is inversely proportional to H , and therefore the centrifugal force is proportional to the first power of H and not to the second one.

velocity \mathbf{o} . To completely prove Larmor's theorem, it would also be necessary to show that this possibility is *indeed carried out* upon the (sufficiently slow) switching on of the field \mathbf{H} . We shall take on trust this statement whose proof is set out in the atomic theory, but shall conduct a different derivation of Larmor's formula (5.63). On our way, we shall obtain a number of formulas which will be needed in following sections.

3. The law of conservation of angular momentum holds for any system of particles moving in the central field of forces. Particularly, the total angular momentum of electrons relative to the atomic nucleus, equal to

$$\mathbf{L} = m \sum_{i=1}^n [\mathbf{R}_i \mathbf{v}_i] \quad (5.64)$$

remains constant in time in the absence of external fields (the interaction of electrons does not violate the constancy of \mathbf{L}). Here \mathbf{R}_i is the distance from the nucleus to the i -th electron, and \mathbf{v}_i is its velocity.

On the other hand, Eq. (4.133) for the magnetic moment of an atom, i.e.

$$\boldsymbol{\mu} = \frac{1}{2c} \int_V [\mathbf{R} \mathbf{j}] dV$$

can be transformed as follows. The microscopic current density \mathbf{j} at each point of space can be expressed through the microscopic volume density ρ of the charges at this point and through their velocity \mathbf{v}^* :

$$\mathbf{j} = \rho \mathbf{v} \quad (5.65)$$

Indeed, \mathbf{j} is evidently parallel or antiparallel to the velocity of the charges \mathbf{v} depending on the sign of ρ . Further, all the charges in a cylinder having the altitude \mathbf{v} and a base that is a unit area perpendicular to \mathbf{v} will pass through this base, whence we get Eq. (5.65). Introducing Eq. (5.65) into Eq. (4.133), we get

$$\boldsymbol{\mu} = \frac{1}{2c} \int [\mathbf{R} \mathbf{v}] \rho dV \quad (5.66)$$

* Formula (5.65) cannot be directly carried over into the macroscopic theory because the mean value of the product $\rho \mathbf{v}$ does not equal the product of the mean values of the multipliers. For example, in a conductor through which a current is flowing, the mean density of the charges ρ may equal zero although the mean current density \mathbf{j} differs from zero because charges of only one sign (electrons) are moving in the conductor. Equation (5.65) may be applied to this case if by ρ in it we mean the density of only the moving charges—electrons. When this condition is observed it coincides with Eq. (4.18), where ne is the macroscopic value of the density ρ of the electron charges.

From the viewpoint of Bohr's atomic theory, this form, which in general has been derived for closed steady currents, may be used for the motion of electrons in orbits inside of atoms that is averaged in time. We may consider with sufficient accuracy that the value of the product $[\mathbf{Rv}]$ is identical at all points of each separate electron and, consequently, it may be put outside the integral:

$$\boldsymbol{\mu} = \frac{1}{2c} [\mathbf{Rv}] \int \rho dV = \frac{e}{2c} [\mathbf{Rv}]$$

where $e = \int \rho dV$ is the charge of an electron. If an atom contains several electrons, we get, accordingly,

$$\boldsymbol{\mu} = \frac{e}{2c} \sum_{i=1}^n [\mathbf{R}_i \mathbf{v}_i] \tag{5.67}$$

Comparing this expression with Eq. (5.64), we see that the angular momentum of the electrons \mathbf{L} is proportional to the magnetic moment $\boldsymbol{\mu}$ created by the motion of the electrons in their orbit:

$$\boldsymbol{\mu} = \eta \mathbf{L} \tag{5.68}$$

Here $\boldsymbol{\mu}$ is directed oppositely to \mathbf{L} because the proportionality factor

$$\eta = \frac{e}{2mc} \tag{5.69}$$

is negative since the charge of an electron $e < 0$.

4. In the absence of external fields, as we have already mentioned, the angular momentum of an atom \mathbf{L} and, consequently, the magnetic moment of the atom $\boldsymbol{\mu}$ proportional to it are constant in time. When an external magnetic field is present, an atom is acted upon by a couple of forces whose moment \mathbf{N} equals [Eq. (4.127)]

$$\mathbf{N} = [\boldsymbol{\mu} \mathbf{H}]$$

If an atom had no angular momentum, under the action of this couple of forces its magnetic axis would tend to set itself up along the direction of the field \mathbf{H} . The presence of the angular momentum \mathbf{L} , however, makes the atom similar to a top (gyroscope) in the mechanical respect. It is known that if a couple of forces whose moment is perpendicular to the axis of a top begins to act on a rotating top, this axis begins to *precess* about the direction of the forces, the *angle of inclination of the axis* to the direction of the forces *remaining unchanged* (cf. the precession of a heavy top in a gravitational field).

Indeed, according to a well-known theorem of mechanics, under the action of a couple of forces having the moment \mathbf{N} , the head of the angular momentum vector \mathbf{L} of a material system moves with a linear velocity $d\mathbf{L}/dt$ equal to \mathbf{N} :

$$\frac{d\mathbf{L}}{dt} = \mathbf{N} = [\boldsymbol{\mu} \mathbf{H}]$$

Using in this equation the value of μ from Eq. (5.68), we get

$$\frac{d\mathbf{L}}{dt} = \eta[\mathbf{L}\mathbf{H}]$$

Since, according to Eqs. (5.63) and (5.69), we have

$$\mathbf{o} = -\eta\mathbf{H} \quad (5.70)$$

we can also write

$$\frac{d\mathbf{L}}{dt} = [\mathbf{o}\mathbf{L}] \quad (5.71)$$

It follows from this equation that the vector \mathbf{L} and, consequently, μ will rotate about the direction of \mathbf{o} with the angular velocity o or, in other words, will rotate with an angular velocity determined in magnitude and direction by the vector \mathbf{o} . This means that upon the switching on of a magnetic field \mathbf{H} the electron shell of the atom will begin to *precess* about the direction of the field with the angular velocity o , *the angle of inclination of the magnetic axis of the atom to the direction of the field remaining unchanged.*

For example, if we mentally replace a complex of intratomic currents with a line plane current loop (corresponding in Bohr's theory to an electron orbit), then the vector \mathbf{L} will be perpendicular to the plane of this loop (the plane of the orbit), and the precession

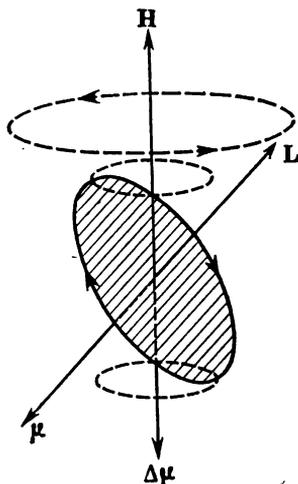


Fig. 66

of this vector will correspond to the precession of the plane of the loop (orbit) with the same angular velocity o (Fig. 66; the vector $\Delta\mu$ in the figure will be considered in Sec. 5.10).

This is the content of *Larmor's theorem* which we have thus proved in two different ways. A shortcoming of the second proof is that it

cannot be applied to atoms whose resultant magnetic moment μ in the absence of an external field equals zero.

5. Everything set out in this section strictly relates to isolated atoms of *gaseous* magnetics. Qualitatively, however, the results of this section may be applied whenever the electron shell of atoms or ions can rotate more or less freely about the nucleus of the atom. If the electron shell of an atom or ion has spherical symmetry (as, for instance, in the atoms of the noble gases or in ions having the same number of electrons as the noble gases do), then free rotation of this shell may also occur in liquids and solids.

At any rate, experiments show in accordance with a theory which is more accurate than the one we are setting out that the main results of Secs. 5.9-5.11 with a sufficient degree of accuracy may be applied both to polyatomic gases and to many liquid and solid para- and diamagnetics.

5.10 Diamagnetism

1. In the absence of an external magnetic field, the molecular currents in (non-ferromagnetic) magnetics are distributed quite randomly, and therefore the magnetization of a magnetic (i.e. the magnetic moment of a unit of its volume) equals zero. When a magnetic is introduced into a magnetic field, however, *Larmor precession* of the electron shells of the atoms appears that is equivalent to the additional rotation of the electrons with the angular velocity ω about an axis that is parallel to \mathbf{H} and passes through the nucleus of the atom. Owing to this rotation of the electrons, a diamagnetic atom acquires a certain magnetic moment $\Delta\mu$. Since ω is parallel to the field [because $\eta < 0$, see Eq. (5.69)], then the direction of the additional rotation of the electrons forms a *right-handed* system with the vector \mathbf{H} . Therefore, the direction of the "additional" current $\Delta\mathbf{I}$ equivalent to this precession motion of the *negative* electrons forms a *left-handed* system with the vector \mathbf{H} . Hence, the magnetic moment $\Delta\mu$ acquired by an atom in a magnetic field owing to precession is directed *oppositely* to the magnetic field \mathbf{H} (see Fig. 66). Thus, as a result of precession, the atoms acquire a magnetic moment opposite to the field, i.e. the body is magnetized in a direction opposite to the field. This is exactly the explanation of *diamagnetism*.

This diamagnetic effect also occurs in paramagnetic bodies, but as we shall show in the following section, is completely hidden in them by the considerably stronger paramagnetic effect that is opposite in its sign.

2. To determine the numerical value of the diamagnetic susceptibility, we shall note that owing to Larmor's precession of the electron shell

of an atom, each volume element ρdV of the charge of this shell acquires the additional velocity

$$\Delta \mathbf{v} = [\mathbf{oR}]$$

where \mathbf{R} is the distance from the nucleus of the atom to the element dV . According to Eq. (5.66), the additional magnetic moment of an atom corresponding to this precession equals

$$\Delta \boldsymbol{\mu} = \frac{1}{2c} \int [\mathbf{R} \cdot \Delta \mathbf{v}] \rho dV = \frac{1}{2c} \int [\mathbf{R}[\mathbf{oR}]] \rho dV \quad (5.72)$$

Further,

$$[\mathbf{R}[\mathbf{o}, \mathbf{R}]] = \mathbf{o}R^2 - \mathbf{R}(\mathbf{oR})$$

Assume that the z -axis runs in the direction of \mathbf{o} coinciding with the direction of \mathbf{H} . Hence, the component of this vector along the z -axis will be

$$oR^2 - z(oz) = o(R^2 - z^2) = o(x^2 + y^2)$$

and the components along the x - and y -axes will be, respectively, $-zox$ and $-zoy$.

Upon introduction into the integral and averaging the value of $\Delta \boldsymbol{\mu}$ over the precession period, the components along the x - and y -axes will vanish because owing to the precession of the electrons the atom will have cylindrical symmetry with respect to the z -axis on an average in time. Indeed, owing to this symmetry, for any element of charge ρdV with the coordinates x , y , and z there will be found an equal element of charge with the coordinates $-x$, y , and z , so that the sum of the expressions $-zox$ for this pair of elements will become equal to zero.

Thus, the mean value of $\Delta \boldsymbol{\mu}$ in time equals

$$\overline{\Delta \boldsymbol{\mu}} = \frac{\mathbf{o}}{2c} \int \overline{(x^2 + y^2)} \rho dV$$

The last integral obviously equals the product of the sum of the charges of the electrons in an atom Ze (where Z signifies the number of electrons in an atom) and the mean value of the square of the distance from the electrons to the z -axis:

$$\int \overline{(x^2 + y^2)} \rho dV = Ze \overline{(x^2 + y^2)}$$

Introducing, finally, the value of \mathbf{o} from Eq. (5.63), we get

$$\overline{\Delta \boldsymbol{\mu}} = \frac{1}{2c} Zoe \overline{(x^2 + y^2)} = -\frac{Ze^2}{4mc^2} \overline{(x^2 + y^2)} \mathbf{H} \quad (5.73)$$

The mean value in time of x^2 , i.e. $\overline{x^2}$, will differ for different atoms. The mean value for differently oriented atoms will obviously be

$$\overline{x^2} = \overline{y^2} = \frac{1}{3} \overline{R^2}$$

where $\overline{R^2}$ is the mean square of the distance from the electrons to the nucleus of an atom. Hence,

$$\overline{\Delta \mu} = -\frac{Ze^2}{6mc} \overline{R^2} \mathbf{H}$$

and the magnetization of a unit volume of a diamagnetic equals

$$\mathbf{M} = N \cdot \overline{\Delta \mu} = -\frac{NZe^2}{6mc^2} \overline{R^2} \mathbf{H} \quad (5.74)$$

where N is the number of atoms in a unit volume.

Consequently, the susceptibility of a diamagnetic equals

$$\chi = -\frac{NZe^2}{6mc^2} \overline{R^2} \quad (5.75)$$

At constant volume (i.e. at a constant value of N) it does not depend on the temperature.

If we use quantum mechanics to calculate the mean square of the distance from the electrons to a nucleus for different diamagnetic atoms, then Eq. (5.75) gives a value of χ that agrees very well with experimental data.

It must be noted that \mathbf{H} in Eq. (5.74) in its meaning denotes the mean intensity of a microscopic field $\mathbf{H}_{\text{micro}}$ (if we disregard its difference from the mean effective field acting on a molecule) and that it would therefore be more correct, in accordance with Eq. (5.23), to write \mathbf{B} instead of \mathbf{H} in Eq. (5.74). Making this change in accordance with Eq. (5.36), instead of Eq. (5.75) we get

$$\frac{\chi}{\mu} = \frac{\chi}{1 + 4\pi\chi} = -\frac{NZe^2}{6mc^2} \overline{R^2} \quad (5.76)$$

The difference of the permeability μ from unity in diamagnetics is so small, however, that Eq. (5.76) does not virtually differ from Eq. (5.75).

5.11 Paramagnetism

1. If a magnetic consists of atoms or molecules whose magnetic moment μ in the absence of an external field \mathbf{H} equals zero ($\mu = 0$), then the action of the magnetic field on the magnetic consists only

in the diamagnetic moment that was considered in the preceding section, and this magnetic is diamagnetic ($\chi < 0$). If the magnetic moment μ of the atoms and molecules of a medium in the absence of an external field differs from zero ($\mu \neq 0$), then in addition to the diamagnetic effect [the appearance of an additional moment of the atoms $\Delta\mu$, Eq. (5.72)], the magnetic field also causes a *redistribution of the directions* of the magnetic moments μ of the atoms and molecules in the medium. In the absence of an external magnetic field, the magnetic moments of the atoms are oriented quite randomly so that the magnetization of the medium equals zero. When an external magnetic field \mathbf{H} is present, the directions of the magnetic moments of the atoms that are close to the direction of the field gain predominance. This magnetization of a magnetic in the direction of the field (positive magnetization, *paramagnetism*) always considerably exceeds the diamagnetic effect when $\mu \neq 0$. Therefore, all magnetics with $\mu \neq 0$ are paramagnetics ($\chi > 0$).

2. As we have seen in Sec. 5.9, a magnetic field does not directly change the angle of inclination of the magnetic moment of an atom to the direction of the field, but only causes precession of the magnetic axis of the atom about the direction of the field with the same angle of inclination to it. If prior to induction of the field the direction of the axes of the atoms were distributed randomly, then in the magnetic field too the vector sum of the moments of the separate atoms must remain equal to zero. Consequently, the *direct* action of a field on a magnetic consists in the diamagnetic effect considered in Sec. 5.10.

This is true, however, only as long as we limit ourselves to a consideration of free or isolated atoms or molecules and take no account whatsoever of their interaction. In the simplest case of an ideal gas, the interaction of molecules consists in their colliding with one another. What new important facts are introduced when we take these collisions into account?

Upon each collision, the direction of the axis of the molecules will change. The collision of molecules is so intricate that we cannot watch all the details of this occurrence. We can take the influence of the collisions into consideration, however, by resorting to the general principles of statistical mechanics, namely, to *Boltzmann's theorem*. The latter states that the probability of a given state of a molecule is the greater, the smaller is its energy*. We shall now prove that with a given magnetic moment of an atom

* Boltzmann's theorem relates to systems in a state of *thermodynamic equilibrium*, and an essential condition for its applicability is the possibility itself of a change in the states of molecules (in the given case a change in the direction of their axes). We have seen that in a magnetic field the direction of the axes of molecules can change only when interaction (particularly *collisions*) of the molecules is present: otherwise Boltzmann's theorem could not be applied.

the kinetic energy of the electrons in it will be the smaller, the smaller is the angle between the magnetic moment of the atom and the direction of the magnetic field. Consequently, according to Boltzmann's theorem, in the presence of an external field the collisions of the atoms should result in the predominance of the direction of the magnetic axes of the atoms close to the direction of \mathbf{H} , and the body should become magnetized (the paramagnetic effect).

3. Thus, we have to determine the change in the kinetic energy of the electrons in atoms under the action of the external magnetic field \mathbf{H} . The precession of the electrons in the magnetic field with the angular velocity \mathbf{o} causes a change in the velocity \mathbf{v}_i of the i -th electron in an atom by the quantity

$$\Delta \mathbf{v}_i = [\mathbf{o} \mathbf{R}_i] \tag{5.77}$$

where \mathbf{R}_i is the distance from the i -th electron to the nucleus of the atom. Accordingly, the kinetic energy T of the electrons in the atom changes by

$$\Delta T = \frac{m}{2} \sum_{i=1}^n \{(\mathbf{v}_i + \Delta \mathbf{v}_i)^2 - \mathbf{v}_i^2\} = \frac{m}{2} \sum_{i=1}^n \{2\mathbf{v}_i \Delta \mathbf{v}_i + (\Delta \mathbf{v}_i)^2\} \tag{5.78}$$

where m = mass of an electron
 n = number of electrons in an atom.

Since $\Delta \mathbf{v}_i$ in all available fields is much smaller than \mathbf{v}_i , we may ignore the square of $\Delta \mathbf{v}_i$:

$$\Delta T = m \sum_{i=1}^n \mathbf{v}_i \Delta \mathbf{v}_i = m \sum_{i=1}^n \mathbf{v}_i [\mathbf{o} \mathbf{R}_i] = m \mathbf{o} \sum_{i=1}^n [\mathbf{R}_i \mathbf{v}_i]$$

Using Eq. (5.64), we get

$$\Delta T = \mathbf{o} \mathbf{L}$$

where \mathbf{L} equals the value of the angular momentum of the electron shell of the atom undisturbed by the field. With the aid of Eqs. (5.70) and (5.68), we finally get

$$\Delta T = - \eta \mathbf{L} \mathbf{H} = - \boldsymbol{\mu} \mathbf{H} \tag{5.79}$$

Thus, the change in the *kinetic energy* of the electrons * in a magnetic field \mathbf{H} is numerically equal to the *potential energy* in this field of a *magnetic dipole* whose moment $\boldsymbol{\mu}$ equals the magnetic moment of an atom [see Eq. (4.125)]. Here the equivalence of an elementary current loop and a magnetic dipole, which we have repeatedly drawn attention to, again manifests itself.

* Naturally, potential energy of an electron in a non-conservative magnetic field is out of the question.

4. To determine the magnetization of paramagnetics, it remains for us only to use Boltzmann's theorem. We already used it in Sec. 2.10 when we applied it to the conservative field of electric forces. Boltzmann's theorem also remains applicable to a non-conservative field of magnetic forces if only in the formulation of this theorem given in Sec. 2.10 we replace the potential energy of a molecule V with the increment ΔU of the *total* (potential and kinetic) energy of the molecule in the field of forces being considered. We thus arrive at the following formulation of Boltzmann's theorem: in the conditions of thermodynamic equilibrium, the law of distribution of molecules by their different states when an external field of forces is present differs from the law of their distribution in the absence of this field by the factor $\exp(-\Delta U/kT)$, where T is the absolute temperature, and, naturally, the dependence of ΔU on the state of a molecule is of significance. In other words, Boltzmann's theorem states that the probability of a given state of a molecule is the greater, the smaller is the energy of this state.

In the case we are considering, $\Delta U = \Delta T = -\boldsymbol{\mu}\mathbf{H}$, so that the energy of a molecule is the smaller, the smaller is the angle between the direction of its moment $\boldsymbol{\mu}$ and that of the field \mathbf{H} . Thus, the magnetization of paramagnetics is absolutely similar to the polarization of dielectrics with solid dipoles: the Boltzmann factor $\exp(-\Delta U/kT)$ determining the distribution of the axes of the molecules in the external field equals $\exp(\boldsymbol{\mu}\mathbf{H}/kT)$ for paramagnetic molecules in the magnetic field \mathbf{H} and $\exp(\mathbf{p}\mathbf{E}/kT)$ for solid electric dipoles in the electric field \mathbf{E} [see Eq. (2.68)]. Therefore, in the theory of paramagnetism, we may directly use the results of the theory of dielectrics with solid dipoles.

For this purpose, it is sufficient to substitute the magnetic quantities $\boldsymbol{\mu}$ and \mathbf{M} for the relevant electrical quantities \mathbf{p}_0 and \mathbf{P} in the formulas of Sec. 2.10. As regards the electric field intensity \mathbf{E} , it may be replaced with the magnetic induction \mathbf{B} instead of \mathbf{H} because it is exactly \mathbf{B} that equals the mean value of the microscopic intensity $\mathbf{H}_{\text{micro}}$ [Eq. (5.23)]. Making this substitution in Eq. (2.71), we get

$$\mathbf{M} = \frac{N\mu^2}{3kT} \mathbf{B}$$

whence on the basis of Eq. (5.36)

$$\frac{\chi}{\mu_p} = \frac{\chi}{1 + 4\pi\chi} = \frac{N\mu^2}{3kT} \quad (5.80)$$

Here T naturally stands for the absolute temperature of the paramagnetic, unlike Eqs. (5.78) and (5.79), where it denotes the kinetic energy, and μ_p stands for the permeability. It is simple to determine the susceptibility χ from Eq. (5.80). The susceptibility of non-ferromagnetic bodies is so small, however, that the term $4\pi\chi$ may be

disregarded in comparison with unity * and we may assume, as is usually done, that

$$\chi = \frac{N\mu^2}{3kT} \quad (5.81)$$

5. Thus, unlike the susceptibility of diamagnetics [Eq. (5.75)], the susceptibility of paramagnetics χ at constant volume (i.e. at constant N) should change inversely proportional to the absolute temperature (compare the two classes of dielectrics, Sec. 2.10). This nature of the temperature dependence of χ was discovered experimentally by P. Curie before the corresponding theory was developed and is called *Curie's law*. This law is well observed experimentally for gaseous paramagnetics and also for a number of solid paramagnetics (for example for salts of the rare-earth metals). On the other hand, for many liquid and solid paramagnetics, the above elementary theory assuming the free precession of the magnetic moments of atoms about the direction of a field is found to be inadequate, and Curie's law is violated in these paramagnetics.

In the paramagnetics to which this assumption is applicable, however, deviations from Curie's law ought to be and are observed in very strong fields and at very low temperatures (of the order of magnitude of several kelvins). These deviations correspond quite well to the expectations of the theory being treated and are explained by the circumstance that the applicability of Eqs. (5.80) and (5.81) is limited by the condition

$$a = \frac{\mu H}{kT} \ll 1 \quad (5.82)$$

[cf. the corresponding equation (2.74) and Sec. 2.10]. When $a \approx 1$, *saturation of magnetization* of paramagnetics is observed. This consists in violation of the proportionality between \mathbf{M} and \mathbf{H} , the magnetization M tending to a constant limit $M_{\text{sat}} = N\mu$ when the field grows. This maximum possible magnetization corresponds to the arrangement of the magnetic moments of *all* the atoms along the direction of the field \mathbf{H} (see Sec. 5.13).

Equation (5.80) makes it possible to calculate the value of the magnetic moment μ according to the data obtained in measuring the quantities χ and N at different temperatures T . The values of the magnetic moment of paramagnetic atoms and molecules deter-

* In essence, it would not be consistent to take into account the difference between \mathbf{B} and \mathbf{H} in the above formulas and at the same time disregard, as we have done everywhere in this chapter, the difference between the field acting on a molecule and the mean field (Sec. 2.9), the difference between the mean square of the field intensity and the square of its mean intensity, etc., because all these differences are of the same order of magnitude.

mined in this way agree quite well with the conclusions of the quantum theory of the atom.

6. We shall note in conclusion that the following question arises quite lawfully in connection with Eqs. (5.78) and (5.79). The forces of a magnetic field are perpendicular to the velocity of an electron and therefore perform no work. How can the appearance of a magnetic field change the kinetic energy of the electrons?

The answer is that any change in the intensity of a magnetic field, particularly the appearance of this field, induces an electric field (Sec. 6.10). The change in the energy of an electron when a magnetic field is induced is exactly due to the work done by the forces of this electric field. For completeness, we shall give the corresponding calculations here, although we shall have to use some postulates that will be proved in the following chapter. Therefore, these calculations may be omitted when reading the book the first time.

By averaging the motion of electrons in an atom with respect to the time of their revolution in their orbits, we can reduce the motion of atomic electrons to the corresponding system of closed steady currents (quantum mechanics directly reduces the magnetic field of an atom to the field of such a system of currents). Further, an arbitrary system of closed steady currents can be resolved into a combination of closed current filaments. We may therefore limit ourselves to the consideration of one such current filament or, which is simpler, a line closed current I . The work W done by the forces of an electric field on the current I during the time from t_1 to t_2 equals [cf. Eq. (3.5)]

$$W = \int_{t_1}^{t_2} I dt \oint E_s ds \quad (5.83)$$

where the line integral should be taken over the current contour L . Using Eq. (6.6), we get

$$\oint_L E_s ds = \mathcal{E}_{\text{ind}} = -\frac{1}{c} \frac{d}{dt} \int_S H_n dS = -\frac{1}{c} \frac{d\Phi}{dt}$$

where the surface integrals should be taken over the surface S resting on the contour of the current, and Φ is the magnetic flux through this contour. Using this expression in Eq. (5.83) and assuming that the change in the current I under the influence of the induction is so insignificant that it may be disregarded, and I may be put outside the integral with respect to time, we get

$$W = -\frac{1}{c} \int_{t_1}^{t_2} I \frac{d\Phi}{dt} dt = -\frac{I}{c} [\Phi(t_2) - \Phi(t_1)]$$

If at the initial moment there were no magnetic field and, consequently, $\Phi(t_1)$ equalled zero, and if the magnetic field on the surface S had the constant value \mathbf{H} at the moment t_2 , then

$$\Phi(t_1) = 0, \quad \Phi(t_2) = \mathbf{H}S$$

and according to Eq. (4.122)

$$W = -\frac{I\mathbf{H}S}{c} = -\mu\mathbf{H} \quad (5.84)$$

Thus, the work of the electric forces induced when a magnetic field is generated does indeed equal the change in the kinetic energy of the electrons [Eq. (5.79)].

5.12 More Precise Definition and Additions to the Theory of Magnetization. The Part of Spin. Gyromagnetic Phenomena

In Secs. 5.9-5.11 devoted to the theory of magnetization of para- and diamagnetics, for purposes of simplification we did not consider a number of circumstances part of which we shall now deal with additionally.

1. First, in our previous treatment, we proceeded from the assumption that all molecules (or all atoms) of a given substance have (in the absence of an external magnetic field) a quite definite magnetic moment μ undergoing a quite definite change $\Delta\mu$ [Eq. (5.72)] when the external field \mathbf{H} is induced. The magnitude of this increment depends only on the orientation of an atom relative to the field \mathbf{H} because we consider that the mean distance from the electrons to the atomic nucleus in Eq. (5.73) is the same for all atoms.

It is not difficult to generalize the results of the preceding sections for a mixture of different species of atoms or molecules (for example a mixture of gases, a solution, a mixture of unexcited and excited atoms, etc.); the susceptibility of the mixture will evidently equal the sum of the susceptibilities of its components. The assumption that the possible numerical values of the magnetic moment of atoms μ and the mean square of the distance from the electrons to the atomic nucleus \bar{R}^2 [Eq. (5.74)] form a *discrete* set, however, has an absolutely decisive significance. This assumption corresponds quite well to the quantum theory, but cannot be kept within the confines of classical physics. If we assume in accordance with the ideas of classical physics that both the moment of each atom μ and the quantity \bar{R}^2 can have any value from 0 to ∞ , then the *susceptibility* of a substance *identically equals zero* *. No such difficulty appears for electric polarization, however.

* See, for instance, D. C. Mattis. *The Theory of Magnetism*. New York, Harper and Row (1965).

Thus, although the reasoning of the preceding sections appeared to be within the confines of the classical theory, strictly speaking, the consistent electron theory of magnetization is absolutely impossible within the confines of classical physics.

2. Second, in Secs. 5.9-5.11 we took absolutely no account of the *spin* of electrons, and must now fill in this gap.

Limiting ourselves to a consideration of the so-called *orbital* magnetic moment μ_{orb} and *orbital* angular momentum L_{orb} of an atom due to translational motion of the electrons in the atom *, we established in Sec. 5.9 that these quantities are proportional to each other [Eq. (5.68)]:

$$\mu_{\text{orb}} = \eta L_{\text{orb}} \quad (5.85)$$

the proportionality constant having a universal value [Eq. (5.69)]:

$$\eta = \frac{e}{2mc} < 0 \quad (5.86)$$

The total magnetic moment and mechanical angular momentum of an atom are not exhausted, however, by these orbital quantities, but consist of the sum of them and the spin magnetic moment μ' and angular momentum L' of the electrons. A direct proportion also exists between the quantities μ' and L' for each separate electron:

$$\mu' = \eta' L' \quad (5.87)$$

According to Eq. (4.144), however, the proportionality constant here differs from η :

$$\eta' = \frac{e}{mc} = 2\eta \quad (5.88)$$

Therefore, the proportionality constant η'' between the resultant magnetic moment μ and angular momentum L for an entire atom as a whole

$$\mu = \eta'' L \quad (5.89)$$

is not a universal quantity but depends on the ratio between the orbital and spin moments and angular momenta in the given atom**. It must obviously have a negative value within the limits

$$2\eta \leq \eta'' \leq \eta \quad (5.90)$$

* We have now introduced the subscript "orb" to distinguish these quantities from their spin counterparts.

** As a matter of fact, owing to the difference between η and η' it is not self-evident that the resultant magnetic moment of an atom $\mu = \mu_{\text{orb}} + \sum \mu'$ is always directed oppositely to the resultant angular momentum $L = L_{\text{orb}} + \sum L'$. If μ_{orb} and the sum of the spin magnetic moments of the electrons $\sum \mu'$, however, are at an angle to each other, then the forces of magnetic interaction

Will this circumstance affect the results of Secs. 5.10-5.11? When deriving Curie's law (5.80) for paramagnetics, we in essence only based our reasoning on the fact that the change in the energy of an atom with the moment μ upon the appearance of the magnetic field \mathbf{H} is

$$\Delta U = -\mu H \quad * \quad (5.91)$$

We proved this relationship in Sec. 5.11 for the case when the moment μ is created by the orbital motion of the electrons, and ΔU is determined by the change in their *kinetic* energy. Equation (5.91), however, also holds for a solid magnetic dipole; here ΔU equals the *potential* energy of this dipole in the field \mathbf{H} [see Eq. (4.125)]. Thus, Eq. (5.91) has a universal nature, and Curie's law (5.80) following from it remains applicable with any relationship between the orbital and the spin components of the total magnetic moment of an atom μ .

As regards diamagnetic atoms, their total magnetic moment and angular momentum in the absence of a magnetic field equal zero, and we can therefore expect that consideration of the spin moment of an electron will not affect the results of Sec. 5.10. Indeed, quantum mechanical calculations lead to the conclusion that Eq. (5.75) for the susceptibility of diamagnetics also remains in force when the spin of the electrons is taken into consideration.*

3. Thus, consideration of the spin of electrons does not introduce any changes into the fundamental relationships of the theory of magnetization of dia- and paramagnetics. There also exist, however, such macroscopic phenomena (apart from ferromagnetism which will be discussed below) for whose proper explanation spin must be taken into account. These are the so-called *magnetomechanical* or *gyromagnetic* phenomena consisting, first, in that upon the magnetization of para- or ferromagnetic ** bodies the latter begin to rotate about the direction of magnetization, and, second, in that upon the rotation of these bodies magnetization parallel to the axis of rotation appears in them. These phenomena are due to the circumstance that according to Eq. (5.89) the magnetic moment and angular momentum of each atom are proportional to each other. Therefore, the magnetization of a substance, i.e. the appearance of a resultant

between μ_{orb} and $\Sigma\mu'$ (the so-called "spin orbit coupling") call forth precession of each of these quantities about the vector \mathbf{L} whose direction and magnitude in an isolated atom, according to the law of conservation of angular momentum, cannot change. As a result, the mean value of the resultant magnetic moment of the atom over the period of this "internal" precession is directed oppositely to \mathbf{L} .

* In this book we pay no attention whatsoever to the question of the dia- and paramagnetism of the free electrons in metals which requires special treatment (see, for example, the book by D. Mattis referred to earlier in this section).

** Although we have not considered ferromagnetics up to now, both Eqs. (5.85)-(5.90) and all the following content of this section are of an absolutely general nature and are equally applicable to para- and ferromagnetics.

magnetic moment of a unit volume, is connected with the appearance of the relevant angular momentum, and vice versa.

Let us first consider the angular momentum of an atom relative to a fixed (relative to an inertial frame) point O . It is known from mechanics that the angular momentum of a system of material points (in our case an atom) equals the sum of the angular momentum which the system would have if all its mass were concentrated at its centre of inertia and the angular momentum corresponding to the motion of the points of the system relative to its centre of mass. The centre of mass of an atom may be considered to coincide with its nucleus. Therefore, the total angular momentum of an atom \mathbf{L} will equal the sum of the momentum \mathbf{L}_a corresponding to the motion of the centre of mass (i.e. the nucleus) of the atom, and the momentum \mathbf{L}_e corresponding to the motion of the electrons relative to the nucleus:

$$\mathbf{L} = \mathbf{L}_a + \mathbf{L}_e$$

The total angular momentum of a unit volume of a body will be

$$\sum \mathbf{L} = \sum \mathbf{L}_a + \sum \mathbf{L}_e$$

where the summation sign signifies summation over all the atoms in a unit volume. Taking into account the relationship between the angular momentum and the magnetic moment of the electron shell of an atom, i.e. Eq. (5.89), we get

$$\sum \mathbf{L} = \sum \mathbf{L}_a + \eta'' \sum \boldsymbol{\mu} = \sum \mathbf{L}_a + \eta'' \mathbf{M}$$

because the sum $\sum \boldsymbol{\mu}$, by definition, equals the magnetization of a body*.

Let us now consider the magnetization of para- and ferromagnetics. The direct action of a magnetic field on the electron shell of an atom results only in precession of the shell; we disregard the diamagnetic effect of this precession owing to its small value. A change in the directions of the magnetic axes of the atoms, which is the cause of para- and ferromagnetic magnetization, occurs, as we have seen, only upon collision of the atoms or, in general, when there is interaction between the atoms. Only as a result of this interaction does turning of the axes of the atoms in the direction of the field \mathbf{H} occur, i.e. a change in the direction of both the magnetic moment $\boldsymbol{\mu}$ and of the angular momentum \mathbf{L}_e of the electrons in the atoms directly related to it. Since the law of conservation of angular momentum must be observed in the interaction of the atoms, then the sum $\sum \mathbf{L}$ should

* The motion of an atom as a whole ($\sum \mathbf{L}_a$) does not create a magnetic field if the atom is neutral (not ionized).

remain constant * and a change in $\sum L_e$ must be compensated by a corresponding change in $\sum L_a$.

Assume that prior to magnetization, not only $\sum L_e$, but also $\sum L_a$ and, consequently, $\sum L$ had equalled zero. In this case, according to the last equation, the appearance of the magnetization M proportional to $\sum L_e$ is attended by the simultaneous appearance of the moment $\sum L_a$ equal to

$$\sum L_a = -\eta'' M \tag{5.92}$$

Assume further that a solid para- or ferromagnetic is subjected to magnetization. The atoms of the solid body cannot move separately, and the presence of the angular momentum of the atoms of the body $\sum L_a$ signifies in this case rotation of the entire body as a whole. The angular velocity of rotation ω which the body should acquire when the magnetization M appears is determined from the equation

$$|\sum L_a| = I\omega = -\eta'' M \tag{5.93}$$

where I is the moment of inertia of the body relative to an axis parallel to the direction of magnetization and passing through the centre of mass of the body. Consequently, if we magnetize, for instance, a freely suspended iron bar, then the latter should acquire rotation about the axis of magnetization with the angular velocity ω (the *Einstein-de Haas effect*). Since ω , I and M can be measured directly, then Eq. (5.93) makes it possible to determine the ratio of the angular momentum and the magnetic moment of an atom η'' **.

It is simple to show that conversely, if we rapidly rotate the bar, then magnetization should appear in it whose magnitude depends on the speed of rotation and on η'' (the *Barnett effect*). Without stopping to consider in detail the theory of this effect, we shall only note that it is quite similar to the following well-known mechanical phenomenon: if we secure a gyroscope (which the electron orbits correspond to in our case) on a support (the crystalline skeleton of the solid) and rotate the latter, then the axis of rotation of the gyroscope will tend to arrange itself in the direction of the axis of rotation of the support (which magnetization corresponds to in our case).

A number of experimental investigations have confirmed the existence of the gyromagnetic effects predicted by the electron theory. Particularly, the fact was confirmed that magnetism is due to the motion of *negative* electric charges (electrons) because the values of

* For the magnetization of a body of revolution whose axis is parallel to the external magnetic field H , it is easy, on the basis of considerations of symmetry and of Eq. (6.74) to directly prove that the moment N of the forces of the external field applied to the body being magnetized does actually equal zero and that, consequently, $\sum L$ must remain constant upon magnetization.

** This experiment is run in practice as follows: a bar is suspended on a thread so that its axis coincides with that of a coil through which an alternating current flows. This current periodically remagnetizes the bar, imparting to it each time a definite angular momentum $\sum L_a$. Knowing I , M , and the torsional elasticity of the thread, we can determine η'' according to the amplitude of the torsional oscillations of the bar.

the coefficient η'' were found to be negative [the positive direction of the axis of rotation of a body was found to coincide with the direction of its magnetization — cf. Eq. (5.92)].

As regards the numerical values of the ratio of the angular momentum and magnetic moments of an atom η'' , the values of η'' , as should be expected, were found to be between those of the universal constants η and η' determined by Eqs. (5.86) and (5.88).

It is very significant that for all the studied *ferromagnetics* (iron, nickel, cobalt, a number of alloys), the coefficient η'' was found to equal η' . This shows that *the magnetism of ferromagnetics is due only to the spin of electrons* and not to their orbital motion.

5.13 Ferromagnetism. Weiss Molecular Field

1. Like paramagnetism, the magnetization of ferromagnetics is explained by order setting in in the orientation of the magnetic moments of the ferromagnetic atoms. The exceedingly complicated nature of ferromagnetic phenomena is due to the very appreciable forces of interaction between adjacent atoms of a ferromagnetic that depend on the relative orientation of their magnetic axes. In comparison with these forces, the relevant forces of interaction in a paramagnetic are absolutely negligible. These forces of interaction explain the absence of proportionality between the magnetization of a ferromagnetic and the external magnetic field, residual and spontaneous magnetization, etc.

The nature of these forces of interaction (the so-called “*exchange forces*” between the electrons of the ferromagnetic atoms) does not absolutely lend itself to explanation within the confines of classical physics, and only quantum mechanics has brought along an explanation of the true nature of ferromagnetism.

Already the purely formal introduction of the forces of interaction between atoms depending on the orientation, however, made it possible within the limits of classical physics to gain an understanding of many fundamental laws of ferromagnetism. Therefore, we shall first treat the fundamentals of the classical theory of ferromagnetism worked out by P. Weiss, and only at the end of the section shall we touch on the question of the true nature of the “molecular field of forces” which he introduced when dealing with the subject.

2. According to Weiss’s theory, the field of the forces acting on the magnetic moment of a ferromagnetic atom can be reduced to the sum of the magnetic field \mathbf{H} and a “*molecular field*” which takes into consideration the action of adjacent atoms of the ferromagnetic on a given atom that is proportional to its magnetization \mathbf{M} . In other words, we can say that the “effective” magnetic field \mathbf{H}_{eff} in a ferromagnetic equals the sum of the true magnetic field \mathbf{H} and the molecular field $b\mathbf{M}$:

$$\mathbf{H}_{\text{eff}} = \mathbf{H} + b\mathbf{M} \quad (5.94)$$

where b is a positive constant characterizing the properties of the given ferromagnetic.

The effective magnetic field is also expressed by a formula of the same kind when there is no special molecular field, and the forces of interaction between the atoms which depend on their relative orientation are only magnetic forces. Indeed, in Sec. 2.9 we showed that with known assumptions the effective, i.e. acting on a dipole, electric field in dielectrics with quasi-elastic dipoles is expressed by Eq. (2.63). Substituting the corresponding magnetic quantities \mathbf{H} and \mathbf{M} for the electrical quantities \mathbf{E} and \mathbf{P} in this equation, we get

$$\mathbf{H}_{\text{eff}} = \mathbf{H} + \frac{4\pi}{3} \mathbf{M} \quad (5.95)$$

As we shall see below, however, the experimental determination of the Weiss constant b in ferromagnetics results in such great values of this constant that the Weiss molecular field cannot in any way be reduced to the magnetic interaction of atoms.

According to Eq. (5.94), the part of the energy of an atom that depends on the direction of its magnetic moment will no longer be expressed by Eq. (5.79) or (5.91), but by the formula

$$U = -\mu \mathbf{H}_{\text{eff}} = -\mu(\mathbf{H} + b\mathbf{M}) \quad (5.96)$$

It makes absolutely no difference for our following treatment whether this energy is potential or kinetic, or is partly potential and partly kinetic (see Sec. 5.12, p. 345).

According to Eq. (5.96), the energy of an atom, other conditions being equal, will be the smaller, the closer is the direction of its magnetic moment to that of the magnetization \mathbf{M} of the body. In other words, the presence of a strong molecular field should manifest itself in the tendency of all the atoms to orient themselves in the same direction, i.e. in the tendency to spontaneous magnetization of the body.

Having made the main assumption expressed by Eq. (5.95), we may use Boltzmann's theorem in the following and in general repeat our reasoning of Secs. 2.10 and 5.11. The number of atoms in a unit volume of a ferromagnetic the angle of whose axes with the direction of the effective field \mathbf{H}_{eff} ranges from θ to $\theta + d\theta$, according to Eq. (2.68), is

$$dN = c \exp(a \cos \theta) \sin \theta d\theta$$

where we have introduced the symbol [cf. Eq. (2.69)]*

$$a = \frac{\mu H_{\text{eff}}}{kT} = \frac{\mu(\mathbf{H} + b\mathbf{M})}{kT} \quad (5.97)$$

* The absolute value of the quantity H_{eff} equals that of the quantity $\mathbf{H} + b\mathbf{M}$ only if the vector \mathbf{H} is parallel to \mathbf{M} . Although ferromagnetics are anisotropic and therefore the vectors \mathbf{H} and \mathbf{M} are parallel in them only with a definite orientation of these vectors, for simplicity's sake we shall limit ourselves only to this case.

In considering dielectrics with solid dipoles, we limited ourselves to the case that is virtually always encountered when $a \ll 1$, and correspondingly simplified all our calculations. For ferromagnetics, however, the condition $a \ll 1$, generally speaking, is not observed, and we have to perform all our calculations without any simplifications.

The proportionality constant c can be determined from the condition that the total number of atoms in a unit volume must equal N :

$$N = \int dN = c \int_0^{\pi} \exp(a \cos \theta) \sin \theta \, d\theta = \frac{c}{a} (e^a - e^{-a})$$

Solving this equation relative to c , we get

$$c = \frac{aN}{e^a - e^{-a}} \quad (5.98)$$

We shall now determine the resultant magnetic moment of a unit volume of the body, i.e. its magnetization \mathbf{M} . We consider the vector \mathbf{M} to be parallel to the effective field \mathbf{H}_{eff} (see the first footnote to p. 315), therefore its numerical value will equal the sum of the projections of the moments of all N atoms on the direction of \mathbf{H}_{eff} . The total moment of dN atoms whose axes are between θ and $\theta + d\theta$ equals μdN , and the projection of this moment onto the direction of \mathbf{H}_{eff} equals $\mu dN \cos \theta$. Consequently, the magnetization of a body equals

$$\begin{aligned} M &= \int \mu \cos \theta \, dN = c\mu \int_0^{\pi} \exp(a \cos \theta) \cos \theta \sin \theta \, d\theta = \\ &= c\mu \left(\frac{e^a + e^{-a}}{a} - \frac{e^a - e^{-a}}{a^2} \right) \end{aligned}$$

or upon substituting for c its value from Eq. (5.98), we get

$$M = N\mu \left(\coth a - \frac{1}{a} \right) \quad (5.99)$$

where \coth stands for the hyperbolic cotangent of a determined by the equation

$$\coth a = \frac{e^a + e^{-a}}{e^a - e^{-a}}$$

3. The combination of Eqs. (5.86) and (5.90) makes it possible to determine the magnetization M . Before applying them to ferromagnetics, it will be instructive to employ them for paramagnetism

characterized by the absence of a molecular field. For this purpose it is sufficient to assume in Eq. (5.86) that the constant b equals zero*. Equation (5.99) was first obtained by P. Langevin exactly for this particular case and bears his name, while the function $(\coth a - 1/a)$ is called the *Langevin function*.

When $a \ll 1$, we can expand $\coth a$ into a series by the powers of a :

$$\coth a = \frac{1}{a} + \frac{a}{3} - \frac{a^3}{45} + \dots$$

Introducing this expression into Eq. (5.99), limiting ourselves to the first two terms of the series, and taking into account Eq. (5.97), we get for $b = 0$

$$M = N\mu a = \frac{N\mu^2}{3kT} H$$

Hence, the susceptibility χ in this case equals

$$\chi = \frac{N\mu^2}{3kT}$$

This coincides with Eq. (5.80) which we obtained previously (Curie's law). If a is compatible with unity or is greater than unity, it is naturally necessary to use the accurate equation (5.99) instead of these approximate equations. When a tends to infinity (a strong magnetic field at a low temperature), $\coth a$ tends to unity, and, consequently, according to Eq. (5.99) the magnetization M asymptotically tends to the limiting value:

$$M_0 = N\mu \tag{5.100}$$

corresponding to the arrangement of the axes of all the atoms in one direction (*saturated magnetization*).

4. Returning to ferromagnetics ($b \gg 1$), let us first assume that the magnetic field H is either entirely absent or is so small that it may be disregarded in Eq. (5.97) in comparison with the molecular field bM , and we may assume, using the symbol introduced in Eq. (5.100), that

$$a = \frac{\mu b M}{kT} = \frac{N\mu b M}{NkT} = \frac{bM_0 M}{NkT} = \left(\frac{bM_0^2}{NkT} \right) \frac{M}{M_0}$$

If we introduce the symbol

$$\theta = \frac{bM_0^2}{3Nk} \tag{5.101}$$

* According to Eq. (5.95), we have $b = 4\pi/3$. In paramagnetics, however, the term bM may be ignored because for them $M \ll H$.

then this relationship can be written as follows:

$$\frac{M}{M_0} = \left(\frac{T}{3\theta} \right) a \quad (5.102)$$

It must be noted that the quantity θ characterizing the properties of a ferromagnetic has the dimension of temperature. On the other hand, the equation of the Langevin curve (5.99) can be written as follows on the basis of Eq. (5.100):

$$\frac{M}{M_0} = \coth a - \frac{1}{a} \quad (5.103)$$

Introducing a from Eq. (5.102) into Eq. (5.103), we can obtain an (implicit) functional dependence of M/M_0 on the quantity T/θ . It is simpler, however, to resort to the graphical method. Figure 67 shows the dependence of M/M_0 on a , the curve corresponding to Eq. (5.103), and the straight line AO to Eq. (5.102) (with a definite value of the ratio T/θ). It is obvious that the actual relative magnetization of a body M/M_0 corresponding to a given value of T/θ (i.e. to a given slope of the straight line OA) is determined by the points of intersection of the universal (not depending on the properties and state of a body) Langevin curve ($\coth - 1/a$) with the straight line OA .

In Fig. 67, there are two such points of intersection: one corresponds to the absence of magnetization ($M=0$), and the other (the point A) to magnetization up to about 0.8 of saturation. To determine the magnetization which a body will actually have at a given value of

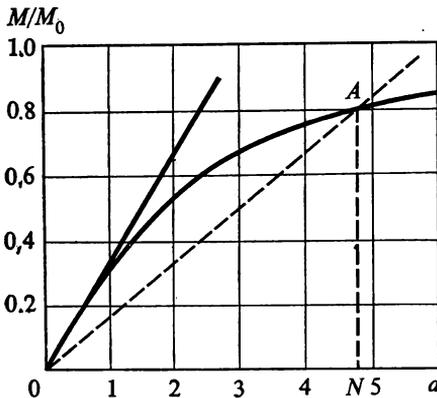


Fig. 67

T/θ , we must establish which of these two states of the body is *stable*. To explain the main properties of ferromagnetics, we must assume that the point A corresponding to the magnetized state of the body is stable, whereas the unmagnetized state $M = 0$ is unstable and is not realized in actual conditions. This assumption is supported by

quantum mechanical calculations according to which the magnetized state of ferromagnetics corresponds to a minimum of their free energy.

Everything said above remains in force for all the values of the ratio T/θ at which the straight line corresponding to Eq. (5.102) has two common points with the Langevin curve corresponding to Eq. (5.103). The origin of coordinates ($M = a = 0$) is always such a common point. As regards the second intersection of the straight line [Eq. (5.102)] with the Langevin curve, the shape of the latter shows that there will be such a point only if the inclination of this straight line to the axis of abscissas is smaller than the inclination of a tangent to the Langevin curve at the origin of coordinates (and shown in Fig. 67). The slope of this tangent will be

$$\left[\frac{d}{da} \left(\coth a - \frac{1}{a} \right) \right]_{a=0} = \frac{1}{3}$$

while the slope of the straight line according to Eq. (5.102) equals $T/3\theta$. Consequently, the straight line will intersect the Langevin curve twice if $T/3\theta$ is less than $1/3$ and, therefore, $T < \theta$.

It thus follows from the above theory that if $T < \theta$ only the magnetized state is stable and, consequently, a ferromagnetic should be magnetized even in the absence of an external magnetic field (spontaneous magnetization). The magnitude of the magnetization is determined by the second point of intersection of the straight line according to Eq. (5.102) with the Langevin curve (the point A in Fig. 67). If $T > \theta$, then in the absence of an external magnetic field the magnetization M should equal zero, i.e. at elevated temperatures a body should lose its ability to become magnetized spontaneously.

The latter circumstance corresponds to the experimental fact that all ferromagnetics when heated lose their ferromagnetic properties at a definite temperature and become paramagnetics. This critical temperature characteristic of each ferromagnetic (758 °C for iron, 374 °C for nickel, etc.) is called the "Curie temperature" or the "Curie point". Thus, the constant θ determined by Eq. (5.101) physically corresponds to the Curie point.

The existence of the Curie point above which a body loses its ferromagnetic properties is evidently due to the following. With adequate heating, the disordering influence of thermal motion on the orientation of the magnetic axes of atoms must in the long run become great enough to overcome the forces of interaction of the atoms tending to arrange their magnetic axes parallel to one another. 5. In our preceding treatment, we silently passed by some very significant questions. First of all, Weiss's theory, as we have seen, leads to the conclusion that at temperatures below the Curie point all ferromagnetics should spontaneously become magnetized even in the absence of an external magnetic field. At the same time, the usual state, for example, of iron at $T < \theta$ is in general a non-magnetic

state although, naturally, there do exist so-called permanent magnets, residual magnetization, and the like.

This apparent contradiction is explained by the fact that any ferromagnetic breaks up in a magnetic respect into a number of exceedingly small microscopic autonomous regions or domains*. Each such *Weiss domain* is always magnetized to a value corresponding to theory (the point *A* in Fig. 67). In ordinary conditions, however, the different Weiss domains are magnetized in different directions. As a result, the mean magnetic moment of the entire body equals zero, and the magnetization of the separate domains remains unnoticeable**.

When an external magnetic field is switched on, the number and dimensions of the domains whose magnetization is parallel to the field (or close to parallelity) grow at the expense of the domains magnetized in the opposite direction, and the body as a whole becomes magnetized. If the external field is now switched off, then the order in the directions of magnetization of the separate domains that has once appeared is partly retained (residual magnetization) until it is destroyed by a new factor such as heating or switching on of a field of the opposite direction exceeding a known minimum (called the coercive force).

Proceeding from these general ideas and also taking into consideration a number of other significant circumstances*** which we cannot stop to treat here, we shall find it possible, at least qualitatively, and sometimes also quantitatively, to explain all the other

* These domains must not be confused with the separate microcrystals which polycrystalline ferromagnetics usually consist of. An iron monocrystal, no matter how carefully it is prepared, also breaks up in a magnetic respect into such domains.

** The magnetization of an entire ferromagnetic in one definite direction in the absence of an external field is not advantageous from an energy viewpoint. This follows from the circumstance that according to quantum mechanics, the forces of interaction between atoms corresponding to the Weiss molecular field diminish very rapidly with the distance and virtually act only between adjacent atoms. Hence, when a magnetic breaks up into Weiss domains the potential energy increases only in the atoms adjoining the interfaces of the Weiss domains: only these atoms have neighbors with a non-parallel magnetic moment at the other side of the interface (the forces of interaction between neighboring atoms tend to arrange their magnetic moments parallel to each other). On the other hand, when a magnetic breaks up into Weiss domains the energy of the magnetic field induced by the ferromagnetic diminishes because the fields of differently oriented Weiss domains to a considerable extent compensate one another. Thus, the breaking of the Weiss domains results, on one hand, in an increase in the area and, consequently, in the potential energy of the interfaces, and on the other, in a reduction in the volume energy of the magnetic field. The actual (mean) dimension of the Weiss domains is determined by the minimum of the summary free energy of a ferromagnetic.

*** Magnetic anisotropy connected with the crystalline structure of ferromagnetics and manifesting itself in that the readiness with which they become magnetized in different directions differs; internal tensions and distortions of the crystal lattice that are always present in real bodies; magnetostriction, etc.

laws observed in ferromagnetics (the dependence of M on H , the hysteresis loop, etc.).

6. Owing to the lack of space, we are forced to limit ourselves to the consideration of only one additional question. In setting out the mathematical part of Weiss's theory, we assumed that the magnetic field H may be disregarded in comparison with the molecular field bM . What is the magnitude of this field, however? Having determined experimentally the Curie point θ and the saturation magnetization M_0^* , we can use Eq. (5.101) to find bM . It was established that, for example, in iron at normal temperatures bM reaches values of the order of magnitude of 7×10^6 Gs, i.e. it indeed quite considerably exceeds the intensity of practically achievable magnetic fields. Here values ranging from 4000 to 30 000 are obtained for the constant b .

This exclusively great magnitude of the molecular Weiss field was the main obstacle in the path of the classical theory of ferromagnetism and condemned to failure all attempts to reduce this field to the magnetic interaction of the atoms of a ferromagnetic. Indeed, Eq. (5.95) relating to quasi-elastic dipoles and leading to the value of $b = 4\pi/3$ cannot naturally be applied to ferromagnetics. In a magnetic too, however, if its atoms have a constant magnetic moment μ , the maximum possible intensity of the effective field due to the magnetic interaction of atoms cannot with respect to its order of magnitude appreciably exceed the intensity μ/R^3 of the field of the dipole μ at the centre of the adjacent atom (here R stands for the distance between adjacent atoms of a ferromagnetic). As regards their order of magnitude, the magnetic moment of an atom μ is about 10^{-20} Gs·cm³ and R is about 2×10^{-8} cm, so that μ/R^3 is about 10^3 Gs, i.e. one-thousandth of the Weiss field intensity. All the numerous attempts to circumvent this difficulty within the confines of the classical theory failed. Only quantum mechanics made it possible to solve the problem of the nature of the Weiss molecular field.

This explanation briefly consists in the following. If we take as the basis the conventional Coulomb law of interaction of the charges of electrons and atomic nuclei, but use the laws of quantum mechanics to determine the motion of electrons, the result would be the same as we would get on the basis of the laws of motion of classical mechanics with the presence of certain additional forces of interaction between the electrons apart from the Coulomb forces. Thus, if we want to use the notions of classical physics, we must introduce into consideration the corresponding additional forces, which have been called exchange forces. Their part in the phenomena of magnetism consists in that in known conditions relating to the electron structure of atoms, the structure of a crystal lattice, etc., these exchange forces

* To determine the latter, naturally, there is absolutely no need to achieve complete saturation experimentally. It is quite sufficient to find a number of values of M for comparatively great values of a and extrapolate along a Langevin curve to $a = \infty$.

tend to set up the spins of the electrons in adjacent atoms of a magnetic parallel to one another, i.e. tend to magnetize a body, which upon the observance of these conditions is ferromagnetic. In agreement with this, the studying of gyromagnetic effects in ferromagnetics showed (see Sec. 5.12) that ferromagnetism is due to the spin of electrons and not to their orbital motion.

Theoretical calculations show that the exchange interaction of atoms can be taken into account with sufficient accuracy by introducing into consideration a certain "molecular" field bM equivalent to it. Thus, the formal Weiss theory of ferromagnetism has received a physical substantiation on the basis of the general laws of quantum mechanics without any special assumptions and hypotheses.

Of course, apart from the exchange interaction following from Coulomb's law, there also exists the conventional classical magnetic interaction of atoms equivalent to the interaction of the corresponding magnetic dipoles. As already indicated, however, it is several thousand times weaker than the exchange interaction. Nevertheless, such phenomena as magnetic anisotropy and magnetostriction are explained exactly by the magnetic interaction of atoms.

5.14 Equations of the Field in Idealized Ferromagnetics (Conventional Variant). Permanent Magnets

1. The results of Secs. 5.1-5.3 are of an absolutely general nature and can be applied to any magnetics. In Sec. 5.4, however, we assumed that the magnetization of a medium \mathbf{M} is proportional to the intensity of the magnetic field \mathbf{H} , and thus excluded ferromagnetics from our consideration. Now we have to fill in this gap.

The phenomena of hysteresis in ferromagnetics mean that there is no unambiguous relationship between \mathbf{M} and \mathbf{H} . But even in such ferromagnetics in which hysteresis may be ignored owing to its insignificant value, the non-linear nature of the dependence of \mathbf{M} on \mathbf{H} introduces exceedingly great complications into theory. To avoid these complications, we shall limit ourselves to a consideration of the so-called "*idealized ferromagnetics*" or "*permanent magnets*" assuming that the magnetization \mathbf{M} of these magnetics consists of the sum of the *induced magnetization*, which is a linear function of the field intensity, and the *permanent magnetization* \mathbf{M}_0 , which does not depend at all on the field intensity. In this section, we shall proceed from the generally adopted assumption that the induced magnetization is proportional to the *intensity* of the magnetic field, i.e. that

$$\mathbf{M} = \mathbf{M}_0 + \chi\mathbf{H} \quad (5.104)$$

where \mathbf{M}_0 is a given position function, and χ is independent of \mathbf{H} . (A somewhat different variant of the theory of permanent magnets will be considered in the following section.) It is evident that Eq.

(5.104) includes as a particular case ($\mathbf{M}_0 = 0$) Eq. (5.32) holding for para- and diamagnetics. We shall call ferromagnetic bodies for which \mathbf{M}_0 differs from zero *permanent magnets*.

Thus, the relationship between \mathbf{M} and \mathbf{H} in idealized ferromagnetics remains *linear*, and we obtain the possibility of using the principle of superposition of the fields.

Although the properties of real ferromagnetics differ quite appreciably from those of our idealized ferromagnetics (as is shown by the very possibility of making permanent magnets from nonmagnetic steel, the demagnetization of magnets, etc.), in some cases Eq. (5.104) is nevertheless a certain approximation to actual conditions. For example, it may be applied for hard steel magnetized almost to saturation on condition that the changes in the external field are sufficiently small. At the same time, consideration of idealized ferromagnetics permits us to understand the old theories of magnetism operating with notions of magnetic charges or poles.

The notion of idealized ferromagnetics which we have introduced consists in essence according to Eq. (5.104), in the replacement of a ferromagnetic with a combination of a permanent magnet (in the strict meaning of this term) and a paramagnetic.

2. On the basis of Eqs. (5.26), (5.104), and (5.34), we have

$$\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M} = \mu\mathbf{H} + 4\pi\mathbf{M}_0 \quad (5.105)$$

Here and below μ is the permeability.

Since the previously derived differential equations of a field (5.24) and (5.25) remain true for an arbitrary medium, then the total system of equations of a field with permanent magnets present acquires the form

$$\text{curl } \mathbf{H} = \frac{4\pi}{c} \mathbf{j}, \quad \text{div } \mathbf{B} = 0, \quad \mathbf{B} = \mu\mathbf{H} + 4\pi\mathbf{M}_0 \quad (5.106)$$

(for brevity's sake we are not writing out the boundary conditions unambiguously following from the differential equations). It differs from the system of equations (C) in Sec. 5.5 only in the term $4\pi\mathbf{M}_0$ in the last equation and makes it possible to *unambiguously* determine the magnetic field if we know the distribution of the electric currents \mathbf{j} in the conductors, the distribution of the *permanent* magnetization \mathbf{M}_0 in the ferromagnetics, and, finally, the value of the permeability at each point of the medium.

It is the simplest to convince oneself that this unambiguity is true by dividing the field intensity \mathbf{H} into the sum of the intensities of a vortex and vortex-free fields:

$$\mathbf{H} = \mathbf{H}' + \mathbf{H}'', \quad \mathbf{B} = \mathbf{B}' + \mathbf{B}'' \quad (5.107)$$

and assuming that

$$\text{curl } \mathbf{H}' = \frac{4\pi}{c} \mathbf{j}, \quad \text{div } \mathbf{B}' = 0, \quad \mathbf{B}' = \mu\mathbf{H}' \quad (5.108)$$

and

$$\text{curl } \mathbf{H}'' = 0, \quad \text{div } \mathbf{B}'' = 0, \quad \mathbf{B}'' = \mu \mathbf{H}'' + 4\pi \mathbf{M}_0 \quad (5.109)$$

Equations (5.108) for the field of currents \mathbf{H}' completely coincide with the system (C) in Sec. 5.5 and as proved in the same Sec. 5.5 unambiguously determine \mathbf{H}' if we know the values of \mathbf{j} and μ for each point of space. Equations (5.109) for the field \mathbf{H}'' of permanent magnets can be written as follows:

$$\text{curl } \mathbf{H}'' = 0, \quad \text{div } \mu \mathbf{H}'' = 4\pi \rho_m^0 \quad (5.110)$$

where we have introduced the notation

$$\rho_m^0 = -\text{div } \mathbf{M}_0 \quad (5.111)$$

The quantity ρ_m^0 is customarily called the density of *permanent magnetic charges*. This historically established term is very convenient, although from the point of view of modern physics it is absolutely conditional.

When the equations of a field are written in this way, we can trace complete analogy between the magnetic field \mathbf{H}'' of permanent magnets and the electric field \mathbf{E} of electric charges: the field \mathbf{H}'' , being a vortex-free one, similar to \mathbf{E} has a single-valued scalar potential ψ :

$$\mathbf{H}'' = -\text{grad } \psi \quad (5.112)$$

Further, the sources of the vector $\mu \mathbf{H}''$ (which differs from the magnetic induction \mathbf{B}'' by $4\pi \mathbf{M}_0$) are the permanent magnetic charges ρ_m^0 similar to how the *free* electric charges ρ are the sources of the vector $\epsilon \mathbf{E}$ (equal to the electric displacement \mathbf{D}). Thus, there is complete agreement between \mathbf{H}'' and \mathbf{E} , between $\mu \mathbf{H}'' = \mathbf{B}'' - 4\pi \mathbf{M}_0$ and $\epsilon \mathbf{E} = \mathbf{D}$, and, finally, between ρ_m^0 and ρ . Consequently, all the results obtained when studying dielectrics can be applied to the field of permanent magnets (with substitution of the relevant magnetic quantities for their electric counterparts).

Particularly, it follows from the results of Sec. 2.3 that the field \mathbf{H}'' is unambiguously determined by setting the permeability μ as the position function and by setting the density ρ_m^0 of the permanent magnetic charges, which in turn is determined by setting \mathbf{M}_0 as the position function.

Thus, the magnetic field \mathbf{H} can be *unambiguously* represented as the superposition of two independent fields—that of the currents \mathbf{H}' and that of the magnets \mathbf{H}'' . We already considered the field of currents on an earlier page, so that now we can limit ourselves to considering the field of permanent magnets \mathbf{H}'' .

3. Using the relationship $\mu \mathbf{H}'' = \mathbf{H}'' + 4\pi \chi \mathbf{H}''$, we can write the second of equations (5.110) as follows:

$$\text{div } \mathbf{H}'' = 4\pi \rho_m^0 - 4\pi \text{div } \chi \mathbf{H}''$$

Thus, ρ_m^0 corresponds to the density of *free* electric charges (by setting ρ_m^0 we *determine* the field \mathbf{H}''), whereas

$$-\text{div } \chi \mathbf{H}'' = \rho_m' \quad (5.113)$$

corresponds to the density of the bound electric charges ($\rho_{\text{bound}} = -\text{div } \mathbf{P} = -\text{div } \alpha \mathbf{E}$): the quantity ρ'_m can be called the density of the *bound* magnetic charges *induced* in a magnetic by the field \mathbf{H}' . (Knowing ρ_m^0 and μ , we can determine \mathbf{H}' , and knowing \mathbf{H}' , we can determine ρ'_m ; particularly, if $\rho_m^0 = 0$, then both $\mathbf{H}' = 0$ and $\rho'_m = 0$.)

In a somewhat different respect, however, ρ_m^0 corresponds to the density not of the free, but of the *bound* electric charges: unlike free electric charges, permanent magnetic charges cannot be separated from one another. Indeed, let us consider an arbitrary body surrounded by a non-ferromagnetic medium. Let us conduct in this medium a closed surface S enveloping this body and confining the volume V . Integrating Eq. (5.111) over this volume and using Gauss's theorem (A.17), we get

$$\int_V \rho_m^0 dV = - \oint_S M_{0n} dS = 0 \tag{5.114}$$

because on the surface S , i.e. in a non-ferromagnetic medium, the permanent magnetization \mathbf{M}_0 identically equals zero. Thus, the total "magnetic charge" of any permanent magnet always equals zero.

When a field contains surfaces of discontinuity of the vectors \mathbf{H}' and \mathbf{M}_0 and of the permeability μ , the preceding equations must be supplemented with the following boundary conditions unambiguously following from them (cf. Secs. 1.6 and 4.8):

$$\left. \begin{aligned} \text{Curl } \mathbf{H}' &= 0, \quad \text{Div } \mu \mathbf{H}' = \mu_2 H'_{2n} - \mu_1 H'_{1n} = 4\pi \sigma_m^0 \\ \sigma_m^0 &= -\text{Div } \mathbf{M}_0 = -(M_{0,2n} - M_{0,1n}), \quad \sigma'_m = -\text{Div } \chi \mathbf{H}' \end{aligned} \right\} \tag{5.115}$$

where σ_m^0 and σ'_m signify the surface density of the permanent and induced magnetic charges. Equation (5.114), on the other hand, must be supplemented with an integral over the surfaces of discontinuity S' of the vector \mathbf{M}_0 inside of the volume V , i.e. with the sum of the surface magnetic charges σ_m^0 in this volume:

$$\int_V \rho_m^0 dV + \oint_{S'} \sigma_m^0 dS = 0 \tag{5.116}$$

Assume that a permanent magnet is uniformly magnetized over its entire volume ($\mathbf{M}_0 = \text{const}$). Hence, all the permanent magnetic charges will be concentrated on the surface of the magnet ($\rho_m^0 = 0$). If, particularly, the magnet has the shape of a right cylinder and if the magnetization \mathbf{M}_0 is parallel to the axis of the cylinder, then all the permanent magnetic charges will be distributed over the bases of the cylinder with the surface density

$$\sigma_m^0 = \pm M_0 \tag{5.117}$$

(because on the side surface of the cylinder $M_{0n} = 0$, while outside of the magnet $\mathbf{M}_0 = 0$). Thus, such a magnet having the cross section S can be considered as a combination of two magnetic charges equal in magnitude:

$$m_0 = \pm M_0 S$$

and of the opposite sign distributed over the bases of the magnet (the so-called *poles of the magnet*). The field of such a magnet can be considered as the superposition of the fields induced by each of its poles separately. It is exactly for this reason that the theories of the 18th and 19th centuries used the concept of magnetic charges or poles.

If, however, we cut the magnet in half, then each cut surface acquires the nature of a surface of discontinuity of the vector \mathbf{M}_0 , i.e. a surface magnetic charge of the same density $\sigma_m^0 = \pm M_0$ "appears" on it so that the sum of the charges of each separate piece equals zero.

4. It follows from the correspondence between the magnetic field of permanent magnets and the electric field of free electric charges established above that in a *homogeneous* magnetic ($\mu = \text{const}$) the scalar potential for the magnets equals [cf. Eq. (2.48)]

$$\psi = \frac{1}{\mu} \int \frac{\rho_m^0}{R} dV + \frac{1}{\mu} \int \frac{\sigma_m^0}{R} dS$$

and the intensity of the magnetic field \mathbf{H}'' equals

$$\mathbf{H}'' = \frac{1}{\mu} \int \frac{\mathbf{R}}{R^3} \rho_m^0 dV + \frac{1}{\mu} \int \frac{\mathbf{R}}{R^3} \sigma_m^0 dS \quad (5.118)$$

Thus, *the field intensity of permanent magnets in a homogeneous medium* is inversely proportional to the permeability of the medium. Unlike this, as we have seen in Sec. 5.5, the intensity of the magnetic field of currents does not at all depend on the permeability of the medium when the latter is homogeneous.* We shall find out why the field of magnets differs from that of currents in the next section.

5. It is generally adopted in the theory of permanent magnets to proceed from the assumption that the density \mathbf{f} of the ponderomotive forces acting on permanent magnetic charges, by analogy with electrostatics, is expressed by the formula

$$\mathbf{f} = \rho_m^0 \mathbf{H} \quad (5.119)$$

We shall prove the truth of this formula in Sec. 5.16. Meanwhile, we shall note that from the complete analogy of Eqs. (5.118) and (5.119) with the corresponding equations of electrostatics it follows

* The homogeneity of the medium, strictly speaking, means that μ has an identical value at all the points of the field of the magnets, including those inside the material which the magnets themselves are made from.

that the force of interaction of point (permanent) magnetic charges in a homogeneous medium is determined by *Coulomb's law*:

$$F = \frac{mm'}{\mu R^2} \quad (5.120)$$

Here, naturally, by a "point" magnetic charge or pole we must obviously understand a charge having a sufficiently small volume ΔV of a permanent magnet:

$$m = \int_{\Delta V} \rho_m^0 dV \quad (5.121)$$

It is exactly this Coulomb law (5.120) that played the part of the mean postulate in the theories of magnetism of the 18th and 19th centuries.

According to Eq. (5.119), the ponderomotive forces acting on magnetic charges in a magnetic field are determined by the intensity \mathbf{H} of this field, whereas the forces acting on electric currents, according to Eq. (5.41), are determined by the magnetic induction \mathbf{B} .

At the same time, as we have just noted, the intensity of the field of magnets in a homogeneous medium is inversely proportional to the permeability of this medium μ , whereas the intensity of the field of the currents in a homogeneous medium does not depend on μ . Comparing these results, we arrive at the following conclusions: *in a homogeneous magnetic medium ($\mu = \text{const}$) the ponderomotive forces of interaction of permanent magnets are inversely proportional to μ , the forces of interaction of a current and a permanent magnet do not depend on μ , and, finally, as already noted in Sec. 5.6, the forces of interaction of currents are directly proportional to μ .*

We shall find the reason for this difference between currents and permanent magnets in the following section.

6. We obtained the equations for a magnetic field at the beginning of this chapter from the notion that the properties of magnetics are due to the presence of molecular currents in them. Without changing these equations and only formally introducing the concept of the density of magnetic charges determined by Eq. (5.111), we showed that the theory of permanent magnets can be represented in a form corresponding to the notion of the existence of real magnetic charges in the molecules of magnetics.

Although actually no magnetic charges exist, it is nevertheless convenient in a number of cases to use this form of theory, which allows us to directly use the results of the theory of dielectrics in that of magnetism.

At the same time, the proved greatly extending equivalence of the theory of molecular currents and that of magnetic charges makes it easy to understand the success of the theory of magnetism that appeared in the 18th and 19th centuries and use the concept of

magnetic charges or poles. Indeed, owing to the equivalence of elementary currents and magnetic dipoles considered in Sec. 4.15, all the equations of the *macroscopic* field in any magnetics, and not only in permanent magnets, can be formally interpreted both from the viewpoint of the modern electron theory and on the basis of notions of the existence of magnetic charges in the molecules of magnetics*. If, however, we pass beyond the scope of the macroscopic theory which considers the permeability μ as a given characteristic of a magnetic and raise the questions of the mechanism of magnetization, of the dependence of the values of μ on other characteristics of a body, etc., we shall immediately see that the old theories of magnetism are not only incompatible with today's information on the structure of atoms, but that even in the field of macroscopic phenomena they are absolutely helpless, for instance, to explain diamagnetism or gyromagnetic phenomena (see Sec. 5.12).

5.15 Another Variant of the Equations of the Magnetic Field in Idealized Ferromagnetics.

The Equivalence of Electric Currents and Permanent Magnets

1. In the preceding section, we set out the generally adopted theory of permanent magnets according to which both the active and the passive characteristics of these magnets (i.e. both the field they induce and the forces acting on them in an external magnetic field) depend in an absolutely different way on the permeability μ of a medium than the corresponding magnetic characteristics of electric currents. In the present section, we shall try to find out the meaning and the cause of this difference.

First of all, we must note that a number of conclusions of the preceding section related to the hypothetic case of a medium that is *strictly homogeneous* in the magnetic respect (μ has an identical value *at all* points of a field). It is clear that this case has no practical significance.

Indeed, the permeability of the material which magnets are made of differs from that of the medium surrounding them, and thus the condition of the constancy of μ in the entire field is not observed. But even if it were observed upon a certain definite choice of the medium surrounding the magnets, then to verify the results obtained in Sec. 5.14 it would be necessary to vary the permeability μ of the medium and compare the field intensities, the magnitudes of the ponderomotive forces, etc. in media having different permeabilities. Here, however, the permeability of the permanent magnets them-

* If we segregate ourselves from the difficulties connected with the fact that the susceptibility χ of diamagnetics is negative.

selves would necessarily remain unchanged, i.e. the constancy of the value of μ in the entire field would be violated.

It is thus necessary to distinguish the *hypothetic* case of a *strictly homogeneous medium* and the practically single interesting case of the *homogeneity of the external medium* (i.e. the medium outside of the magnets). Therefore, the question raised at the beginning of the section must be divided into two: (1) how is the difference between magnets and currents in a *strictly homogeneous* medium established in Sec. 5.14 explained? and (2) how does the field of magnets and the forces acting on them depend on the permeability of a homogeneous *external medium*?

Let us consider these questions one at a time.

2. The difference between magnets and currents in a *strictly homogeneous* medium established in Sec. 5.14 is explained quite simply by the fact that, following the historical tradition (whose origin will be treated below), we proceeded in Sec. 5.14 from Eq. (5.104):

$$\mathbf{M} = \chi\mathbf{H} + \mathbf{M}_0$$

i.e. we assumed that the *induced* magnetization \mathbf{M}_{ind} of permanent magnets is proportional to the field *intensity* in them:

$$\mathbf{M}_{\text{ind}} = \chi\mathbf{H} \quad (5.122)$$

For dia- and paramagnetics, this formula is evidently equivalent to the formula

$$\mathbf{M}_{\text{ind}} = \frac{\chi}{\mu} \mathbf{B} \quad (5.123)$$

In permanent magnets, however, $\mathbf{H} \neq \frac{1}{\mu} \mathbf{B}$, and the equivalence of Eqs. (5.122) and (5.123) is violated. Therefore, in attempting to establish a *linear* system of equations of the field in idealized ferromagnetics which at $\mathbf{M}_0 = 0$ would transform into the equations for the field in dia- and paramagnetics that we already know, we could proceed with equal right from Eq. (5.123) instead of Eq. (5.122), i.e. instead of Eq. (5.104) we could assume that

$$\mathbf{M} = \frac{\chi}{\mu} \mathbf{B} + \mathbf{M}_0 \quad (5.124)$$

We can choose one of the Eqs. (5.104) and (5.124) only on the basis of a more detailed analysis of them, which we shall do somewhat later. Meanwhile we shall consider the variant of the equations of a field based on substituting Eq. (5.124) for Eq. (5.104). We shall call it the “new variant” to distinguish it from the “conventional variant”.

First of all in the new variant, Eq. (5.105) will be replaced by the formula

$$\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M} = \mathbf{H} + \frac{4\pi\chi}{\mu}\mathbf{B} + 4\pi\mathbf{M}_0$$

whence on the basis of Eq. (5.34) we have

$$\mathbf{B} = \mu\mathbf{H} + 4\pi\mu\mathbf{M}_0 \quad (5.125)$$

This equation differs from Eq. (5.105) in the appearance of the factor μ in the last term of the right-hand side.

The corresponding change must also be introduced into Eqs. (5.106) and (5.109); further, in Eqs. (5.111) and (5.115) it is necessary in the new variant to change the determination of the volume and surface density of the permanent magnetic charges and assume that

$$\rho_m^0 = -\operatorname{div} \mu\mathbf{M}_0 \quad \text{and} \quad \sigma_m^0 = -\operatorname{Div} \mu\mathbf{M}_0 \quad (5.126)$$

All the remaining (numbered) equations of Sec. 5.14, except for Eqs. (5.117) and (5.121), as can easily be seen, also remain true in the new variant.

Let us now consider a *strictly homogeneous* medium. Here in Eqs. (5.126) we can put μ outside the spatial derivative:

$$\rho_m^0 = -\mu \operatorname{div} \mathbf{M}_0 \quad \text{and} \quad \sigma_m^0 = -\mu \operatorname{Div} \mathbf{M}_0 \quad (5.127)$$

Thus in the new formulation, unlike the conventional one, *the density of permanent magnetic charges (at a given permanent magnetization \mathbf{M}_0) is proportional to the permeability μ .*

Introducing Eq. (5.127) into Eq. (5.118), we see that *the intensity of the field of permanent magnets (at a given value of \mathbf{M}_0) does not depend on the permeability of the (strictly homogeneous) medium*, i.e. that the previous difference in this respect between magnets and currents disappears in the new variant.

Finally, introducing Eq. (5.127) into Eq. (5.119) and assuming that the permeability μ is constant, if not in the entire medium, then at least in the sections of a ferromagnetic where \mathbf{M}_0 differs from zero, we get

$$\mathbf{f} = \rho_m^0\mathbf{H} = -\mu\mathbf{H} \operatorname{div} \mathbf{M}_0 \quad (5.128)$$

Thus, in the new variant, *the forces acting on magnetic charges are determined (at a given value of \mathbf{M}_0) not by the field intensity \mathbf{H} , but by the vector $\mu\mathbf{H}$.* Consequently, owing to \mathbf{H} being independent of μ , the ponderomotive forces of interaction of magnets in a *strictly homogeneous* medium, like the forces of interaction of currents, are directly proportional to μ , i.e. in this respect too the new variant eliminates the difference between permanent magnets and currents present in the conventional variant.

3. Which of the variants of the theory of permanent magnets should be preferred?

The conventional variant of the theory can be characterized either by Eq. (5.104) or by Eq. (5.111) equivalent to it according to which the density of the permanent magnetic charges is determined by the permanent magnetization vector \mathbf{M}_0 regardless of the permeability μ of the substance of the magnet. On the other hand, the new variant can be characterized either by Eq. (5.124) or by Eq. (5.126) equivalent to it according to which ρ_m^0 with \mathbf{M}_0 constant changes in proportion to the permeability μ of the magnet.

If there were a way of changing the permeability μ of the substance which a given magnet is made of without changing the permanent magnetization \mathbf{M}_0 , it would be possible to establish experimentally which of the variants of the theory corresponds to actual facts. Since there is no such way, however, then *the difference between the two variants is purely a terminological one*: the entire difference between the second and the first variants consists in the replacement of Eq. (5.105), $\mathbf{B} = \mu\mathbf{H} + 4\pi\mathbf{M}_0$, with Eq. (5.125), $\mathbf{B} = \mu\mathbf{H} + 4\pi\mu\mathbf{M}_0$. Since, by definition, the values of \mathbf{M}_0 and μ at each point of a permanent magnet are constant in time, no conclusions of theory that can be verified experimentally can depend on whether the difference $\mathbf{B} - \mu\mathbf{H}$ is denoted by $4\pi\mathbf{M}_0$ or by $4\pi\mu\mathbf{M}_0$.

4. Thus, the difference between the dependence of the active and passive characteristics of magnets and currents on the permeability of a *strictly homogeneous* medium established in Sec. 5.14 is, in essence, a terminological one. The terminology of the conventional variant of the theory has been taken from the theories of magnetism of the 19th century. They proceeded from the notions of the existence of real magnetic charges in the molecules of a magnetic and of the interaction of these charges according to Coulomb's law and correspond quite well to these notions. The terminology of the new variant, on the other hand, corresponds to the modern notions of the nature of magnetism. This tells, first, in that it eliminates the unjustified terminological difference between the dependence of the active and passive characteristics of magnets and currents on μ ; second, in that it relates the constancy of the magnetization \mathbf{M}_0 not to the constancy of the fictitious charges ρ_m^0 , but to the *constancy of the molecular currents* \mathbf{j}_{mol} that this magnetization is due to.

Indeed, according to modern notions, the permanent magnetization \mathbf{M}_0 must owe its existence to steady molecular currents whose (mean) density is related to \mathbf{M}_0 by the expression [see Eqs. (5.13) and (5.14)]

$$\mathbf{j}_{\text{mol}}^0 = c \text{curl } \mathbf{M}_0 \quad \text{and} \quad \mathbf{i}_{\text{mol}}^0 = c \text{Curl } \mathbf{M}_0 \quad (5.129)$$

Using Eqs. (5.129) and (5.125), we obtain

$$\text{curl} \left(\frac{\mathbf{B}}{\mu} \right) = \text{curl } \mathbf{H} + 4\pi \text{curl } \mathbf{M}_0 = \frac{4\pi}{c} (\mathbf{j} + \mathbf{j}_{\text{mol}}^0) \quad (5.130)$$

Consequently, according to the new variant, the steady molecular currents are absolutely equivalent to the conduction currents \mathbf{j} .

The replacement of Eq. (5.124) with Eq. (5.122), however, i.e. the transition to the conventional variant, violates this equivalence, namely, in this case the latter equation is replaced by the following one:

$$\operatorname{curl} \left(\frac{\mathbf{B}}{\mu} \right) = \operatorname{curl} \mathbf{H} + 4\pi \operatorname{curl} \frac{\mathbf{M}_0}{\mu} = \frac{4\pi}{c} \left(\mathbf{j} + \frac{1}{\mu} \mathbf{j}_{\text{mol}}^0 \right) + 4\pi \left[\nabla \frac{1}{\mu} \mathbf{M}_0 \right]$$

In general, it must be noted that although in the new variant too we can operate with the fictitious density of the magnetic charges ρ_m^0 and σ_m^0 , in this variant, unlike the conventional one, the distribution of the steady molecular currents $\mathbf{j}_{\text{mol}}^0$ and $\mathbf{i}_{\text{mol}}^0$ is a rational characteristic of permanent magnets and not ρ_m^0 and σ_m^0 . Knowing this distribution, we can determine the magnetic field of a permanent magnet with the aid of Eq. (5.129), and the ponderomotive forces acting on a magnet with the aid of the equation

$$\mathbf{f} = \frac{1}{c} [\mathbf{j}_{\text{mol}}^0 \cdot \mathbf{B}]$$

The equivalence of this expression and Eq. (5.119) which we used up to now will be proved in the following section.

5. Whereas the entire question of magnets and currents in a *strictly homogeneous* medium is of a purely formal nature, the question of the active and passive characteristics of magnets and currents in a *homogeneous external* medium, which we shall now pass over to, has a direct physical content, and the answer to it can be verified experimentally. It must be noted that the answer to the last question will be absolutely the same in both the conventional and the new variants of the theory.

Let us first consider a permanent magnet made of a homogeneous material having the permeability μ_i placed in a homogeneous external medium having the permeability μ_e . It is simple to see that there is no unambiguous functional dependence of the field intensity of the magnet in the external medium on the permeability of this medium μ_e , and that this dependence is determined by the *geometrical* shape of the magnet.

Let us consider, particularly, a uniformly magnetized ($\mathbf{M}_0 = \text{const}$) ellipsoid—this is one of the few cases when we are able to find an analytical expression in closed form for the field of the magnet. For our purposes, it is sufficient to consider an ellipsoid of revolution magnetized parallel to its axis of symmetry. Let us denote the lengths of the principal axes of the ellipsoid by a , b , and c , and let $b = c$.

We can show that the intensity \mathbf{H}_e of the field of such a magnet at an arbitrary point of the external medium depends on μ_e as follows:

$$\mathbf{H}_e = \frac{\mathbf{h}(x, y, z)}{\mu_i + \left(\frac{1}{u} - 1\right) \mu_e} \quad (5.131)$$

where the vector $\mathbf{h}(x, y, z)$ does not depend on μ_e , while the constant u depends only on the ratio a/c of the ellipsoid axes:

$$u = \frac{1}{2} \int_0^\infty \frac{d\xi}{(1 + \xi)^{3/2} \left(1 + \frac{a^2}{c^2} \xi\right)} \quad (5.132)$$

It is easy to obtain this result by using the solution of the problem of the field of a uniformly magnetized ellipsoid that can be found in a number of courses on mathematical physics or the theory of the potential*. We shall give this solution without its proof. Let us place the origin of Cartesian coordinates at the centre of the ellipsoid and direct the x -axis along the axis of symmetry of the ellipsoid, i.e. in the direction of magnetization. The scalar potential ψ of the magnetic field of the ellipsoid inside of it will equal

$$\psi_i = \alpha x$$

where α is a constant, while the potential outside of the ellipsoid will be

$$\psi_e = \frac{\beta x}{2} \int_{\eta/a^2}^\infty \frac{d\xi}{(1 + \xi)^{3/2} \left(1 + \left(\frac{a^2}{c^2} \xi\right)\right)}$$

where β is a constant, and η is a function of the coordinates x , y , and z determined by the equation

$$\frac{x^2}{a^2 + \eta} + \frac{y^2 + z^2}{c^2 + \eta} = 1$$

On the basis of this expression, we can show for ψ_e , first, that on the surface of the ellipsoid

$$\psi_e = \beta u x$$

where the constant u is determined by Eq. (5.132), and, second, that on the surface of the ellipsoid the derivative of ψ_e with respect

* See, for instance, Stratton, J. A. *Electromagnetic Theory*. New York, McGraw Hill (1941).

to the normal \mathbf{n} to this surface is

$$\frac{\partial \psi_e}{\partial n} = \beta(u - 1) \frac{\partial x}{\partial n}$$

The continuity of the tangents that are the components of the intensity \mathbf{H} of the magnetic field on the surface of the ellipsoid is equivalent to the continuity of the potential ψ on this surface. This yields

$$\alpha = \beta u$$

Further, the continuity of the normal component of the magnetic induction \mathbf{B} on the same surface, in view of Eq. (5.105), is expressed by the equation

$$-\mu_e \frac{\partial \psi_e}{\partial n} = -\mu_i \frac{\partial \psi_i}{\partial n} + 4\pi M_{0n}$$

Since $\partial \psi_i / \partial n = \alpha (\partial x / \partial n)$ and $M_{0n} = M_0 (\partial x / \partial n)$, then using the above expression for $\partial \psi_e / \partial n$ and cancelling $\partial x / \partial n$, we get

$$-\mu_e \beta(u - 1) = -\mu_i \alpha + 4\pi M_0$$

Finally, excluding the constant α from the last two equations, we get

$$\beta = \frac{4\pi M_0}{\mu_e(1 - u) + \mu_i u} = \frac{\text{const}}{\mu_i + \mu_e \left(\frac{1}{u} - 1 \right)}$$

Since ψ_e and, consequently, \mathbf{H}_e depend on μ_e only through the coefficient β , then formula (5.131) has been proved.

The integral of Eq. (5.132) is expressed in elementary functions, namely:

when $\gamma = a/c < 1$

$$u = \frac{1}{1 - \gamma^2} \left[1 - \frac{1}{\lambda} \left(\frac{\pi}{2} - \arctan \frac{1}{\lambda} \right) \right]$$

where

$$\lambda = \frac{\sqrt{1 - \gamma^2}}{\gamma}$$

when $\gamma = a/c > 1$

$$u = \frac{1}{\gamma^2 - 1} \left(\frac{1}{2\lambda} \ln \frac{1 + \lambda}{1 - \lambda} - 1 \right)$$

where

$$\lambda = \frac{\sqrt{\gamma^2 - 1}}{\gamma}$$

For our purposes it is sufficient to consider three cases.

(1) The ellipsoid is stretched out, and its axis of symmetry is much longer than the other two axes: $a \gg c$. In other words, the magnet is a long thin bar. In this case, the value of u is very small $\left(u \sim \frac{c^2}{a^2} \ln \frac{a}{c}\right)$, the factor at μ_e in the denominator of Eq. (5.131) is very great, and H_e in a first approximation is inversely proportional to μ_e :

$$H_e \sim \frac{1}{\mu_e} \tag{5.133}$$

(2) The ellipsoid degenerates into a sphere: $a = c$. In this case (which will also be directly considered in Problem 34 at the end of this section), $u = 1/3$, and Eq. (5.131) becomes

$$H_e \sim \frac{1}{\mu_1 + 2\mu_e} \tag{5.134}$$

(3) Finally, the ellipsoid is flattened, and its axis of symmetry is much shorter than the other two axes: $a \ll c$. In other words, the magnet is a flat disk magnetized perpendicular to its plane of symmetry. In this case, the value of u is close to unity $\left(u \sim 1 - \frac{\pi a}{2c}\right)$, the factor at μ_e in Eq. (5.131) is very small, and H_e in the first approximation does not at all depend on μ_e :

$$H_e \sim \frac{1}{\mu_1} \tag{5.135}$$

Thus, the shape of a magnet plays a decisive part, and there is no universal dependence of the field of a magnet on the permeability μ_e of the external medium. The frequently encountered statement that *the intensity of the field of a magnet is inversely proportional to μ_e* holds *only for long bar-shaped magnets*. Disk-shaped magnets, however, are similar to electric currents in the respect that the intensity of their field in the external medium does not practically depend on its permeability μ_e .

Everything said about permanent magnets is obviously also true for *electromagnets with a magnetic core*—the molecular currents responsible for permanent magnetism create a field according to the same law as the conduction currents, while for the applicability of the preceding reasoning it is only of significance that the permeability μ_1 of the material over whose surface the currents are circulating remains constant when μ_e changes.

Matters are different with the field of a *line current contour* or the field of an (open) solenoid. Here the external medium fills the space between the current turns, and the intensity of the magnetic field does not virtually depend on its permeability μ_e (as with a strictly

homogeneous medium). Indeed, in this case the closed (or helical—see Sec. 4.12) lines of force of the magnetic field are mainly arranged *completely in the homogeneous external medium*. Upon magnetization of the medium, the magnetic dipoles of this medium, orienting themselves in the direction of the field, will form closed (or helical deprived of a beginning and an end) chains, and as a result the charges of adjacent dipoles will neutralize one another. Such magnetization of the medium will obviously not cause a change in the field.

Conversely, when a core is present, virtually every magnetic line of force intersects the interface between the core and the external medium. The jump in the magnetization on this surface is equivalent to the appearance on it of induced surface magnetic charges [see Eq. (5.115)] having the density

$$\sigma'_m = -\text{Div } \chi \mathbf{H} = \pm (\chi_e H_{en} - \chi_i H_{in})$$

where χ_e and χ_i are the susceptibilities of the external medium and the magnet, respectively. The field of these induced charges obviously alters the resultant field of the current.

6. It remains for us to consider the dependence of the passive characteristics of magnets and currents on the permeability of a homogeneous external medium. We shall show in the following section that the forces acting on the source of a magnetic field (magnet and current) in a given external magnetic field with the induction \mathbf{B} can be unambiguously determined if we know the “intrinsic” field \mathbf{H}' induced by this source. Therefore, the forces with a given induction of the external field \mathbf{B} change with changes in μ_e in the same way as their intrinsic field \mathbf{H}' . Thus, everything said about the active characteristics of magnets and currents can be directly applied to their passive characteristics. Particularly, there is no universal dependence of both the passive and the active characteristics of magnets and currents on the permeability μ_e of a homogeneous *external* medium. In this respect, there is also no difference between permanent magnets and electromagnets with *cores*: if these magnets have an elongated bar shape, the force acting on them is determined by the intensity of the external field \mathbf{H} (i.e. inversely proportional to μ_e with a given value of \mathbf{B}); if they have the shape of a disk, then the force acting on them is determined by the induction of the external field \mathbf{B} (i.e. with a given \mathbf{B} it does not depend on μ_e). The latter case also holds for line currents and electromagnets without a core.

We can now understand the conventional statement that when measuring a magnetic field according to the magnitude of a couple of forces acting on a *magnetic pointer or a current loop* placed in it, in the first case (an elongated bar) we measure the *intensity* \mathbf{H} of the field, and in the second its *induction* \mathbf{B} .

Problem 34. Determine the field of a uniformly magnetized permanent magnet having a spherical shape in a homogeneous external medium with the permeability μ_e .

5.16 Ponderomotive Forces Acting on Permanent Magnets in an External Magnetic Field

1. In Secs. 5.14 and 5.15, we used Eq. (5.119) to calculate the forces acting on a magnet and wrote it in the same way as for electrostatics. Now we shall give a proper substantiation of this equation.

A strict derivation of the expression for the ponderomotive forces acting in a magnetic field will be given in Sec. 6.8. This derivation, however, will be based on a definite expression for the energy of a magnetic field that holds only for non-ferromagnetic media. Therefore, the results of Sec. 6.8 cannot be directly applied to permanent magnets. If, however, we abandon the consideration of the internal tensile forces induced in permanent magnets by an external magnetic field and limit ourselves to determining, first, the resultant \mathbf{F} of the forces applied to a magnet and, second, the resultant moment \mathbf{N} of these forces, then the results obtained in Secs. 2.14 and 6.9 will be quite sufficient for a single-valued solution of this problem.

Indeed, the ponderomotive forces of an electromagnetic field can be reduced to tensile stresses (see Secs. 2.14 and 6.9). Therefore, the components of the resultant \mathbf{F} and the couple of forces \mathbf{N} applied to a magnet should equal [see Eqs. (2.112) and (2.119)]

$$F_x = \oint_S T_{nx} dS, \quad N_x = \oint_S (yT_{zn} - zT_{yn}) dS \tag{5.136}$$

where the closed surface of integration S envelops the magnet and adheres to its surface from its *outer* side. In other words, S is completely in a non-ferromagnetic medium. Hence, the components of the stress tensor \mathbf{T} in Eq. (5.136) should be determined by formulas that hold for a non-ferromagnetic medium.

It will be proved in Sec. 6.9 that the components of the stress tensor of a magnetic field in a non-ferromagnetic medium equal

$$T_{xx} = \frac{\mu}{4\pi} \left(H_x^2 - \frac{1}{2} H^2 \right), \tag{5.137}$$

$$T_{xy} = \frac{\mu}{4\pi} H_x H_y, \quad T_{xz} = \frac{\mu}{4\pi} H_x H_z$$

and similar expressions are obtained for T_{yy} , T_{yz} , etc *. It must be noted that from the viewpoint of the theory based on the notion of the existence of magnetic charges, Eq. (5.137) can be directly obtained from the relevant electrostatic equation (2.122) by substituting \mathbf{H} for \mathbf{E} and μ for ϵ .

* We limit ourselves here to a consideration of only the Maxwellian stress tensor denoted in Sec. 6.9 by \mathbf{T}' and discard the tensor \mathbf{T}'' because the latter changes only the distribution of the tensile stresses in a medium, but does not affect \mathbf{F} and \mathbf{N} .

Using Eq. (5.137) in Eq. (5.136), we can calculate the resultant force \mathbf{F} and the resultant moment \mathbf{N} of the forces acting on a magnet in the magnetic field \mathbf{H} .

2. If when introducing Eq. (5.137) into Eq. (5.136) we transform the integrals in Eq. (5.136) with the aid of Eqs. (2.113) and (2.117), then we get the formulas

$$\mathbf{F} = \int_V \mathbf{f}_{\text{eq}} dV \quad \text{and} \quad \mathbf{N} = \int_V [\mathbf{Rf}_{\text{eq}}] dV \quad (5.138)$$

where the following notation has been introduced:

$$f_{x, \text{eq}} = \frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} + \frac{\partial T_{xz}}{\partial z} \quad (5.139)$$

It is assumed that the components T_{xx} , T_{xy} , etc. of the tensor \mathbf{T} are expressed by Eqs. (5.137), which hold for a non-ferromagnetic medium. The vector \mathbf{f}_{eq} determined for this condition can be called the “equivalent density” of the ponderomotive forces in a permanent magnet. Indeed, it follows from the above that we shall get the correct value of the resultant force \mathbf{F} and resultant moment \mathbf{N} of the forces applied to a magnet if we introduce Eq. (5.139) into Eqs. (5.138) and integrate over the entire volume of the magnet. The true distribution of the ponderomotive forces \mathbf{f} over the volume of a magnet, however, may differ absolutely from the distribution of the “equivalent” forces \mathbf{f}_{eq} because the stress tensor \mathbf{T} is expressed by Eqs. (5.137) only *outside* of the magnet. We do not know its components inside the magnet, and they may depend in an absolutely different way on the vectors \mathbf{H} and \mathbf{B} . We can therefore only state that if the forces \mathbf{f} actually applied to the elements of volume of a magnet are replaced by the forces \mathbf{f}_{eq} , then this will change neither the resultant nor the resultant moment of all the forces applied to the magnet.

Thus, the equivalent density of the forces \mathbf{f}_{eq} may be used only when we are not interested in the distribution of the ponderomotive forces over the volume of a magnet. On the other hand, the introduction of the equivalent forces \mathbf{f}_{eq} is very convenient in these cases because, as we shall now show, \mathbf{f}_{eq} can be expressed quite simply through the density of the permanent magnetic charges ρ_m^0 and the density of the conduction currents \mathbf{j} in a magnet.

3. Using Eqs. (5.137) in Eq. (5.139) and differentiating, after the appropriate regrouping of the terms we get

$$\begin{aligned} 4\pi f_{x, \text{eq}} = & H_x \operatorname{div}(\mu \mathbf{H}) + \mu H_y \left(\frac{\partial H_x}{\partial y} - \frac{\partial H_y}{\partial x} \right) + \\ & + \mu H_z \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) - \frac{1}{2} H^2 \frac{\partial \mu}{\partial x} \end{aligned} \quad (5.140)$$

Since

$$\begin{aligned} \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} &= \text{curl}_z \mathbf{H} = \frac{4\pi}{c} j_z \quad \text{and} \quad \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = \\ &= \text{curl}_y \mathbf{H} = \frac{4\pi}{c} j_y \end{aligned}$$

then the sum of the second and third terms is

$$\frac{4\pi}{c} (j_y \cdot \mu H_z - j_z \cdot \mu H_y) = \frac{4\pi}{c} [\mathbf{j} \cdot \mu \mathbf{H}]_x$$

Therefore, Eq. (5.140) is the component of the following vector equation along the x -axis:

$$\mathbf{f}_{\text{eq}} = \frac{1}{4\pi} \mathbf{H} \text{div} (\mu \mathbf{H}) + \frac{1}{c} [\mathbf{j} \cdot \mu \mathbf{H}] - \frac{1}{8\pi} H^2 \nabla \mu$$

Using Eq. (5.110), which may be applied in both the conventional and the new variants of the theory, we get a final expression for the equivalent density of the ponderomotive forces in a permanent magnet:

$$\mathbf{f}_{\text{eq}} = \rho_m^0 \mathbf{H} + \frac{1}{c} [\mathbf{j} \cdot \mu \mathbf{H}] - \frac{1}{8\pi} H^2 \nabla \mu \quad (5.141)$$

The first term in the right-hand side expresses the force acting on the permanent magnet charges and coincides with Eq. (5.119) which we have thus proved.

The second term differs from the conventional expression (5.41) for the force acting on conduction currents only by the replacement of the vector \mathbf{B} with the vector $\mu \mathbf{H}$ that differs from \mathbf{B} only in permanent magnets [see. Eq. (5.105)]. Finally, the last term in the right-hand side expresses the force depending on the non-uniformity of the permeability of a magnet. Up to this substitution of the vector $\mu \mathbf{H}$ for the vector \mathbf{B} , the group including the last two terms in Eq. (5.141) coincides with Eq. (6.77) for the ponderomotive forces in non-ferromagnetic media*.

4. Equation (5.141) corresponds as regards its form to the notion of the existence of magnetic charges in permanent magnets. We can, however, also obtain another equivalent expression for \mathbf{f}_{eq} in which the permanent magnets are characterized by the distribution in them of constant molecular currents instead of magnetic charges.

Instead of transforming Eq. (5.141) as required, it is more convenient for this purpose to return to the initial equations (5.137). Taking advantage of the fact that $\mathbf{M}_0 = 0$ on the surface S over which integration is performed in Eq. (5.136) and therefore $\mathbf{B} = \mu \mathbf{H}$, we

* Equation (6.77), apart from the forces \mathbf{f}' , also includes the striction forces \mathbf{f}' that correspond to the striction tensor \mathbf{T}'' which we have discarded (see the footnote on p. 178).

can write Eqs. (5.137) in the following form:

$$T_{xx} = \frac{1}{4\pi\mu} \left(B_x^2 - \frac{1}{2} B^2 \right), \quad T_{xy} = \frac{1}{4\pi\mu} B_x B_y, \quad (5.142)$$

$$T_{xz} = \frac{1}{4\pi\mu} B_x B_z$$

Using these expressions in Eq. (5.139) and differentiating, after regrouping terms we get

$$4\pi f_{x, \text{eq}} = \frac{1}{\mu} B_x \operatorname{div} \mathbf{B} + B_y \left\{ \frac{\partial}{\partial y} \left(\frac{B_x}{\mu} \right) - \frac{\partial}{\partial x} \left(\frac{B_y}{\mu} \right) \right\} +$$

$$+ B_z \left\{ \frac{\partial}{\partial z} \left(\frac{B_x}{\mu} \right) - \frac{\partial}{\partial x} \left(\frac{B_z}{\mu} \right) \right\} - \frac{B^2}{2\mu^2} \frac{\partial \mu}{\partial x}$$

The first term in the right-hand side equals zero because $\operatorname{div} \mathbf{B} = 0$. The following two terms can be transformed with the aid of Eq. (5.130):

$$\operatorname{curl} \left(\frac{\mathbf{B}}{\mu} \right) = \frac{4\pi}{c} (\mathbf{j} + \mathbf{j}_{\text{mol}}^0)$$

in absolutely the same way in which we transformed similar terms in Eq. (5.140). As a result, we get the following expression for \mathbf{f}_{eq} :

$$\mathbf{f}_{\text{eq}} = \frac{1}{c} [\mathbf{j} + \mathbf{j}_{\text{mol}}^0 \cdot \mathbf{B}] \frac{B^2}{8\pi\mu^2} \nabla \mu \quad (5.143)$$

The last term in this expression corresponds to the last term in the previous formula (5.141) and coincides with it when $\mathbf{M}_0 = 0$. The main first term of Eq. (5.143) expresses the circumstance that the ponderomotive action of the external field \mathbf{B} on constant molecular currents $\mathbf{j}_{\text{mol}}^0$ is the same as on the conduction currents \mathbf{j} , and that permanent magnets are completely characterized by the distribution of these currents $\mathbf{j}_{\text{mol}}^0$ in them.

5. In concluding, we shall make two remarks. Equations (5.141) and (5.143) for the equivalent density of the ponderomotive forces of the external field in permanent magnets are equivalent to each other in the sense that, as follows from the derivation of these expressions, they both after introduction into Eq. (5.138) lead to the same value of the resultant \mathbf{F} and the resultant moment \mathbf{N} of the forces acting on a magnet in a magnetic field. These equations (5.141) and (5.143), however, are by no means equal to each other, i.e. they correspond to different distributions of the tensile stresses and volume forces over the volume of a magnet. We cannot even say with certainty that at least one of these distributions corresponds to actual facts because the method we have used makes it possible to determine only the resultant \mathbf{F} and the resultant moment \mathbf{N} of the forces applied

to a magnet. The distribution of these forces over the volume of a magnet can be obtained only by a much deeper analysis of the entire problem as a whole.

In this section, we limited ourselves for brevity's sake to a consideration of only space magnetic charges ρ_m^0 and space molecular currents \mathbf{j}_{m0l}^0 . Expressions for the surface density of the ponderomotive forces applied to the surface charges and currents σ_m^0 and \mathbf{i}_{m0l}^0 will obviously be obtained from the corresponding terms of Eqs. (5.141) and (5.143) by replacing ρ_m^0 with σ_m^0 and \mathbf{j}_{m0l}^0 with \mathbf{i}_{m0l}^0 .

It is also not difficult to take account of the forces acting on the surface of discontinuity of the permeability μ .

6. It remains for us to prove the statement made at the end of Sec. 5.15 that the forces acting on the source of a magnetic field in an external magnetic field having the given induction $\mathbf{B}^{(e)}$ (a magnet) can be determined if we know the "intrinsic" field $\mathbf{H}^{(i)}$ induced by this source (magnet) and that upon changes in the permeability μ_e of the external medium surrounding the magnet these forces change in the same way as the intrinsic field $\mathbf{H}^{(i)}$ of the magnet.

Owing to the assumption made in Sec. 5.14 on the linear nature of the equations of a magnetic field, the resultant field in the external medium is

$$\mathbf{H} = \mathbf{H}^{(e)} + \mathbf{H}^{(i)} = \frac{1}{\mu_e} \mathbf{B}^{(e)} + \mathbf{H}^{(i)}$$

According to Eq. (5.136), the forces acting on a magnet are determined by the stress tensor \mathbf{T} for the surrounding non-ferromagnetic medium. According to Eq. (5.137), the components of this tensor are quadratic relative to \mathbf{H} , so that, for example,

$$T_{xy} = \frac{\mu_e}{4\pi} [H_x^{(e)}H_y^{(e)} + H_x^{(e)}H_y^{(i)} + H_x^{(i)}H_y^{(e)} + H_x^{(i)}H_y^{(i)}]$$

The tensile stress $\frac{\mu_e}{4\pi} H_x^{(e)}H_y^{(e)}$ determines the forces that would act on the substance of a magnet when permanent magnetization inducing the field $\mathbf{H}^{(i)}$ is absent in it. These forces are determined by formulas derived for non-ferromagnetic bodies, and at present they do not interest us. The tensile stresses $\frac{\mu_e}{4\pi} H_x^{(i)}H_y^{(i)}$ express the action of different elements of a permanent magnet on one another and add nothing to the resultant \mathbf{F} and the moment \mathbf{N} of these forces. Thus, the forces acting on a magnet in an external field are determined by tensile stresses of the kind

$$T_{xy} = \frac{\mu_e}{4\pi} [H_x^{(e)}H_y^{(i)} + H_x^{(i)}H_y^{(e)}] = \frac{1}{4\pi} [B_x^{(e)}H_y^{(i)} + H_x^{(i)}B_y^{(e)}]$$

which proves the above statement.

In the preceding discussion, we took no account of the circumstance that a change in the permeability μ_e of the external medium with an unchanging permeability μ_i of the magnet is inevitably connected with a change in the induction of the external field $\mathbf{B}^{(i)}$ near the magnet. It is not difficult to show, however, that if the external medium is homogeneous and if the external field is also uniform far from the magnet, then this circumstance does not violate the correctness of the proved statement.

6

Quasistationary Electromagnetic Field

6.1 Induction of Currents in Moving Conductors

Having revealed the main properties of a stationary electromagnetic field whose intensity is independent of time, we shall pass over to studying a varying field. Turning first of all to studying the induction of currents, we shall preliminarily set out a number of considerations that will help us find out the nature of these phenomena and relate them to facts that we already know.

1. Let us assume that a closed metal conductor L which no extraneous e.m.f.'s are applied to is moving in an external magnetic field* \mathbf{H} with a certain velocity \mathbf{v} . Let \mathbf{u} be the velocity of a "free" conduction electron of this conductor relative to the conductor L . Hence, the total velocity of the electron will be

$$\mathbf{v}' = \mathbf{v} + \mathbf{u}$$

and, according to Eq. (4.20), the electron will be acted on by the Lorentz force

$$\mathbf{F}' = \frac{e}{c} [\mathbf{v}'\mathbf{H}] = \frac{e}{c} [\mathbf{v}\mathbf{H}] + \frac{e}{c} [\mathbf{u}\mathbf{H}] \quad (6.1)$$

The second term in the right-hand side is perpendicular to the velocity \mathbf{u} of the electron relative to the conductor, and therefore the force corresponding to this term will not change the magnitude of the velocity \mathbf{u} , but will only curve the path of the electron in the conductor. The influence of this force on the distribution of currents and charges in a conductor (the Hall effect) was already treated in Sec. 4.4, and we shall not be interested in it any longer.

As regards the first component of the force \mathbf{F}' [Eq. (4.20)]

$$\mathbf{F} = \frac{e}{c} [\mathbf{v}\mathbf{H}]$$

that differs from zero only in moving conductors, this force \mathbf{F} , generally speaking, is not perpendicular to \mathbf{u} and can therefore accelerate

* We shall assume for simplicity in the present section that the permeability of a conductor $\mu = 1$, i.e. that $\mathbf{H} = \mathbf{B}$ in it.

(and retard) the motion of the electrons relative to the conductor, i.e. can induce electric currents. For example, in the conductor portion of the length L_1 (Fig. 68), the Lorentz force applied to the negative

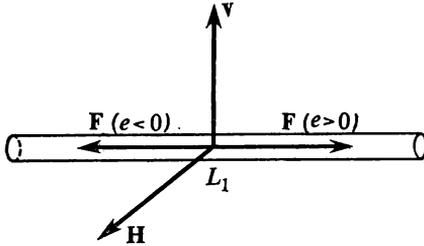


Fig. 68

electrons ($e < 0$) will force them *to the left* along the conductor. Hence, a current appears in the conductor. Since the direction of a current is conditionally considered to be opposite that of motion of negative charges, then this current will be found to flow from left to right. This is exactly the phenomenon of the *induction* of currents when a conductor moves in a magnetic field. It thus receives a very simple interpretation on the basis of the laws of a constant field which we have previously established.

2. Let us calculate the intensity of the induced current. We shall note for this purpose that the force (4.20) acting in a magnetic field H on an electron moving together with the conductor at the velocity v equals the force acting on the electron in an electric field having the intensity E' if

$$E' = \frac{1}{c} [vH] \quad (6.2)$$

It follows from Kirchoff's second law that the action of the field E' and, therefore, the action of the field H equivalent to E' in the closed contour L should induce a current whose value I is determined from Eq. (3.28):

$$IR = \oint_L E' ds = \mathcal{E}_{\text{ind}} \quad (6.3)$$

where R = resistance of the contour L

\mathcal{E}_{ind} = circulation of the vector E' around the contour L .

The latter quantity is called the *induced (motional) e.m.f.* According to Eq. (6.2), it equals

$$\mathcal{E}_{\text{ind}} = \frac{1}{c} \oint [vH] ds = -\frac{1}{c} \oint v [ds H]$$

where the last equation has been written on the basis of the known identity of vector algebra

$$[\mathbf{ab}] \mathbf{c} = - \mathbf{a}[\mathbf{cb}]$$

To determine the value of the last integral, we shall note that $\mathbf{v} = d\mathbf{q}/dt$, where $d\mathbf{q}$ is the displacement of the element ds of the contour L being considered during the time dt . Consequently,

$$\mathcal{E}_{\text{ind}} = - \frac{1}{c} \oint \frac{d\mathbf{q}}{dt} [d\mathbf{s} \mathbf{H}] \quad (6.4)$$

On the other hand, from a comparison of Eq. (4.67) with the expression for δW given on p. 249, it follows that

$$\oint \delta\mathbf{q} [d\mathbf{s} \mathbf{H}] = \delta\Phi = \delta \int H_n dS \quad (6.5)$$

where $\delta\mathbf{q}$ = virtual displacement of the element ds of the current contour

$\delta\Phi$ = increment of the magnetic flux Φ through this contour due to the displacement $\delta\mathbf{q}$.

Replacing $\delta\mathbf{q}$ in Eq. (6.5) with $d\mathbf{q}$ and comparing it with Eq. (6.4), we get

$$\mathcal{E}_{\text{ind}} = - \frac{1}{c} \frac{d\Phi}{dt} = - \frac{1}{c} \frac{d}{dt} \int H_n dS \quad (6.6)$$

This equation expresses the well-known *law of electromagnetic induction* in moving conductors: the e.m.f. induced in a conductor equals the rate of the change of the magnetic flux through its contour (divided by c). The minus sign in Eq. (6.6) signifies that if the magnetic flux through the contour of the current grows numerically, then the direction of the induction e.m.f. in this contour forms a *left-handed* system with the direction of the flux instead of a right-handed one. Conversely, when Φ diminishes, the directions of Φ and \mathcal{E}_{ind} form a right-handed system.

3. It follows from Eq. (6.6), particularly, that if a closed conductor is moving in a magnetic field so that the magnetic flux Φ through its contour remains constant, then notwithstanding this motion, no current will be induced in the conductor. The same conclusion can be arrived at directly from Eq. (4.20) in a number of particular cases. Assume, for instance, that the plane contour L is moving in the field \mathbf{H} so that its plane permanently remains parallel to \mathbf{H} . In this case, Φ always equals zero and, therefore, by Eq. (6.6), \mathcal{E}_{ind} also vanishes. It follows directly from Eq. (4.20), on the other hand, that the ponderomotive Lorentz force is directed perpendicular to \mathbf{v} and \mathbf{H} , i.e. in our case perpendicular to the plane of the contour and, therefore, cannot cause motion of the electrons *along* the contour, i.e. cannot induce a current.

Let us further consider the *translational* motion of a current-conducting contour $ABCD$ (Fig. 69) in a *uniform* magnetic field (\mathbf{H} is the same at all points of the field). Here, all points of the contour

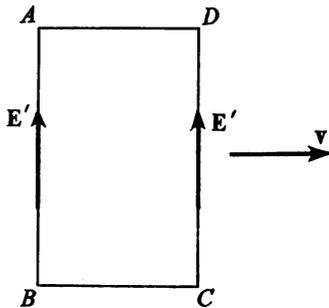


Fig. 69

have the same velocity \mathbf{v} . The flux Φ through the contour of the conductor in these conditions will evidently remain unchanged, and, therefore, \mathcal{E}_{ind} will vanish. It follows directly from the constancy of the vectors \mathbf{H} and \mathbf{v} that the forces \mathbf{E}' identical in value and direction will act on the electrons in the parallel sides AB and CD of the conductor. These forces will tend to induce circulation of the electrons (a current) through the conductor in opposite directions, and their action will mutually be balanced.

4. We shall now show that Eq. (4.15) for the density of the ponderomotive forces \mathbf{f} acting on a current-carrying conductor in a magnetic field, i.e.

$$\mathbf{f} = \left[\frac{1}{c} [\mathbf{j}\mathbf{H}] \right] \quad (6.7)$$

can be applied not only to stationary conductors, but also to moving ones. According to Eq. (6.1), the sum of the forces acting in the magnetic field \mathbf{H} on all the conduction electrons in a unit volume of the conductor is

$$\mathbf{f}' = \frac{ne}{c} [\mathbf{v}\mathbf{H}] + \frac{ne}{c} [\bar{\mathbf{u}}\mathbf{H}]$$

where n = number of electrons in a unit volume

$\bar{\mathbf{u}}$ = mean velocity of the electrons relative to the conductor.

The magnetic field acts not only on the electrons, however, but also on the positive ions forming the solid crystalline skeleton of the conductor and moving, according to the initial conditions, with the velocity \mathbf{v} . The sum of the forces acting on the ions in a unit volume of the conductor is

$$\mathbf{f}_+ = - \frac{ne}{c} [\mathbf{v}\mathbf{H}]$$

because when the conductor is neutral the sum of the charges of the ions must equal the sum ne of the charges of the electrons with the opposite sign. Consequently, the resultant of all the forces applied to a unit volume of the conductor is

$$\mathbf{f} = \mathbf{f}' + \mathbf{f}_+ = \frac{ne}{c} [\bar{\mathbf{u}}\mathbf{H}] \quad (6.8)$$

Since by Eq. (4.18) the density of the current \mathbf{j} in a conductor is $\mathbf{j} = ne\mathbf{u}^*$, then Eq. (6.8), as was to be proved, is equivalent to Eq. (4.15) or (6.7).

5. On the basis of the results of this section, we acquire the possibility of eliminating the seeming intrinsic contradiction of the theory that was mentioned in Sec. 4.4. It consisted in that in accordance with Eq. (4.20), the force acting on an electric charge in a magnetic field is perpendicular to its velocity and therefore can do no work, whereas upon the motion of a current-carrying conductor (moving charges!) the ponderomotive forces of the magnetic field undoubtedly do a certain amount of work. Namely if the velocity of a conductor is \mathbf{v} , then the ponderomotive forces of the magnetic field applied to a unit volume of the conductor, according to Eq. (6.7), during the time dt do the work

$$W = \mathbf{f}\mathbf{v} dt = \frac{1}{c} [\mathbf{j}\mathbf{H}] \mathbf{v} dt \quad (6.9)$$

This seeming contradiction is explained by the fact that the work of the *ponderomotive* forces of a magnetic field \mathbf{f} is only a portion of the total work done by the field. This work would be completely accounted for by Eq. (6.9), i.e. by the work of the resultant of all the forces \mathbf{f} only if all the charges in the conductor travelled with the identical velocity \mathbf{v} . Since the velocity of the electrons is $\mathbf{v}' = \mathbf{v} + \mathbf{u}$ instead of \mathbf{v} , then the magnetic field does on each of them the additional work

$$\mathbf{f}\mathbf{u} dt = \frac{e}{c} \{[\mathbf{v}\mathbf{H}] + [\mathbf{u}\mathbf{H}]\} \mathbf{u} dt = \frac{e}{c} [\mathbf{v}\mathbf{H}] \mathbf{u} dt$$

not taken into account in Eq. (6.9) and, consequently, on all the electrons in a unit volume the additional work

$$W' = \frac{ne}{b} \bar{\mathbf{u}}[\mathbf{v}\mathbf{H}] dt = \frac{\mathbf{j}}{c} [\mathbf{v}\mathbf{H}] dt \quad (6.10)$$

* This expression for \mathbf{j} also holds in a moving *neutral* conductor because the transfer of a charge through a unit area inside the conductor that is stationary with respect to the observer (done both by the electrons and by the ions stationary relative to the conductor) in a unit time equals $\mathbf{j} = ne(\mathbf{v} + \bar{\mathbf{u}}) + (-ne)\mathbf{v} = ne\bar{\mathbf{u}}$.

Thus, the total work of the forces of a magnetic field

$$W + W' = \frac{1}{c} [\mathbf{jH}] \mathbf{v} dt + \frac{1}{c} \mathbf{j}[\mathbf{vH}] dt = 0 \quad (6.11)$$

does indeed equal zero.

Using the notation (6.2), we can write Eq. (6.10) as follows:

$$W' = \mathbf{jE}' dt \quad (6.12)$$

where \mathbf{E}' is the intensity of the e.m.f. induced in a conductor when it moves in a magnetic field. Thus, W' is nothing but the work done by the induced e.m.f. when the current \mathbf{j} flows through a conductor [cf. Eq. (3.5)]. In the macroscopic theory, the e.m.f.'s inducing currents in conductors are not listed among the ponderomotive forces, and the ponderomotive force is determined as the resultant of all the forces of a field acting on an infinitely small volume of the body. Hence, from the viewpoint of the macroscopic theory, we can sum up the results obtained as follows: when a conductor moves in a magnetic field *the work (6.9) of the ponderomotive forces of this field equals in value and is opposite in sign to the work (6.12) of the e.m.f.'s induced in the conductor* owing to its motion in the field. Therefore, the total work of the forces of the magnetic field equals zero.

6.2 Law of Electromagnetic Induction. Ohm's Law for Varying Currents

1. In the preceding section we considered the induction of currents when a closed conductor L_1 moved in a given external magnetic field \mathbf{H} . Let us now assume, first, that this field \mathbf{H} is induced by a current circulating in a stationary contour L_2 and, second, that the contour L_1 is moving uniformly with a constant velocity \mathbf{v} . Let us further take into account that the concepts of *rest and uniform motion are relative* and that, according to the principle of relativity, the interaction of two uniformly moving bodies (in our case of the contours L_1 and L_2) can depend only on the relative, and not on the absolute velocity of these bodies. Hence, it follows that if, for example, the contour L_1 is stationary and the contour L_2 is moving with a constant velocity, then the induction of currents in the contour L_1 should remain the same as previously.

Consequently, the magnitude of the e.m.f. induced in the contour L_1 should also be determined by Eq. (6.6) when this contour remains stationary, and the change in the magnetic flux Φ through it is due to the *uniform* motion of the current I_2 inducing the field \mathbf{H} .

2. We shall now repeat the above reasoning in a more developed form. Since we are dealing with the principle of relativity, we must

establish first of all the frame or frames of reference in which the laws of electrodynamics treated in the preceding chapters hold.

These laws were established on the basis of experiments in which, as in the majority of physical experiments, motion was measured relative to the Earth. More accurate experiments*, however, showed that the laws of electrodynamics, like those of Newtonian mechanics, hold only approximately in the Earth frame of reference and hold strictly only in the so-called *inertial* frames of reference, i.e. in frames in which motion by inertia occurs in accordance with Newton's first law. In particular, a frame of reference related to the "fixed" stars can be considered with a sufficient degree of accuracy to be an inertial frame.

Further, according to the principle of relativity of motion, all frames of reference moving *uniformly* relative to one another are absolutely equivalent, and the laws describing all physical phenomena should be identical when using readings made in any of these frames. Hence, apart from the frame of the fixed stars, all the frames of reference moving uniformly relative to it are also inertial and the same laws of electrodynamics must hold in all of them. In the following, unless otherwise indicated, we shall always assume that the measurements of an electromagnetic field and the reading of the position of charges, conductors, etc. are performed in a certain *inertial* frame of reference K .

After this digression, let us return to our problem. We assumed that in a certain inertial frame K the contour L_2 along which a steady current I_2 is circulating is stationary, while the contour L_1 is moving with the constant velocity v (the first case). We then considered a second case when L_1 is stationary in the frame K and L_2 is moving with the constant velocity v . Assume that the inertial frame of reference K' is moving relative to K with the same velocity v . The conditions of the second case when it is observed relative to the frame K' are identical with those of the first one relative to the frame K . Consequently, the phenomena of induction in L_1 in both cases being considered should be absolutely the same if in the first case the measurements are performed in the frame of reference K , and in the second in the frame K' . Finally, the results obtained when measuring the induction in the second case relative to the frame K can be determined from the results of measuring the same case in the frame K' by recalculation according to the rules established in the theory of

* These include, for example, A. Michelson's experiment of 1925 (do not confuse it with his experiment of 1881) in which the time needed for a light beam (electromagnetic waves) to travel over a closed path on the Earth's surface in two opposite directions was compared. This experiment is the optical (i.e. electromagnetic) analogue of J. Foucault's mechanical experiment and makes it possible to detect the rotation of the Earth [see, for example, S. Vavilov, *Ekspierimental'nye osnovaniya teorii otноситel'nosti* (Experimental Substantiation of the Theory of Relativity), GIZ (1928), or A. Sommerfeld, *Optik*, Wiesbaden (1950)].

relativity. As a result of this recalculation, the phenomena of induction in the two cases being considered when they are measured in both cases in *the same frame of reference* K will be somewhat different. This difference, however, becomes appreciable only at very high velocities v comparable with the velocity of light c . When $v \ll c$, this difference may be completely disregarded, and we arrive at the statement made at the beginning of the section.

3. Thus, the e.m.f. induced in the contour L_1 should be determined by Eq. (6.6) with any uniform motion of the contour L_1 relative to the current I_2 inducing the field \mathbf{H} .

It is quite natural to assume that this equation may also be applied when L_2 moves non-uniformly (relative to an inertial frame). Moreover, we shall assume that Eq. (6.6) also holds if the change in the magnetic flux through the contour L_1 is due not only to the motion of L_1 or the motion of the contour L_2 carrying a steady current, but also to the switching on or off of the current in L_2 , to a change in its intensity I_2 (a varying current), etc. In other words, we shall assume that Eq. (6.6) may be applied regardless of the nature of the causes of a change in the magnitude of the magnetic flux Φ .

This assumption completely corresponds to the spirit of the short-range theory and the theory of the field in general because it, in essence, consists in the assumption that all electromagnetic phenomena in a given body or a given region of space are determined by the values of the field intensity (and by its derivatives with respect to time) in this region and do not at all depend on how the field was induced.

Recapitulation: on the basis of Eq. (4.20) for the Lorentz force established with a view to the results of studying steady currents, we *derived* the law of induction of currents in a contour moving in a constant magnetic field. Generalizing, further, the field of application of this law on the basis of considerations connected with the principle of relativity and with the concept of a field, we assumed that Eq. (6.6) has a universal significance, i.e. that it may be applied regardless of the nature of the causes of a change in the magnetic flux Φ . Experiments show that our assumption *corresponds to reality*.

4. We implicitly assumed in our preceding treatment that the conductors we are considering are in a vacuum far from other bodies. Only for this condition can we state that the phenomena of induction depend only on the relative velocity of the conductors and do not depend, for instance, on the velocity of the conductors relative to the magnetic medium. We shall now abandon this restriction and assume that the space surrounding our conductors is filled with arbitrary magnetics ($\mu \neq 1$) and that the conductors themselves are also capable of becoming magnetized. Now the question is how these factors will affect the form of the induction law [Eq. (6.6)].

We established in the preceding chapter that the magnetic properties of a substance are due to the presence in it of molecular

currents. Hence, if the law of induction [Eq. (6.6)] may be applied in the absence of magnetics, then it should also hold when they are present, provided, of course, that when calculating the flux Φ we also take into consideration the magnetic field of the molecular currents circulating in the magnetic in addition to the magnetic field of the macroscopic currents. In other words, the macroscopic field intensity \mathbf{H} in Eq. (6.6) must be replaced with the mean value of the intensity of the microscopic magnetic field $\overline{\mathbf{H}}_{\text{micro}}$.

We saw in Sec. 5.3 that the mean value of the intensity of the microscopic field in magnetic media equals the magnetic induction vector \mathbf{B} [Eq. (5.23)]:

$$\overline{\mathbf{H}}_{\text{micro}} = \mathbf{B}$$

Therefore, substituting \mathbf{B} for \mathbf{H} in Eq. (6.6) and using the symbol Ψ for the magnetic induction flux [Eq. (5.43)], we get the final form of the law of induction of currents that can be applied to an arbitrary magnetic medium:

$$\mathcal{E}_{\text{ind}} = -\frac{1}{c} \frac{d}{dt} \int B_n dS = -\frac{1}{c} \frac{d\Psi}{dt} \quad (6.13)$$

This law states that *the e.m.f. induced in an arbitrary closed contour numerically equals the rate of change in the magnetic induction flux through this contour divided by c , the direction of \mathcal{E}_{ind} forming a left-handed system with the direction of the increase in the flux Ψ [the minus sign in Eq. (6.13)!]*.

Experimental investigations confirm the correctness of Eq. (6.13).

5. The e.m.f. induced in closed contours is usually evaluated not by direct measurement, but indirectly by measuring the *current* in them. This is based on the assumption that Ohm's and Kirchhoff's laws established for steady currents also hold for varying ones. In the formulation of these laws, naturally, the induced e.m.f.'s must also be taken into account in addition to the extraneous e.m.f.'s due to physicochemical non-uniformities of the conductor (contact, thermo- and other e.m.f.'s). Particularly, if here, unlike Chapter 3, *we shall not include the induced e.m.f. in the concept of extraneous e.m.f.'s*, then the current in an unbranched conductor, according to Eq. (3.28), will be expressed by the formula

$$IR = \mathcal{E}_{\text{ext}} + \mathcal{E}_{\text{ind}} \quad (6.14)$$

where R = resistance of the contour

\mathcal{E}_{ext} and \mathcal{E}_{ind} = circulations along the contour of the forces of an "extraneous" and an induced origin acting on the electric charges [Eqs. (3.29) and (6.3)].

Equation (6.3), which is one of the cornerstones of our preceding reasoning, is a particular case of Eq. (6.14) corresponding to the assumption made at the beginning of the preceding section that $\mathcal{E}_{\text{ext}} = 0$.

We shall convince ourselves in the following that Eq. (6.14) may be directly applied to varying currents only if these currents are quasistationary (Secs. 6.3 and 7.6).

6. Using Eq. (6.13) in Eq. (6.14), we get

$$IR = \mathcal{E}_{\text{ext}} - \frac{1}{c} \frac{d\Psi}{dt} \quad (6.15)$$

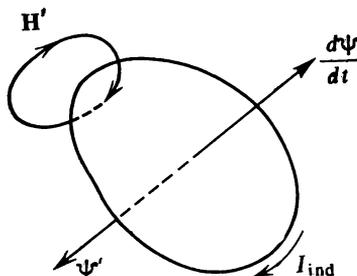


Fig. 70

As we have already mentioned, the minus sign before the term $\frac{1}{c} \frac{d\Psi}{dt}$ in Eqs. (6.13) and (6.15) means that the direction of \mathcal{E}_{ind} and, consequently, that of the induced current I_{ind} form a *left-handed* system with the direction of *growth* of the magnetic induction flux Ψ (Fig. 70)*. The induced current I_{ind} , in turn, induces in the surrounding space the magnetic field \mathbf{H}' that corresponds to its intensity. The direction of the lines of force of the field \mathbf{H}' forms a *right-handed* system with the direction of the current (Sec. 4.12). Hence it follows that the flux Ψ' of the magnetic induction vector $\mathbf{B}' = \mu\mathbf{H}'$ of this field through the current contour is directed opposite to the growth in the flux Ψ of the *external* field inducing the current I_{ind} . Consequently, we have the right to make the following general statement: *upon any change in the magnetic induction flux through a closed current-carrying contour, currents are induced in it in such a direction that the magnetic field of these currents tends to compensate the change in the magnetic induction flux through the contour of the conductor, i.e. tends to retain a constant value of the flux Ψ .*

6.3 Quasistationary Currents. Differential Equations for Varying Currents

1. The law of electromagnetic induction (6.13) plays an exceedingly important part for the entire science of varying currents. In passing

* The flux Ψ is a scalar so that in essence it is impossible to speak of its direction. For brevity, we shall conventionally consider the flux Ψ to be directed along a positive or negative normal to the plane of the current depending on whether it is positive or negative.

over to studying these currents, during this whole chapter we shall limit ourselves to a consideration of *quasistationary currents*.

Varying currents are called quasistationary if we can assume with sufficient accuracy that the *magnetic* field of these currents*, the forces of ponderomotive interaction between them, etc. at each given moment of time have the same value which these quantities would have for steady (stationary) currents of the same intensity as the instantaneous intensity of the varying currents. It is evident that varying currents can comply with the conditions of being quasistationary only if like steady currents they will be *closed* and will have the same intensity in all the cross sections of unbranched portions of the circuit**. In general, this is not an obligatory condition for varying currents (Sec. 6.13).

Further, it must be taken into account that changes in the electromagnetic field due to a change in the currents and a change in the distribution of the electric charges do not propagate instantaneously, but with a finite velocity (Sec. 7.7). Therefore at each given moment t , the intensity of the field of varying currents, strictly speaking, corresponds not to the instantaneous value of these currents, capacitor charges, etc., but to the values that these quantities had at a certain preceding moment of time $t - \tau$. Here τ is the time needed for the propagation of the electromagnetic perturbations from the corresponding portions of the circuit to the point of the field being considered.

Thus, the field of varying currents can comply with the conditions of being quasistationary only within a limited region of space in direct proximity to these currents and, evidently, only if the currents, the charges of capacitors, etc. do not change more or less considerably during the period τ that is needed for the propagation of electromagnetic perturbations between the two remotest points of the system of currents being considered.

Thus, the main condition for a current being quasistationary is a sufficiently slow rate of changes of the field. As we shall see in Sec. 6.13, this also guarantees the approximately closed nature of varying currents.

In practice, the varying (alternating) currents with which power engineering has to do (tens, hundreds and thousands of hertz) comply with a sufficient degree of accuracy to the conditions of being quasistationary. The theory of quasistationary currents, however, cannot be applied to the high frequency electrical oscillations used in radio engineering, or can be applied only with known restrictions.

2. To consider the corollaries following from the law of induction established in the preceding section, it will be sufficient for us to

* Their electric field, generally speaking, differs from the field of steady currents (see p. 427).

** It is obvious that these requirements are also necessary conditions for the applicability of Ohm's law (6.15).

consider two closed unbranched contours L_1 and L_2 in a dia- or paramagnetic medium. We shall meanwhile not consider a ferromagnetic medium.

Let us write Eqs. (6.14) and (6.15) for each of the contours L_1 and L_2 :

$$I_i R_i = \mathcal{E}_{\text{ind}, i} + \mathcal{E}_{\text{ext}, i} = \mathcal{E}_{\text{ext}, i} - \frac{1}{c} \frac{d\Psi_i}{dt} \quad (6.16)$$

where i is 1 or 2.

On the other hand, the induction flux Ψ_1 through, for example, the contour of the first current ($i = 1$) equals the sum of the average flux Ψ_{11} which it itself sends through its own contour (for the meaning of this quantity see what is said about Φ_{11} on p. 261) and of the flux Ψ_{12} that is sent through its contour by the second current I_2 . According to Eq. (5.46), we have

$$\Psi_1 = \Psi_{11} + \Psi_{12} = \frac{1}{c} L_{11} I_1 + \frac{1}{c} L_{12} I_2$$

where L_{11} and L_{12} are the self- and mutual inductances. We can use these expressions, which have been derived for steady currents, for varying currents too if they are quasistationary.

Introducing them into Eq. (6.13), we get the value of the e.m.f. induced in the first contour:

$$\begin{aligned} \mathcal{E}_{\text{ind}, i} = & -\frac{1}{c} \frac{d\Psi_1}{dt} = -\frac{1}{c^2} \left(L_{11} \frac{dI_1}{dt} + \right. \\ & \left. + L_{12} \frac{dI_2}{dt} + I_1 \frac{dL_{11}}{dt} + I_2 \frac{dL_{12}}{dt} \right) \end{aligned} \quad (6.17)$$

and a similar equation for the second contour. Thus, the value of the e.m.f. induced in the first contour will depend both on the rate of change of the currents I_1 and I_2 and on the rate of change of the inductances L_{11} and L_{12} .

In the following, we shall consider the case when the contours L_1 and L_2 are in a *homogeneous* external dia- or paramagnetic medium*. Upon this condition, the values of the inductances L_1 and L_2 are determined by the geometrical configuration of the contours [see Eq. (5.47) and also p. 259]. Hence, the values of the derivatives dL_{11}/dt and dL_{12}/dt depend only on the change in this configuration, i.e. on the nature of displacement of the contours L_1 and L_2 . The derivative dL_{11}/dt depends only on the change in the

* If the external medium is not homogeneous, then changes in the inductances L_{11} and L_{12} may also be due to displacements of this medium, for example the moving into one of the contours of a core capable of becoming magnetized.

shape of the first contour (if the latter does not deform, then L_{11} is constant), and dL_{12}/dt on the change in the mutual arrangement of the contours L_1 and L_2 .

Introducing Eq. (6.17) and the corresponding expression for $\mathcal{E}_{ind, 2}$ into Eq. (6.16), we get two linear differential equations for I_1 and I_2 from which we can determine I_1 and I_2 as functions of time provided that $\mathcal{E}_{ext, 1}$, $\mathcal{E}_{ext, 2}$, L_{11} , L_{12} , and L_{22} are known functions of time. Particularly, if the contours of the two currents are stationary relative each other and are not deformed, the inductances will be constant, and the following system of linear equations with constant coefficients will hold for I_1 and I_2 :

$$\left. \begin{aligned} I_1 R_1 + \frac{1}{c^2} L_{11} \frac{dI_1}{dt} + \frac{1}{c^2} L_{12} \frac{dI_2}{dt} &= \mathcal{E}_{ext, 1} \\ I_2 R_2 + \frac{1}{c^2} L_{22} \frac{dI_2}{dt} + \frac{1}{c^2} L_{12} \frac{dI_1}{dt} &= \mathcal{E}_{ext, 2} \end{aligned} \right\} \quad (6.18)$$

3. Let us consider in conclusion the particular case of the currents induced in the contour L_1 which without deformation ($L_{11} = \text{const}$) is moving in an invariable magnetic field, for example in the field of a steady current I_2 . Strictly speaking, the appearance of an induced current in the contour L_1 leads to the induction of the corresponding magnetic field. The latter, in turn, should induce secondary currents in the contour L_2 , and so on. Thus, the assumption that the external magnetic field is invariable consists, in essence, in the assumption that the secondary current induced in the contour L_2 is so small in comparison with the constant component of the current I_2 that this current may be completely disregarded. Upon this condition, Eq. (6.17) becomes

$$\mathcal{E}_{ind, 1} = -\frac{1}{c^3} \left(I_2 \frac{dL_{12}}{dt} + L_{11} \frac{dI_1}{dt} \right) \quad (6.19)$$

4. Equations (6.17) and (6.18) play an exceedingly important part in the science of varying (alternating) currents in general and in electrical engineering in particular. The field of their application is very broad. It includes the theory of dynamos, transformers, electrical measuring instruments, etc. Referring our reader to special publications for all details, we shall limit ourselves to a consideration of only a few examples in Sec. 6.5 after acquainting ourselves with the magnetic energy of currents.

6.4 Transformations of Energy in the Field of Varying Currents.

Energy of Magnetic Interaction of Currents.

Lenz's Law

In Chapters 4 and 5, we did not mention at all the energy of the magnetic interaction of currents and the energy of a magnetic field

in general. The explanation is that it is possible to consider this question, which is exactly what we shall now do, only by taking into account the phenomena of electromagnetic induction, and we have become acquainted with them only at the beginning of the present chapter.

1. Let us consider the induction interaction of the varying currents I_1 and I_2 from the energy viewpoint. Here the following transformations of energy must be taken into consideration: (1) the liberation of Joule heat Q in the circuit of the currents, (2) the work A of the extraneous e.m.f.'s, (3) the mechanical work W of the ponderomotive forces of the magnetic field done upon motion of the current contours, and, finally, it is necessary, as we shall see, to take into account (4) the energy of the magnetic interaction of the currents U_m and consider the change dU_m in this energy upon changes both in the currents I_1 and I_2 and in their mutual configuration*.

Let us consider the change in all these kinds of energy occurring during an infinitely small period of time dt . We shall begin with mechanical energy. We established in Secs. 4.11 and 5.6 that the work W of the ponderomotive forces of the magnetic field of *steady* currents equals the decrease in the potential function of these currents U [Eqs. (5.45) and (5.48)], namely,

$$W = - (dU)_I$$

$$U = -\frac{1}{c^2} \left(\frac{1}{2} L_{11} I_1^2 + L_{12} I_1 I_2 + \frac{1}{2} L_{22} I_2^2 \right)$$

Thus, for steady currents, we have

$$W = - (dU)_I = \frac{1}{c^2} \left(\frac{1}{2} I_1^2 dL_{11} + I_1 I_2 dL_{12} + \frac{1}{2} I_2^2 dL_{22} \right) \quad (6.20)$$

where the subscript I of dU signifies that when determining the change in the function U the currents I are assumed to be constant (see p. 254), and where dL_{11} , dL_{12} , and dL_{22} are the changes in the inductances during the time dt . But according to the definition of the term "quasistationary", given at the beginning of this section, the forces of interaction of quasistationary currents at each given moment of time equal the forces of interaction of steady currents having the same intensity I_1 and I_2 . Consequently, Eq. (6.20) should remain in force for quasistationary currents (naturally if we choose such a small

* It is sometimes necessary (for example when a circuit of currents includes a capacitor — see Sec. 6.14) to also take the change in the electric energy into consideration. This does not affect the final result of our calculations, however. All these matters will be again considered quite strictly in Sec. 7.2.

period of time dt that the intensity of these currents during this time will remain virtually constant)*.

Let us pass over to determining the change in the other kinds of energy. It directly follows from Joule's law [Eq. (3.33)] that the quantity of Joule heat liberated in both contours during the time dt is

$$Q dt = (I_1^2 R_1 + I_2^2 R_2) dt$$

Further, according to Eq. (3.34), the extraneous e.m.f.'s during this time perform work in both contours equal to

$$A dt = (I_1 \mathcal{E}_{\text{ext}, 1} + I_2 \mathcal{E}_{\text{ext}, 2}) dt$$

If we assume, for instance, that the extraneous e.m.f.'s are induced by galvanic cells or accumulators connected to the contours, then $A dt$ will measure the work done in the current contours *at the expense* of the chemical energy of these e.m.f. sources. Naturally, when using the last equation it is necessary to select the positive direction around each of the contours in a definite way and provide the quantities I_i and $\mathcal{E}_{\text{ext}, i}$ with the relevant signs. Depending on whether, for example, I_1 and $\mathcal{E}_{\text{ext}, 1}$ have the same or different signs (i.e. directions), the work of the e.m.f. $\mathcal{E}_{\text{ext}, 1}$ will be positive or negative.

2. Thus, during the time dt , the mechanical energy (for example the kinetic energy of motion of the conductors) should *grow* by the amount W , the heat energy should *grow* by $Q dt$, and the energy of the sources of extraneous e.m.f.'s (for example the chemical energy of accumulators) should *decrease* by the amount $A dt$. Hence, the total increase Δ of all these kinds of energy is

$$\Delta = W + (Q - A) dt \tag{6.21}$$

By Eq. (6.14), we have

$$\begin{aligned} Q - A &= I_1 (I_1 R_1 - \mathcal{E}_{\text{ext}, 1}) + I_2 (I_2 R_2 - \mathcal{E}_{\text{ext}, 2}) = \\ &= I_1 \mathcal{E}_{\text{ext}, 1} + I_2 \mathcal{E}_{\text{ext}, 2} \end{aligned} \tag{6.22}$$

Introducing Eq. (6.22) into Eq. (6.21) and expressing $\mathcal{E}_{\text{ind}} dt$ and W with the aid of Eqs. (6.17) and (6.20), we get

$$\begin{aligned} \Delta = - \frac{1}{c^2} &\left(L_{11} I_1 dI_1 + \frac{1}{2} I_1^2 dL_{11} + L_{12} I_1 dI_2 + \right. \\ &\left. + L_{12} I_2 dI_1 + I_1 I_2 dL_{12} + L_{22} I_2 dI_2 + \frac{1}{2} I_2^2 dL_{22} \right) \end{aligned}$$

* If we assume that formulas of the type $W = - dU$ hold for quasistationary currents, then it is clear directly that when determining the work W we must take into consideration only the changes in the value of the potential function U that are due to *motion* of the conductors (i.e. to a change in the inductances) because a change in the *currents* in conductors cannot be related to a change in the work of the *ponderomotive* forces of the field. Hence it follows that for quasistationary currents the formula $W = - dU$ must indeed be defined more precisely in the form of the formula $W = - (dU)_I$.

whence

$$\Delta = -\frac{1}{c^2} d\left(\frac{1}{2} L_{11}I_1^2 + L_{12}I_1I_2 + \frac{1}{2} L_{22}I_2^2\right) \quad (6.23)$$

Thus, any change in the configuration of contours and of the currents in them involves a total increase in the sum of the mechanical, thermal and chemical energy by the amount Δ . On the basis of the law of conservation of energy, we must conclude that these processes must be attended by an equivalent reduction of some other kind of energy. What kind is it? To answer this question, it is sufficient to note that only changes in the forces of *magnetic* interaction of currents are *inseparably* linked with the motion of conductors and the change in the currents circulating in them (apart from the mechanical, thermal, and chemical processes which we have already taken into consideration). It is therefore obvious that a certain energy U_m must be ascribed to the magnetic interaction of currents, and that the increment Δ of all the other kinds of energy must occur at the expense of an equivalent change dU_m in the magnetic energy:

$$dU_m = -\Delta = +\frac{1}{c^2} d\left(\frac{1}{2} L_{11}I_1^2 + L_{12}I_1I_2 + \frac{1}{2} L_{22}I_2^2\right) \quad (6.24)$$

Consequently*,

$$U_m = \frac{1}{c^2} \left(\frac{1}{2} L_{11}I_1^2 + L_{12}I_1I_2 + \frac{1}{2} L_{22}I_2^2\right) \quad (6.25)$$

This expression can also be written as follows:

$$U_m = \frac{1}{2c^2} \sum_{ik} L_{ik}I_iI_k \quad (6.26)$$

where $i, k = 1, 2$.

It is clear that the magnetic energy of a system consisting not of two but of an arbitrary number n of currents will be expressed by a formula of the same kind in which the subscripts i and k will take on all the values from 1 to n .

3. Comparing Eq. (6.26) with Eq. (5.48), we get

$$U_m = -U \quad (6.27)$$

Consequently, the magnetic energy of currents U_m equals the potential function of these currents U taken with the *opposite sign*. We thus again become convinced that this potential function, as we have repeatedly stressed, does not at all equal the (potential) energy of currents.

* Strictly speaking, by setting dU_m , the value of U_m is determined only up to an additive constant which we assume to equal zero for the magnetic energy to equal zero in the absence of currents ($I_1 = I_2 = 0$).

The circumstance that the meaning of the quantities U_m and U is absolutely different manifests itself with especial clarity in Eq. (6.27). Indeed, in accordance with the definition of the function U , its decrease $-(dU)_I$ equals the *mechanical* work W done by the ponderomotive forces of a magnetic field. When determining the decrease $-(dU)_I$ by Eq. (6.20), the currents should be considered constant. But the decrease in the magnetic energy $-dU_m$ equals the *sum* of the increments of *all* the other kinds of energy, and not only of the mechanical energy, so that when determining the decrease $-dU_m$ by Eq. (6.24) we must also take into consideration the change in the currents. For this reason, we could not expect *a priori* a simple relationship between the quantities U and U_m . Equation (6.27) shows, however, that such a relationship does actually exist.

For steady currents in stationary conductors, both the mechanical work W and the change in the magnetic energy dU_m obviously equal zero. In this case, we also have $\mathcal{E}_{\text{ind}} = 0$, and by Eq. (6.22) $Q = A$, i.e. the work of the extraneous forces is completely converted into heat.

The mechanical work W can be done, however, only upon motion of the conductors. This, in turn, is connected with the appearance of induced e.m.f.'s, i.e. with violation of the equality between the liberated heat Q and the work of the extraneous e.m.f.'s. The surplus (compared with the liberation of heat) expenditure of energy of the sources of these e.m.f.'s (for instance the chemical energy of accumulators) is sufficient not only for performing the mechanical work W , but also for increasing the magnetic energy U_m .

Indeed, it follows from Eqs. (6.24) and (6.21) that

$$-dU_m = A = W + (Q - A) dt \quad (6.28)$$

which we can write as follows:

$$(A - Q) dt = W + dU_m$$

4. Let us assume as an example that upon the motion of the contours L_1 and L_2 the currents I_1 and I_2 are kept constant in them by a continuous change in the extraneous e.m.f.'s, which balance the influence of induction. In this hypothetical case ensuing from Eqs. (6.20) and (6.27), the relationship

$$W = -(dU)_I = + (dU_m)_I \quad (6.29)$$

acquires the form

$$W = dU_m$$

where dU_m is the total change in the magnetic energy during motion because we have assumed that $I_i = \text{const}$. In other words, the mechanical work W will equal the *increment* of magnetic energy, and

$$W + dU_m = 2W$$

On the other hand, since we have assumed that the currents remain constant during motion, then the quantity of liberated heat Q also remains constant. Hence, during motion, the work A of the extraneous e.m.f.'s should grow by the (positive or negative) value $2W$, and half of it will go to increase the magnetic energy U_m . This growth in the work A is explained by the fact that to maintain the currents constant during motion the extraneous e.m.f.'s must be so changed as to compensate for the influence of the induced e.m.f.'s.

It should be noted that motion of the current contours provided that $I_i = \text{const}$ is quite similar from the energy viewpoint to motion of the plates of a capacitor provided that $\varphi_1 - \varphi_2 = \text{const}$. In the latter case, the work of the ponderomotive forces of an electric field also equals the *increment* of the energy of this field:

$$W = dU_m$$

Both the work W and the increment of the electric energy dU_e are performed at the expense of the work of the (extraneous) e.m.f.'s maintaining a constant potential difference across the capacitor plates (see Sec. 1.18).

Actually, of course, the magnitude of the extraneous e.m.f.'s remains constant in conductors upon their motion instead of the currents in them in the majority of cases. The currents change, and this makes matters much more complicated.

5. As a final remark let us discuss the transformations of energy in the contour L_1 which no extraneous e.m.f.'s are applied to ($\mathcal{E}_{\text{ext}, 1} = 0$), and which moves in an *invariable* magnetic field, for example in the field of the steady current I_2 . If L_{11} remains constant in this case, then the induced e.m.f. in this contour is expressed by Eq. (6.19). Multiplying Eq. (6.14) by I_1 , we get

$$I_1^2 R_1 = I_1 \mathcal{E}_{\text{ind}, 1} = -\frac{1}{c^2} \left(I_1 I_2 \frac{dL_{12}}{dt} + L_{11} I_1 \frac{dI_1}{dt} \right).$$

On the other hand, owing to the assumption that the contour L_2 is stationary ($L_{22} = \text{const}$), from Eq. (6.20) we get

$$W = \frac{1}{c^2} I_1 I_2 dL_{12}$$

Consequently,

$$W = -I_1^2 R_1 dt - \frac{1}{2c^2} L_{11} dI_1^2$$

If there was no current in a conductor at the beginning of its motion, then $dI_1^2 > 0$. Further, L_{11} and $I_1^2 R_1 dt$ are essentially positive quantities, and, consequently, $W < 0$. Therefore, the ponderomotive forces of a magnetic field do *negative* work when the conductor L_1 moves, i.e. *oppose* this motion. Hence follows the so-called *Lenz's*

law: the currents induced in a conductor when it moves in a constant magnetic field are directed so that the ponderomotive forces of the magnetic field acting on these currents *oppose* the motion of the conductor.

6.5 Simple Applications of the Varying Current Theory. Transformer

Before passing over to applying the results obtained to concrete problems, we must note that we can eliminate the factor c making all the formulas of the last sections more complicated by transition to the practical (and also to the electromagnetic) system of units (Sec. 4.18). We shall show how this is done taking Eqs. (6.17) and (6.26) as examples.

Using Table 3 (p. 300), we can write that

$$\mathcal{E} = \frac{1}{300} \mathcal{E}', \quad I = 3 \times 10^9 I', \quad \text{and} \quad L = 10^9 L'$$

where \mathcal{E}' , I' , and L' are the values of the e.m.f., current, and inductance expressed in *practical* units (volt, ampere, henry), whereas \mathcal{E} , I , and L are the values of the same quantities in *absolute* units, which is what we have used up to now. In addition, the practical unit of energy is the joule equal to 10^7 ergs. Hence, using the same symbols,

$$U = 10^7 U'$$

Expressing in Eqs. (6.17) and (6.26) the values of all the quantities in practical units and using the numerical value of the constant $c = 3 \times 10^{10}$ in them, after cancelling we get

$$\begin{aligned} \mathcal{E}'_{\text{ind},1} = & - \left(L'_{11} \frac{dI'_1}{dt} + L'_{12} \frac{dI'_2}{dt} + \right. \\ & \left. + I'_1 \frac{dL'_{11}}{dt} + I'_2 \frac{dL'_{12}}{dt} \right) \end{aligned} \quad (6.30)$$

and

$$U'_m = \frac{1}{2} \sum_{ik} L'_{ik} I'_i I'_k \quad (6.31)$$

We shall often use the practical system of units instead of the absolute one in the following examples and problems. It is customarily used in the applied science of electricity.

Example 1. Transformations of Energy in Closing and Opening of a Current Circuit. For an isolated undeforming current circuit, according to Eqs. (6.14) and (6.30), we can write, using the *practical system of units*

$$I'R' = \mathcal{E}'_{\text{ext}} - L' \frac{dI'}{dt} \quad (6.32)$$

where $L' = \text{const}$ is the self-induction of the circuit. The general solution of this differential equation when $\mathcal{E}'_{\text{ext}}$ is independent of the time is

$$I' = \frac{\mathcal{E}'_{\text{ext}}}{R'} + a \exp\left(-\frac{R'}{L'} t\right)$$

where a is an arbitrary integration constant.

Assume that the previously open circuit is closed at the moment $t = 0$ so that $I' = 0$ when $t = 0$. To comply with this condition, we must assume that

$$a = -\frac{\mathcal{E}'_{\text{ext}}}{R'}$$

whence

$$I' = \frac{\mathcal{E}'_{\text{ext}}}{R'} \left[1 - \exp\left(-\frac{R'}{L'} t\right) \right]$$

Consequently, when the circuit is closed, the current grows exponentially from zero to the maximum value of $\mathcal{E}'_{\text{ext}}/R'$ corresponding to Ohm's law.

Multiplying the initial equation by I' , we get

$$I'^2 R' = I' \mathcal{E}'_{\text{ext}} - L' I' \frac{dI'}{dt}$$

or

$$I' \mathcal{E}'_{\text{ext}} = I'^2 R' + \frac{d}{dt} \left[\frac{1}{2} L' I'^2 \right] \quad (6.33)$$

Thus, the work of the extraneous e.m.f.'s $I' \mathcal{E}'_{\text{ext}}$ is spent when the circuit is closed not only for overcoming the (ohmic) resistance of the circuit R' , i.e. for the liberation of the Joule heat $I'^2 R'$, but also for the growth in the magnetic energy of the current $U_m = \frac{1}{2} L' I'^2$ [cf. Eq. (6.31)]. It is exactly this gradual accumulation of magnetic energy that explains the gradual growth of the current when a circuit is closed.

Conversely, if at the moment $t = 0$, a current I'_0 is circulating in the circuit and if at this moment the source of the e.m.f. is

disconnected from the circuit so that the latter nevertheless remains closed (for example by short-circuiting the source of the e.m.f.), then the current will not immediately vanish, but will diminish according to the exponential law:

$$I' = I'_0 \exp\left(-\frac{R'}{L'} t\right)$$

The equation for the energy in this case ($\mathcal{E}'_{\text{ext}} = 0$) becomes

$$(I')^2 R' = -\frac{d}{dt} \left(\frac{1}{2} L' I'^2 \right) = -\frac{dU_m}{dt}$$

This means that after the e.m.f. is switched off the current in the circuit will nevertheless be maintained at the expense of the energy stored in the magnetic field of the current up to the moment when all this energy is converted into Joule heat.

Accordingly, the rate of growth and diminishing of the current is determined by R'/L' , i.e. by the ratio between the "energy capacity" of the magnetic field of the current determined by the inductance L' and the magnitude of the resistance to the current or "friction" causing its energy to be transformed into heat. The smaller the resistance and the greater the self-inductance, the relatively slower does the current change. Thus, the *self-inductance is a measure*, as it were, of the "electromagnetic inertia" of a current.

From the energy viewpoint, an electric current in a closed conductor can be compared with the rotation of a shaft carrying a flywheel and driven by a motor. When the latter is started (extraneous e.m.f.'s), its work goes both to overcome friction in the bearings (the resistance of the conductor), i.e. to liberate heat (Joule heat) and to impart kinetic energy to the flywheel (the energy of the magnetic field of a current). As a result, the speed (current) grows gradually. When the motor is switched off, the shaft continues to rotate by inertia, and the kinetic energy of the flywheel transforms into heat liberated upon friction.

Example 2. *Self-induction in the circuit of a varying current.*

Let us assume that a source of a varying (alternating) e.m.f., for instance a dynamo, is connected in a closed circuit. We shall consider the e.m.f. of this source as an extraneous (for our circuit) e.m.f. $\mathcal{E}'(t)$ that is a given function of time. Hence, Eq. (6.32) becomes

$$I'R' + L' \frac{dI'}{dt} = \mathcal{E}'(t) \quad (6.34)$$

When considering the phenomena occurring in the circuit, as in a considerable number of problems on following pages, we shall limit ourselves to *periodic* varying currents or *alternating* currents

whose intensity is a *sinusoidal* function of time. This form of the time dependence of currents is of the greatest practical importance and lends itself the easiest to mathematical investigation. The results of studying such currents can also be applied to more complicated cases because it is general knowledge that any periodic function can always be expanded into a Fourier series, each term of which is a sinusoidal function of time.

It is known that operating with periodic functions is very simplified when expressions in complex variables are used. In the following, we shall therefore always express sinusoidal periodic functions in the complex form. In particular, in the case being considered we assume

$$\mathcal{E}'(t) = \mathcal{E}'_0 e^{i\omega t} \text{ and } I'(t) = I'_0 e^{i\omega t} \quad (6.35)$$

where the *amplitudes* of the e.m.f. and the current \mathcal{E}'_0 and I'_0 do not depend on the time, and ω is the *cyclic* frequency of the current and the e.m.f. * The amplitudes \mathcal{E}'_0 and I'_0 themselves, generally speaking, will be complex quantities. Naturally, only the real part of these complex expressions has a direct physical meaning. We can, however, take advantage of the circumstance that the real part of the results obtained in performing *linear* operations with complex expressions coincides with the results of performing these operations with only the real parts of the initial expressions. Therefore the transition to the real part of complex expressions, which is the only part we shall assign a physical meaning to, may be performed both before and after these operations. Only with *non-linear* operations (for instance multiplication) is it necessary to pass over to the real parts of complex expressions before performing these operations with them (because the real part of the product of complex quantities *does not equal* the product of their real parts).

Introducing expressions (6.35) into Eq. (6.34), differentiating with respect to time, and then cancelling the common factor $e^{i\omega t}$ from the equation obtained, we get

$$R'I'_0 + i\omega L'I'_0 = \mathcal{E}'_0$$

or

$$I'_0 = \frac{\mathcal{E}'_0}{R' + i\omega L'}$$

It is good practice to transform the complex denominator of the right-hand side with the aid of the known relationships

* By the *cyclic frequency* is meant the number of periods during a time interval of 2π seconds, whereas the term "frequency" without the adjective means the number of periods a second.

$$R' + i\omega L' = \sqrt{R'^2 + \omega^2 L'^2} e^{i\varphi},$$

$$\cos \varphi = \frac{R'}{\sqrt{R'^2 + \omega^2 L'^2}},$$

$$\sin \varphi = \frac{\omega L'}{\sqrt{R'^2 + \omega^2 L'^2}}, \quad \tan \varphi = \frac{\omega L'}{R'}$$

Using these relationships and symbols, we get

$$I'_0 = \frac{\mathcal{E}'_0}{\sqrt{R'^2 + \omega^2 L'^2}} e^{-i\varphi}$$

The real parts of the current I' and the e.m.f. \mathcal{E}' will be expressed, according to Eq. (6.35), by the formulas (considering the quantity \mathcal{E}'_0 to be real)

$$\mathcal{E}'(t) = \mathcal{E}'_0 \cos \omega t, \quad I'(t) = \frac{\mathcal{E}'_0}{\sqrt{R'^2 + \omega^2 L'^2}} \cos(\omega t - \varphi) \quad (6.36)$$

Thus, the presence of self-induction in the circuit of an alternating current, first, increases, as it were, the resistance of the circuit: the amplitude of the current I'_0 is determined by the quotient obtained in dividing \mathcal{E}'_0 by the “apparent” or “effective” resistance R'_{eff} :

$$I'_0 = \frac{\mathcal{E}'_0}{R'_{\text{eff}}}, \quad R'_{\text{eff}} = \sqrt{R'^2 + \omega^2 L'^2} \quad (6.37)$$

which is always greater than the “ohmic” resistance R' .

Further, the presence of self-induction causes the phase of the current to lag behind that of the e.m.f., the phase shift φ , according to the formula

$$\tan \varphi = \frac{\omega L'}{R'} \quad (6.38)$$

being the greater, the greater is the self-inductance of the circuit and the frequency of the current. The cause of this phase shift is the same as that of the gradual growth of the current when the circuit of a steady current is closed (Example 1): the current does not have time to follow the changes of the e.m.f.’s because any change in the current must be attended by the corresponding gradual change in the store of magnetic energy.

The power used in the circuit, i.e. the work of the source of the e.m.f. related to a unit time is determined by the expression

$$\mathcal{E}'(t) I'(t) = \mathcal{E}'_0 I'_0 \cos \omega t \cdot \cos(\omega t - \varphi)$$

Since

$$\begin{aligned} \cos \omega t \cdot \cos(\omega t - \varphi) &= \cos^2 \omega t \cdot \cos \varphi + \\ &+ \cos \omega t \cdot \sin \omega t \cdot \sin \varphi \end{aligned}$$

and since the mean values of $\cos^2 \omega t$ and $\cos \omega t \sin \omega t$ during a period equal $1/2$ and 0 , respectively, then the mean power used during a period is

$$\overline{\mathcal{E}'(t)I'(t)} = \frac{1}{2} \mathcal{E}'_0 I'_0 \cos \varphi = \frac{1}{2} \frac{\mathcal{E}'_0{}^2 \cos \varphi}{\sqrt{R'^2 + \omega^2 L'^2}} \quad (6.39)$$

Thus, with a given e.m.f., the power used in a circuit is proportional to $\cos \varphi$, i.e. it is the smaller, the greater is the phase shift φ .

In the limiting case when $\varphi = \pi/2$, the power used is zero: during half a period the e.m.f. source does positive work ($\mathcal{E}'I' > 0$) at whose expense the store of magnetic energy grows, while during the next half-period this store of energy is returned to the e.m.f. source that performs negative work ($\mathcal{E}'I' < 0$). These energy transformations occur without any losses for Joule heat because when $\varphi = \pi/2$ we have $\tan \varphi = \infty$ and, therefore, according to Eq. (6.38) in this limiting case the ohmic resistance of the circuit should equal zero (if $\omega L'$ is finite).

Example 3. Elementary theory of a transformer. The simplest transformer consists of two current contours (induction coils) with induction interaction between them. To increase this interaction (i.e. increase the mutual inductance L_{12}) both coils are usually fitted onto a common iron core which considerably increases the permeability of the medium. For simplification, however, we shall assume that there are no ferromagnetics in the field of the coils.

Customarily, one of the coils or windings of a transformer (the primary) is supplied with an alternating current from a source of energy, whereas the currents induced in the other winding (the secondary) are delivered to consumers of electromagnetic energy. We shall treat the source of energy feeding the primary as an extraneous e.m.f. $\mathcal{E}'_1(t)$ that is a given function of time. We shall assume that the extraneous e.m.f. in the secondary equals zero.

Applying Eqs. (6.18) to the transformer and passing over to the practical system of units, we get

$$\left. \begin{aligned} I'_1 R'_1 + L'_{11} \frac{dI'_1}{dt} + L'_{12} \frac{dI'_2}{dt} &= \mathcal{E}'_1(t) \\ I'_2 R'_2 + L'_{22} \frac{dI'_2}{dt} + L'_{12} \frac{dI'_1}{dt} &= 0 \end{aligned} \right\} \quad (6.40)$$

Let us assume that the e.m.f. $\mathcal{E}'_1(t)$ is a sinusoidal function of time with the cyclic frequency ω . It therefore follows from Eqs.

(6.40) that I'_1 and I'_2 must be similar functions of time, so that in the complex form

$$\mathcal{E}'_1(t) = \mathcal{E}'_{10} e^{i\omega t} \text{ and } I'_i = I'_{i0} e^{i\omega t}$$

where \mathcal{E}'_{10} and I'_{i0} do not depend on the time, and $i = 1, 2$.

Using these expressions in Eqs. (6.40), differentiating with respect to time, and then cancelling the common factor $e^{i\omega t}$, we get

$$\left. \begin{aligned} I'_{10}R'_1 + i\omega L'_{11}I'_{10} + i\omega L'_{12}I'_{20} &= \mathcal{E}'_{10} \\ I'_{20}R'_2 + i\omega L'_{22}I'_{20} + i\omega L'_{12}I'_{10} &= 0 \end{aligned} \right\} \quad (6.41)$$

From the last equation, we get

$$\frac{I'_{20}}{I'_{10}} = - \frac{i\omega L'_{12}}{R'_2 + i\omega L'_{22}} = \frac{-\omega L'_{12}(\omega L'_{22} + iR'_2)}{R'^2_2 + \omega^2 L'^2_{22}} \quad (6.42)$$

or, performing the usual transformations,

$$\frac{I'_{20}}{I'_{10}} = \frac{-\omega L'_2}{\sqrt{R'^2_2 + \omega^2 L'^2_{22}}} e^{j\delta}, \quad \tan \delta = \frac{R'_2}{\omega L'_{22}}$$

Particularly, in the important case for practical engineering when the "inductive reactance" $\omega L'_{22}$ of the secondary is considerably greater than its ohmic resistance,

$$\omega L'_{22} \gg R'_2$$

Eq. (6.42) gives us approximately (disregarding R'_2 in comparison with $\omega L'_{22}$)

$$\frac{I'_{20}}{I'_{10}} = - \frac{L'_{12}}{L'_{22}} \quad (6.43)$$

Consequently, by properly choosing the inductances L'_{12} and L'_{22} , we can use a transformer to increase or reduce the current in its secondary in comparison with the primary an arbitrary number of times.

It is quite natural that the use of energy in the secondary increases the consumption of energy from the e.m.f. source connected to the primary. Indeed, it follows from the first of equations (6.41) that

$$I'_{10} \left(R'_1 + i\omega L'_{11} + i\omega L'_{12} \frac{I'_{20}}{I'_{10}} \right) = \mathcal{E}'_{10}$$

If the secondary is absent or is open ($R'_2 = \infty$), then $I'_{20} = 0$, and we return to the example of an isolated alternating current circuit considered above. If the secondary is closed, however, then introducing the value of I'_{20}/I'_{10} from Eq. (6.42) into the last equation, we get

$$I'_{10} \{ [R'_1 + \alpha^2 R'_2] + i\omega [L'_{11} - \alpha^2 L'_{22}] \} = \mathcal{E}'_{10},$$

where

$$\alpha^2 = \frac{\omega^2 L_{12}'^2}{R_2'^2 + \omega^2 L_{22}'^2}$$

Thus, connection of the secondary is equivalent to increasing the ohmic resistance and reducing the self-inductance of the primary. Both factors, according to Eqs. (6.38) and (6.39), cause a reduction in the phase shift in the primary and an increase in the energy consumed from the e.m.f. source $\mathcal{E}_1(t)$.

Problem 35. An undeforming closed flat contour L having the resistance R' and self-inductance L' which no extraneous e.m.f.'s are applied to uniformly rotates in a constant uniform magnetic field \mathbf{H}' . The axis of rotation is perpendicular to \mathbf{H}' and is in the plane of the contour. The number of revolutions a second is $n = \omega/2\pi$. Accordingly, the flux of magnetic induction Ψ' of the external field \mathbf{H}' through the contour L is determined by the expression $\Psi' = \Psi'_0 \cos \omega t$. The values of all the quantities are presumed to be expressed in *practical* units (the primed symbols).

Show that the current I' should be induced in the contour L :

$$I' = \omega \Psi'_0 \frac{1}{\sqrt{R'^2 + \omega^2 L'^2}} \sin(\omega t - \varphi)$$

where φ is the phase shift of the current with respect to the phase of the e.m.f. $\mathcal{E}'_{\text{ind},e}$ induced in the contour by a change in the induction flux Ψ' of the external field \mathbf{H}' :

$$\mathcal{E}'_{\text{ind},e} = - \frac{d\Psi'}{dt} = \Psi'_0 \omega \sin \omega t$$

and

$$\tan \varphi = \frac{\omega L'}{R'}$$

Show further that to maintain the rotation of the contour it is necessary to use external mechanical work amounting to

$$W = \frac{\omega^2 \Psi_0'^2 R'}{2(R'^2 + \omega^2 L'^2)} = \frac{1}{2} I_0'^2 R' \text{ J/s}$$

where I_0' stands for the amplitude of the current I' , and the quantity of Joule heat Q equal to the work W is liberated in the contour L .

If we assume that the closed contour L consists of two parts of which one rotates in a constant magnetic field (an armature in the field of a stator), and the other is stationary (the mains), and the contact

between the two parts of the contour is not violated during rotation, we get a very simple scheme of an alternating current generator to which all the results obtained in solving Problem 35 may be applied. In practice, however, attention must be given to the fact that alternating currents in a rotating armature, in turn, induce currents in the stator winding, and this violates the constancy of the “external” magnetic field \mathbf{H}' . Further, in Problem 35 we limited ourselves to a consideration of steady-state current conditions, leaving without attention the phenomena occurring when the armature starts or stops rotating, the load in the mains changes, etc.

**6.6 Energy of a Magnetic Field.
Energy Meaning of Inductances**

1. The expression for the energy of magnetic interaction of currents in the absence of ferromagnetics which we have obtained [Eq. (6.26)], namely,

$$U_m = \frac{1}{2c^2} \sum_{ik} L_{ik} I_i I_k$$

corresponds in its form to the notion of the long-range magnetic interaction of currents. In this respect, it is quite similar to expression (1.100) for the energy of stationary electric charges:

$$U_e = \frac{1}{2} \sum_{ik} \frac{q_i q_k}{R_{ik}}$$

where $i \neq k$.

Indeed, the term

$$\frac{1}{c^2} L_{12} I_1 I_2$$

in Eq. (6.26) can be interpreted as the energy of magnetic interaction of the currents I_1 and I_2 , and the term

$$\frac{1}{2c^2} L_{11} I_1^2$$

as the “proper” energy of the current I_1 , i.e. as the energy of interaction of infinitely thin current filaments into which this current can be resolved. Further, the mutual inductance L_{12} in Eq. (6.26) is similar to the coefficient $1/R_{12}$ in Eq. (1.100) because according to Eq. (4.78)

$$L_{12} = \oint_{L_1} \oint_{L_2} \frac{ds_1 ds_2}{R_{12}}$$

so that L_{12} is a certain mean distance between the current contours L_1 and L_2 .

2. It is not difficult, however, to express the magnetic energy of currents in the form of an integral over the entire volume of the field of these currents and thus, as for an electric field (Sec. 1.16), obtain the possibility of interpreting the energy U_m in the spirit of the short-range theory as the energy of the *field*, and not as the energy of *interaction* of currents.

For this purpose, we shall use formulas (6.27) and (4.100):

$$U_m = -U = \frac{1}{2c} \int \mathbf{A} \mathbf{j} dV \quad (6.44)$$

Expressing \mathbf{j} according to Eq. (5.24) through $\text{curl } \mathbf{H}$, we get

$$U_m = \frac{1}{8\pi} \int \mathbf{A} \text{curl } \mathbf{H} dV \quad (6.45)$$

But, according to the general formula of vector analysis (A.44),

$$\mathbf{A} \text{curl } \mathbf{H} = \mathbf{H} \text{curl } \mathbf{A} + \text{div} [\mathbf{H}\mathbf{A}]$$

By Eq. (5.27), $\text{curl } \mathbf{A} = \mathbf{B}$. Hence,

$$\mathbf{A} \text{curl } \mathbf{H} = \mathbf{H}\mathbf{B} + \text{div} [\mathbf{H}\mathbf{A}] \quad (6.46)$$

Putting Eq. (6.46) inside the integral in Eq. (6.45) and using Gauss's theorem [Eq. (A.17)], we have

$$\begin{aligned} U_m &= \frac{1}{8\pi} \int \mathbf{H}\mathbf{B} dV + \frac{1}{8\pi} \int \text{div} [\mathbf{H}\mathbf{A}] dV = \\ &= \frac{1}{8\pi} \int \mathbf{H}\mathbf{B} dV + \frac{1}{8\pi} \oint_S [\mathbf{H}\mathbf{A}]_n dS \end{aligned} \quad (6.47)$$

If we extend integration over the entire volume of the total field of the currents, then the integral over the boundary surface of this field will vanish, and the expression for U_m will become

$$U_m = \frac{1}{8\pi} \int \mathbf{H}\mathbf{B} dV \quad (6.48)$$

By the *total* magnetic field of currents in accordance with the definition of the total field of electric charges (see Sec. 1.16), we mean the region of space V enclosing all the interacting currents and the *entire* field of these currents. If, as usually happens, the field of the currents extends to infinity, then by the total field we may and must understand the entire infinite space provided without fail (see Sec. 1.16) that the integrands in the integrals over the boundary surface of the field we are interested in (in the given case $[\mathbf{H}\mathbf{A}]$)

diminish more rapidly than $1/R^2$ when this surface is extended to infinity.

If all the currents are in a finite region of space, then this condition is observed because, when the surface is extended to infinity, $[\mathbf{H}\mathbf{A}]$ diminishes not slower than $1/R^3$ (see Sec. 4.5, p. 237).

3. From the viewpoint of the theory of a field, Eq. (6.48) can be interpreted as follows: the magnetic energy is localized in a field and is distributed over its volume with a quite definite density u_m equal to

$$u_m = \frac{1}{8\pi} \mathbf{H}\mathbf{B} \quad (6.49)$$

In quasistationary magnetic fields, both of the above interpretations of magnetic energy (as the energy of interaction of currents and as the energy of a field) are naturally absolutely equivalent because they follow from the mathematically equivalent expressions (6.26) and (6.48) (cf. Sec. 1.16). When going over to fast-varying electromagnetic fields, however, the equivalence of these expressions is violated, and we shall see in the following chapter that only the notion of the localization of magnetic energy in a field can be made to agree with the results of experiments.

4. It must be noted that our initial formula (6.27), as we have repeatedly mentioned, holds only provided that *there are no ferromagnetics in the field*. Thus, this restricts the sphere of application of all the formulas of this section.

Since in the absence of ferromagnetics

$$\mathbf{B} = \mu\mathbf{H}$$

then Eqs. (6.48) and (6.49) can also be written as follows:

$$U_m = \frac{1}{8\pi} \int \mu\mathbf{H}^2 dV \text{ and } u_m = \frac{1}{8\pi} \mu\mathbf{H}^2 \quad (6.50)$$

Thus, with a given field intensity, the energy of a unit of its volume is proportional to the permeability of the medium. For a field in a vacuum, $\mu = 1$ and

$$u_m = \frac{1}{8\pi} \mathbf{H}^2 \quad (6.51)$$

5. Let us consider the energy of the magnetic field \mathbf{H} of two currents I_1 and I_2 in an arbitrary dia- or paramagnetic medium. If \mathbf{H}_1 and \mathbf{H}_2 are the intensities of the field induced by each of these currents separately, then

$$H^2 = (\mathbf{H}_1 + \mathbf{H}_2)^2 = H_1^2 + 2\mathbf{H}_1\mathbf{H}_2 + H_2^2$$

and the total energy of the field of the currents will be

$$U_m = \frac{1}{8\pi} \int \mu H^2 dV = \frac{1}{8\pi} \int \mu H_1^2 dV + \frac{1}{4\pi} \int \mu \mathbf{H}_1 \mathbf{H}_2 dV + \frac{1}{8\pi} \int \mu H_2^2 dV \quad (6.52)$$

It is evident that the first and the last terms of the right-hand side of this equation (we shall denote them by U_{11} and U_{22}) can be called the *proper energy* of each of the currents I_1 and I_2 , and the second term—the *mutual energy* of these currents U_{12} .

The equations (5.47) for the mutual inductance and self-inductance given in Sec. 5.6, as already indicated, may be applied only for a homogeneous magnetic medium ($\mu = \text{const}$). Comparing Eq. (6.52) with Eq. (6.25) for the energy

$$U_m = \frac{1}{2c^2} (L_{11}I_1^2 + 2L_{12}I_1I_2 + L_{22}I_2^2)$$

we get for the more general case of an arbitrary (but not ferromagnetic) medium:

$$\left. \begin{aligned} U_{11} &= \frac{1}{8\pi} \int \mu H_1^2 dV = \frac{1}{2c^2} L_{11}I_1^2 \\ \text{and} \\ U_{12} &= \frac{1}{4\pi} \int \mu \mathbf{H}_1 \mathbf{H}_2 dV = \frac{1}{c^2} L_{12}I_1I_2 \end{aligned} \right\} \quad (6.53)$$

Since \mathbf{H}_1 and \mathbf{H}_2 with a given configuration of the conductors are proportional to I_1 and I_2 , respectively, then the values of the inductances L_{11} and L_{12} determined with the aid of Eqs. (6.53) depend only on the geometrical configuration of the conductors and on the permeability of the medium, but not on the currents in the conductors. When $\mu = \text{const}$, these values should coincide with those of the inductances determined by Eqs. (5.47).

Equations (6.53) are the *most general*, suitable for $\mu \neq \text{const}$, *definition of the inductances*. It follows from this definition that the inductances are, in essence, a *measure of the energy of the magnetic field of currents* (at a given value of these currents). Both the part of the inductances in determining the ponderomotive forces acting on currents in a magnetic field (Secs. 4.10 and 5.6) and their part in determining the induced e.m.f.'s (Sec. 6.3) are inseparably linked, as can easily be seen, with this most important meaning of the inductances.

We shall now give two examples of calculating the self-inductance in which it is the most convenient to proceed directly from the energy definition (6.53) of this quantity.

second, over *both sides* of the surface of discontinuity of the potential. The last of these surface integrals obviously equals

$$\int_{S'} (\psi_- - \psi_+) B_n dS = -\frac{4\pi I}{c} \int_{S'} B_n dS$$

where B_n is the component of \mathbf{B} along the positive normal to S' (Fig. 71).

Thus,

$$U'_m = -\frac{1}{8\pi} \oint_S \psi B_n dS + \frac{I}{2c} \int_{S'} B_n dS \quad (6.56)$$

Let us now assume for definiteness that the space outside of the conductor is filled with a homogeneous magnetic having the permeability μ' , whereas the permeability of the conductor is μ'' . Let us assume, further, that the radius of the conductor r_0 is very small in comparison with the radius R of the circle it forms and consider a portion of the conductor having the length l complying with the condition that $r_0 \ll l \ll R$. Since $l \ll R$, this portion can be considered to be straight. Since, in addition, $r_0 \ll l$, then the field inside the conductor and in direct proximity to its surface will differ only very insignificantly from that of an infinitely long straight current, and with sufficient accuracy will be determined by the equations*

$$H' = \frac{2I}{cr} \quad (6.57)$$

(when $r \gg r_0$), and

$$H'' = \frac{2Ir}{cr_0^2} \quad (6.58)$$

(when $r \leq r_0$), where r is the distance to the point of the field being considered from the axis of the conductor. Thus, outside of the conductor at a sufficiently small distance from its surface ($r \ll R$) the field of the current we are considering coincides with that of a *line* current having the same intensity and concentrated on the axis of the conductor. On the other hand, the field of the current must also coincide with that of a line current at great distances from the surface of the conductor ($r \gg r_0$) at which the distribution of the current over the cross section of the conductor cannot have any influence. Since any point of the external space

* These equations were derived in Sec. 4.6, Problem 30, assuming that $\mu = 1$. Their proof, however, was based only on Eq. (4.39) that remains true for an arbitrary medium, and on account being taken of the axial symmetry of the field of a straight current.

V' complies with at least one of these conditions (since $r_0 \ll R$, either $r \gg r_0$ or $r \ll R$), then when determining the field in the entire space V' we may consider the current I to be concentrated

on the axis of the conductor. Hence, the integral $\int_{S'} B_n dS$ in the expression for U'_m should equal the induction flux Ψ' sent by this *line* ring current having the radius R through a concentric circle having the radius $R - r_0$ and formed by the intersection of the inner side of the surface of the conductor S with the plane S' . Thus, if L_{12} is the mutual inductance of the two concentric circles having the radii R and $R - r_0$, then according to Eq. (5.46)

$$\int_{S'} B_n dS = \Psi' = \frac{1}{c} I L_{12}$$

Since in the indicated conditions the induction (and the intensity) of the magnetic field at the surface of the conductor S is tangent to this surface, the first term in Eq. (6.56) vanishes and, therefore,

$$U'_m = \frac{I}{2c} \int_{S'} B_n dS = \frac{1}{2c^2} I^2 L_{12}$$

Turning to Eq. (6.55) and using in it the value of the field intensity H'' from Eq. (6.58), we get

$$\begin{aligned} U''_m &= \frac{1}{8\pi} \int_{V''} \mu H^2 dV = \frac{\mu''}{8\pi} \int_0^{2\pi R} \int_{r=0}^{r=r_0} \frac{4I^2 r^2}{c^2 r_0^4} 2\pi r dr dl = \\ &= \frac{\pi \mu'' R I^2}{2c^2} \end{aligned}$$

Thus, the total energy of the field of the current is

$$U_m = U'_m + U''_m = \frac{1}{2c^2} I^2 (L_{12} + \pi \mu'' R)$$

whence on the basis of Eq. (6.53) it follows that

$$L_{11} = \pi \mu'' R + L_{12}$$

The quantity $\pi \mu'' R$ is a measure of the energy U''_m stored inside the conductor and can be called its “internal” self-inductance, while the quantity L_{12} that is a measure of the energy U'_m can be called the “external” self-inductance of the conductor. Denoting the external and the internal self-inductances by L' and L'' , we can write that

$$L' = L_{12}, \quad L'' = \pi \mu'' R, \quad \text{and} \quad L_{11} = L' + L''$$

Thus, in the conditions mentioned above, the internal self-inductance L'' of a conductor having the radius r_0 and forming a circle having the radius R is proportional to its length $2\pi R$, while its external self-inductance L' equals the *mutual* inductance L_{12} of the two concentric circles having the radii R and $R - r_0$. The mutual inductance L_{12} can be calculated with the aid of the general formula (5.47) (see the example of determining the inductance L_{12} for two squares in Sec. 4.10). So as not to bother our reader with purely mathematical operations, we shall give only the final results here (for $r_0 \ll R$):

$$L_{12} = 4\pi\mu R \left(\log \frac{8R}{r_0} - 2 \right)$$

Example 2. *Self-inductance of a unit of cable length.* Let us consider a conductor consisting of two concentric hollow cylinders whose length is very great in comparison with their radii r_1 and r_2 . At both ends of the conductor, its inner and outer cylinders are connected to each other so that the combination of the two cylinders forms a closed conducting circuit through which the current I flows. The direction of the current in the outer cylinder is naturally the opposite of its direction in the inner one. We shall *conditionally* call such a circuit a *cable* here and in Secs. 7.16 and 7.17, although this term, of course, has a much broader meaning.

If the length of the cable is sufficiently great in comparison with its radius, then near its middle portion the field of the current flowing through the cable will be the same as for a cable having an infinite length. The concept of the self-inductance of an *infinite* cable naturally has no meaning because with an increase in the length of the cable the total energy of its field and, consequently, the self-inductance of the cable L_{11} grow to infinity. It is good, however, to introduce the *self-inductance of a unit length* of an infinite cable, understanding this quantity to be a measure of the fraction of the energy of its field confined between two planes at a unit distance apart perpendicular to the cable. If we use an asterisk to note all the quantities relating to a unit length of our cable, then by analogy with Eq. (6.53) we can write

$$U_{11}^* = \frac{1}{2c^2} L_{11}^* I^2$$

The physical meaning of the quantity L_{11}^* is that an increase in the length of a sufficiently long cable by unity is attended by an increase in its self-inductance L_{11} by L_{11}^* units.

Let us assume for simplicity that the cable coatings (i.e. the cylindrical conductors forming the cable) are so thin (in comparison with r_1 and r_2) that in a first approximation they can be considered as infinitely thin surfaces.

The field of a current uniformly distributed over the surface of a cylinder equals zero inside the cylinder; in the outer space it is such as if the current were concentrated on the axis of the cylinder (Problems 29 and 30). Hence, the field of the cable inside the cylinder r_1 is zero, between the cylinders r_1 and r_2 it coincides with the field of the line current I and, finally, outside of the cylinder r_2 it also equals zero (because currents equal in magnitude and opposite in direction flow through the inner and outer casings of the cable). Consequently, the energy U_{11}^* per unit of cable length is concentrated in the space between its casings, i.e. in a hollow cylinder having the length l whose inner and outer radii are r_1 and r_2 . Thus,

$$U_m^* = \frac{1}{8\pi} \int \mu H^2 dV = \frac{\mu}{8\pi} \int_{r_1}^{r_2} \left(\frac{2I}{cr} \right)^2 2\pi r dr =$$

$$= \frac{\mu I^2}{c^2} \ln \frac{r_2}{r_1}$$

where μ stands for the permeability of the medium between the cable casings. Comparing this with the preceding equation, we finally get

$$L_{11}^* = 2\mu \ln \frac{r_2}{r_1} \tag{6.59}$$

6.7 Transformation of Energy in the Magnetization of Para- and Diamagnetics. Free Energy of a Magnetic Field

1. We showed above that the quantity U_m determined by Eq. (6.26) or by Eq. (6.48) equivalent to it suffers a change *when current-carrying conductors move*. This increment equals the decrease of all the other kinds of energy (mechanical, thermal, and chemical) during these movements (as regards electric energy see the footnote on p. 390, and also Sec. 6.14). On these grounds, we arrived at the conclusion that this quantity U_m equals the energy of the magnetic field.

To complete the proof of this last statement, we ought to also show that the changes in the value of U_m *when magnetics move* in a magnetic field also comply with the law of conservation of energy [see Eqs. (6.21) and (6.24)]:

$$dU_m = -\Delta = -W - (Q - A) dt$$

The general proof of this statement, however, would be very complicated and would require an exact knowledge of the pondero-

motive forces acting on magnetics in a magnetic field. The latter could be determined only on the basis of special assumptions on the atomistic structure and the properties of a magnetic. The problem is therefore reversed in the macroscopic theory of the field: the expression for the energy of a magnetic field in the form of Eq. (6.48) is adopted as one of the main *postulates* of the theory, while the ponderomotive forces acting on magnetics are determined on the basis of considerations founded on the law of conservation of energy and on Eq. (6.48) for the magnetic energy.

2. We shall postpone the determination of these forces to Sec. 6.8 and stop here to treat another question connected with Eq. (6.48) for the magnetic energy. It follows from Eq. (6.48) or (6.49) that the density of the magnetic energy in a vacuum is

$$u_m = \frac{1}{8\pi} H^2$$

We could expect that with the same field intensity the energy of this field ought to be the same both in a vacuum and when atoms and molecules of a magnetic are interspersed in the vacuum. In other words, we ought to expect that even in the presence of magnetics the density of the energy of a magnetic field should equal:

$$u'_m = \frac{1}{8\pi} [\bar{\mathbf{H}}_{\text{micro}}]^2 = \frac{1}{8\pi} B^2 \quad (6.60)$$

where we have used Eq. (5.23). By Eq. (6.49), however, it equals

$$u_m = \frac{1}{8\pi} \mathbf{BH} = \frac{1}{8\pi\mu} B^2 \quad (6.61)$$

The difference between Eqs. (6.60) and (6.61) is due to three reasons. First, the mean macroscopic value of the energy of a magnetic field in the proper meaning of this word is determined by the *square of the mean* intensity of the true microscopic field $\bar{\mathbf{H}}_{\text{micro}}$ that, generally speaking, differs from the mean square of this intensity:

$$u'_m = \frac{1}{8\pi} \bar{\mathbf{H}}_{\text{micro}}^2 \neq \frac{1}{8\pi} [\bar{\mathbf{H}}_{\text{micro}}]^2$$

Second, upon the magnetization of a magnetic, the Larmor precession of the electrons in the magnetic appears (see Sec. 5.9), and a *change in the kinetic energy of the electrons* is connected with this precession. Since by the energy of a magnetic field in the macroscopic theory is meant the *entire* energy that must be spent to induce the field, then the total energy U'_m of the field in a magnetic is determined by the sum of the magnetic energy proper U'_m and the kinetic energy of the precessing electrons U_{kin} .

Finally, the magnetization of paramagnetics is also connected with a change in their entropy and the *absorption of heat* (cf. the polarization of dielectrics with solid dipoles, Sec. 2.12). Indeed, as we have seen in Sec. 5.11, the magnetization, for example, of gaseous paramagnetics, i.e. the arrangement of the magnetic moments of the molecules of these paramagnetics in the direction of the external field, occurs as a result of collisions between the molecules. These collisions are attended by exchange between the energy of translational thermal motion of the molecules and the intramolecular kinetic energy of precession of the electrons that depends on the orientation of the magnetic moment of a molecule relative to the magnetic field. The result is the liberation of heat*, which is also taken into account when calculating the energy U_m used to induce the magnetic field.

In calculating the transformations of energy occurring upon the motion of current-carrying conductors, we assumed in Sec. 6.4 that the permeability μ of the medium does not change during this motion. Hence, our calculations relate to the case of *isothermal* processes when all the heat is removed from paramagnetics that is liberated in them during the changes in their magnetization connected with the motion of the conductors. Otherwise upon changes in the temperature of a paramagnetic its susceptibility χ would also change [see Eq. (5.81)]. Further, the work done in an isothermal process is determined by the change in the *free energy* of the system Ψ . Hence, the quantity U_m determined by Eq. (6.26) or Eq. (6.48) equivalent to it is not the “internal”, but the *free energy* of a magnetic field (cf. Sec. 2.12). It is exactly this statement that is a precise formulation of the main postulate of the theory which we mentioned at the beginning of this section.

The density of the “internal” energy of a magnetic field coincides with that of its free energy and is expressed by the same formula (6.49) or (6.61) only when the permeability of the magnetic μ does not depend on the temperature T (with a constant volume of the magnetic), which takes place in diamagnetics. This follows from the well-known formula of thermodynamics

$$E = \Psi - T \left(\frac{\partial \Psi}{\partial T} \right)_V$$

relating the “internal” energy of a system E to its free energy Ψ .

* In the physics of low temperatures, a great part is played by the following “magnetic method” of cooling. Paramagnetics that are in an external magnetic field are cooled in conventional ways to as low a temperature as possible. Next the magnetic field is switched off, the paramagnetic is demagnetized and during this process absorbs heat both from the surrounding bodies and from its own store of thermal energy. As a result its temperature and that of the surrounding bodies lower.

3. The following calculation of the magnetic energy in *diamagnetics*, although not strict, may serve as an illustration of the above considerations. The change in the kinetic energy of the electrons in an atom ΔT due to the Larmor precession is determined by Eq. (5.78) and consists of two terms. The first of them, according to Eq. (5.79), equals $-\boldsymbol{\mu}\mathbf{H}$ and differs from zero only in paramagnetics. Thus, in diamagnetic atoms we have

$$\Delta T = \frac{m}{2} \sum \Delta v_i^2 = \frac{m}{2} \sum [\mathbf{o}\mathbf{R}_i]^2$$

where \mathbf{o} is the angular velocity of the Larmor precession [see Eq. (5.63)]:

$$\mathbf{o} = -\frac{e}{2mc} \mathbf{H}$$

where m = mass of an electron

e = its charge

\mathbf{R}_i = distance from the i -th electron to the nucleus of the atom.

Summation is performed for all the electrons of an atom. Assume that the magnetic field is directed along the z -axis. Hence,

$$[\mathbf{H}\mathbf{R}_i]^2 = H^2(x_i^2 + y_i^2)$$

If the number of electrons in an atom is Z and their number in a unit volume is N , then the kinetic energy of precession of the electrons in a unit volume of a magnetic is

$$u_{\text{kin}} = \frac{NZe^2}{8mc^2} H^2 \overline{(x^2 + y^2)}$$

where the bar over the parenthesis denotes averaging over all the electrons. Further,

$$\overline{x^2} = \overline{y^2} = \frac{1}{3} \overline{R^2}$$

where $\overline{R^2}$ is the mean square of the distance from the electrons to the nucleus of an atom. Consequently,

$$u_{\text{kin}} = \frac{NZe^2 \overline{R^2}}{12mc^2} H^2$$

Using here Eq. (5.74) for the magnetization vector of diamagnetics, we get

$$u_{\text{kin}} = -\frac{1}{2} \mathbf{M}\mathbf{H}$$

Let us recall, finally, that when treating the magnetization of diamagnetics and paramagnetics in Secs. 5.11 and 5.12 we considered that the field

\mathbf{H} acts on the atoms of a magnetic, whereas it would be more correct to consider that the field $\mathbf{H}_{\text{micro}} = \mathbf{B}$ acts on them. Making the corresponding substitution in the last equation, we finally get

$$u_{\text{kin}} = -\frac{1}{2} \mathbf{M}\mathbf{B} \tag{6.62}$$

The sum of this density of the kinetic energy of precession of the electrons and of approximate expression (6.60) for the density of the energy of a magnetic field in the proper meaning of this word* is [see Eq. (5.22)]

$$u = \frac{1}{8\pi} \mathbf{B}^2 - \frac{1}{2} \mathbf{M}\mathbf{B} = \frac{1}{8\pi} \mathbf{B}(\mathbf{B} - 4\pi\mathbf{M}) = \frac{1}{8\pi} \mathbf{B}\mathbf{H}$$

i.e., as we had to prove, it equals the macroscopic density of the magnetic energy (6.61).

6.8 Determination of the Ponderomotive Forces of a Magnetic Field from the Expression for Energy **

1. In Sec. 5.7, we determined the ponderomotive forces acting on magnetics in a magnetic field on the basis of definite notions of the molecular structure of the magnetics. Now we shall give a more general derivation of the expression for the ponderomotive forces of a magnetic field. We shall proceed from the expression for the magnetic energy (6.48) and consider the change in this energy, δU upon an infinitely small virtual displacement \mathbf{q} of the bodies in the field. If upon these virtual displacements the conduction currents remain constant, then by Eq. (6.29) the change in the magnetic energy δU upon these displacements will equal the work W of the mechanical forces:

$$(\delta U)_I = W$$

On the other hand, this work evidently equals

$$W = \int \mathbf{q}\mathbf{f} dV$$

where \mathbf{f} is the density of the ponderomotive forces. Consequently,

$$(\delta U)_I = \int \mathbf{q}\mathbf{f} dV \tag{6.63}$$

* In this expression, no account is taken of the difference between $(\overline{H}_{\text{micro}})^2$ and $\overline{H}_{\text{micro}}^2$.

** When reading the book the first time this section may be omitted.

After computing $(\delta U)_I$ from this relationship, we can obviously also determine the required value of the density of the forces \mathbf{f} .

The energy of a magnetic field can be expressed by one of the equations (6.48) and (6.44) as follows:

$$U_1 = \frac{1}{8\pi} \int \mathbf{H}\mathbf{B} \, dV = \frac{1}{8\pi} \int \frac{1}{\mu} B^2 \, dV$$

$$U_2 = U_1 = \frac{1}{2c} \int \mathbf{A}\mathbf{j} \, dV$$

These expressions for the energy, as we have repeatedly indicated, hold only *in the absence of ferromagnetics*. Therefore, all our conclusions will be applicable only to dia- and paramagnetics.

All our further calculations are very similar to those which we performed in determining the ponderomotive forces of an electric field. Particularly, we shall assume that there are no surfaces of discontinuity in the field and that the volume of integration covers the entire field. Hence, all the surface integrals which we shall encounter in the following will vanish.

2. The change in the energy of a field upon an arbitrary infinitely small displacement \mathbf{q} of the bodies in it is

$$\delta U_1 = \frac{1}{8\pi} \int B^2 \delta \left(\frac{1}{\mu} \right) dV + \frac{1}{4\pi} \int \frac{1}{\mu} \mathbf{B} \delta \mathbf{B} \, dV \quad (6.64)$$

or

$$\delta U_2 = \delta U_1 = \frac{1}{2c} \int \mathbf{A} \delta \mathbf{j} \, dV + \frac{1}{2c} \int \mathbf{j} \delta \mathbf{A} \, dV \quad (6.65)$$

where $\delta(1/\mu)$, $\delta\mathbf{j}$, and $\delta\mathbf{A}$ are the changes in the corresponding quantities due to the displacement \mathbf{q} .

Since $\mathbf{B} = \text{curl } \mathbf{A}$, then $\delta\mathbf{B} = \delta \text{curl } \mathbf{A}$. The sequence of performing the operations of differentiation and variation can be changed without distorting the results, so that

$$\delta \text{curl } \mathbf{A} = \text{curl} (\delta\mathbf{A})$$

Hence,

$$\frac{1}{\mu} \mathbf{B} \cdot \delta \mathbf{B} = \mathbf{H} \text{curl} (\delta\mathbf{A})$$

Further, on the basis of Eqs. (A.44) and (5.24), we have

$$\begin{aligned} \mathbf{H} \text{curl} (\delta\mathbf{A}) &= \text{div} [\delta\mathbf{A} \cdot \mathbf{H}] + \text{curl } \mathbf{H} \cdot \delta\mathbf{A} = \\ &= \text{div} [\delta\mathbf{A} \cdot \mathbf{H}] + \frac{4\pi}{c} \mathbf{j} \cdot \delta\mathbf{A} \end{aligned}$$

Inserting this equation in Eq. (6.64) and using Gauss's theorem (A.17), we get

$$\begin{aligned} \delta U_1 &= \frac{1}{8\pi} \int B^2 \delta \left(\frac{1}{\mu} \right) dV + \frac{1}{4\pi} \oint [\delta \mathbf{A} \cdot \mathbf{H}]_n dS + \\ &+ \frac{1}{c} \int \mathbf{j} \cdot \delta \mathbf{A} dV \end{aligned} \quad (6.66)$$

According to the conditions set out at the beginning of this section, the surface integral in this expression vanishes. Finally, since $\delta U_1 = \delta U_2$, the energy increment δU can obviously also be written as follows:

$$\delta U = 2\delta U_2 - \delta U_1 = \frac{1}{c} \int \mathbf{A} \delta \mathbf{j} \cdot dV - \frac{1}{8\pi} \int B^2 \delta \left(\frac{1}{\mu} \right) dV \quad (6.67)$$

Thus, we have reduced the calculation of δU to determining the change in the density of the conduction currents \mathbf{j} and the permeability of the medium μ upon a virtual displacement \mathbf{q} of the bodies.

3. A local change in the quantity $1/\mu$ will evidently be expressed by an equation similar to the expression for $\delta \epsilon$ [see Sec. 2.13, Eq. (2.104)]:

$$\delta \frac{1}{\mu} = - \frac{\partial}{\partial \tau} \left(\frac{1}{\mu} \right) \cdot \tau \operatorname{div} \mathbf{q} - \mathbf{q} \operatorname{grad} \frac{1}{\mu} \quad (6.68)$$

where τ stands for the density of the medium (the mass of a unit volume)*.

As regards the virtual change in the density of the current $\delta \mathbf{j}$, it can be determined from the condition noted at the beginning of this section that upon a virtual displacement \mathbf{q} the current through an arbitrary surface should remain constant (if only this surface moves *together* with the medium).

Assume that the current I flows through an arbitrary surface S before the displacement \mathbf{q} :

$$I = \int_S j_n dS = \int_S \mathbf{j} dS$$

The displacement \mathbf{q} may be attended, first, by a change of $\delta \mathbf{j}$ in the density of the current at different points of the surface S , and, second,

* As for dielectrics, we shall disregard for simplicity the relationship between the permeability μ of solid magnetics and deformations not connected with a change in the density of the medium.

by deformation of the contour L of this surface, as a result of which its total area will grow or diminish. If ds is an element of the contour L , then upon the displacement \mathbf{q} this element will cover the area $\delta S = [\mathbf{q} ds]$ (see Sec. 4.9, particularly Fig. 50). The increment of the area S resting on the contour consists in the sum of the areas δS covered by all of its elements ds . Consequently, the total change in the current I through an arbitrary surface S undergoing deformation that must equal zero is

$$\delta I = \int_S \delta j_n dS + \oint_L \mathbf{j}[\mathbf{q} ds] = 0 \quad (6.69)$$

Transforming the last integral in accordance with the Stokes theorem [Eq. (A.27)], we obtain

$$\oint_L \mathbf{j}[\mathbf{q} ds] = \oint_L [\mathbf{j}\mathbf{q}] ds = \int_S \text{curl}_n [\mathbf{j}\mathbf{q}] dS$$

Introducing this expression into Eq. (6.69) owing to the arbitrary nature of the surface S , we finally get

$$\delta \mathbf{j} = - \text{curl} [\mathbf{j}\mathbf{q}] \quad (6.70)$$

4. Introducing the values of $\delta(1/\mu)$ and $\delta \mathbf{j}$ from Eqs. (6.68) and (6.70) into Eq. (6.67) we get

$$\begin{aligned} \delta U = & - \frac{1}{c} \int \mathbf{A} \text{curl} [\mathbf{j}\mathbf{q}] dV + \\ & + \frac{1}{8\pi} \int B^2 \left\{ \frac{\partial}{\partial \tau} \frac{1}{\mu} \tau \text{div} \mathbf{q} + \mathbf{q} \text{grad} \frac{1}{\mu} \right\} dV \end{aligned} \quad (6.71)$$

On the basis of Eq. (A.44), we have

$$\mathbf{A} \text{curl} [\mathbf{j}\mathbf{q}] = \text{div} [[\mathbf{j}\mathbf{q}] \mathbf{A}] + [\mathbf{j}\mathbf{q}] \text{curl} \mathbf{A}$$

The last term can be transformed as follows:

$$[\mathbf{j}\mathbf{q}] \text{curl} \mathbf{A} = [\mathbf{j}\mathbf{q}] \mathbf{B} = - \mathbf{q}[\mathbf{j}\mathbf{B}]$$

Further,

$$B^2 \frac{\partial}{\partial \tau} \frac{1}{\mu} \tau \text{div} \mathbf{q} = \text{div} \left(B^2 \frac{\partial}{\partial \tau} \frac{1}{\mu} \tau \mathbf{q} \right) - \mathbf{q} \text{grad} \left(B^2 \frac{\partial}{\partial \tau} \frac{1}{\mu} \tau \right)$$

Introducing these values into Eq. (6.71) and taking into consideration that the volume divergence integral can be transformed into a surface integral and therefore, according to our condition, vanishes, we get

$$\delta U = \int \mathbf{q} \left\{ \frac{1}{c} [\mathbf{jB}] - \frac{1}{8\pi} \text{grad} \left(B^2 \frac{\mu}{\partial \tau} \tau \right) + \right. \\ \left. + \frac{1}{8\pi} B^2 \text{grad} \frac{1}{\mu} \right\} dV \quad (6.72)$$

Comparing this equation with Eq. (6.63) and taking into account that \mathbf{q} is an arbitrary position function, we get the following expression for the density of the ponderomotive forces:

$$\mathbf{f} = \frac{1}{c} [\mathbf{jB}] - \frac{1}{8\pi} \text{grad} \left(B^2 \frac{\mu}{\partial \tau} \tau \right) + \frac{1}{8\pi} B^2 \text{grad} \frac{1}{\mu} \quad (6.73)$$

Thus, the ponderomotive forces consist of forces having the density $\mathbf{f}^{(1)} = \frac{1}{c} [\mathbf{jB}]$ acting on conductors carrying a current and coinciding with the previous formula (5.41), and of the forces of density

$$\mathbf{f}^{(2)} = \frac{1}{8\pi} B^2 \text{grad} \frac{1}{\mu} - \frac{1}{8\pi} \text{grad} \left(B^2 \frac{\mu}{\partial \tau} \tau \right) \quad (6.74)$$

acting on the magnetics in the field.

5. It is not difficult to show that the density of these forces $\mathbf{f}^{(2)}$ coincides with the previously derived expression (5.54). The latter expression, as noted in Sec. 5.7, holds only for weakly magnetizing magnetics (cf. Sec. 2.13). In this case, according to Eqs. (5.76) and (5.80), the quantity χ/μ is proportional to the number of molecules in a unit volume of the medium, i.e. is proportional to the density of the medium τ . Hence,

$$\frac{\chi}{\mu} = \frac{\mu - 1}{4\pi\mu} = c\tau$$

where c is a constant not depending on the density of the medium. Consequently,

$$\frac{1}{\mu} = 1 - 4\pi c\tau$$

and

$$\tau \frac{\partial \frac{1}{\mu}}{\partial \tau} = -4\pi c\tau = \frac{1}{\mu} - 1$$

Using this expression in Eq. (6.74), we get

$$\begin{aligned} \mathbf{f}^{(2)} &= -\frac{1}{8\pi} \text{grad} \left\{ B^2 \left(\frac{1}{\mu} - 1 \right) \right\} + \frac{1}{8\pi} B^2 \text{grad} \frac{1}{\mu} = \\ &= \frac{1}{8\pi} \left(1 - \frac{1}{\mu} \right) \text{grad} B^2 = \frac{\mu - 1}{8\pi\mu} \nabla B^2 \end{aligned}$$

which indeed coincides completely with Eq. (5.54).

6. If we replace \mathbf{B} with $\mu\mathbf{H}$ in Eq. (6.74) and perform simple transformations, it acquires the form

$$\mathbf{f}^{(2)} = \frac{1}{8\pi} \text{grad} \left(H^2 \tau \frac{\partial \mu}{\partial \tau} \right) - \frac{1}{8\pi} H^2 \text{grad} \mu \quad (6.75)$$

which is absolutely similar to Eq. (2.108) for the ponderomotive forces acting on dielectrics in an electric field. Equation (6.75) is obtained from Eq. (2.100) by replacing \mathbf{E} with \mathbf{H} and ε with μ .

In this circumstance, we find a manifestation of the fact noted at the end of Sec. 5.14 that owing to the equivalence of elementary currents and magnetic dipoles, the entire *macroscopic* theory of magnetism can be interpreted with the same success both on the basis of modern theory and from the viewpoint of the old theories of magnetism proceeding from the assumption of the existence in magnetics of real magnetic charges (dipoles), i.e. from the assumption of the complete conformity of the electrical properties of dielectrics to the magnetic properties of magnetics. It is exactly from the viewpoint of these theories that there is correspondence between the vectors \mathbf{E} and \mathbf{H} and between the quantities ε and μ , whereas actually the vector $\mathbf{B} = \bar{\mathbf{H}}_{\text{micro}}$ corresponds to the vector \mathbf{E} instead of the vector \mathbf{H} , and $1/\mu$ corresponds to the permittivity ε instead of μ .

It should be noted that J. Maxwell and a number of other authors (for instance M. Abraham, W. Kohn, etc.) did not take into account the dependence of the susceptibility on the density of the medium. Hence, the expression for the ponderomotive forces in magnetics which they used differed from Eq. (6.75) in the absence of the first term, i.e. of the striction forces

$$\mathbf{f}'' = \frac{1}{8\pi} \text{grad} \left(H^2 \tau \frac{\partial \mu}{\partial \tau} \right) \quad (6.76)$$

It must be noted in passing that we can prove in absolutely the same way as for dielectrics (see Secs. 2.13 and 2.15) that the compo-

ment \mathbf{f}'' of the density of the forces $\mathbf{f}^{(2)}$ not taken into consideration by Maxwell tells only on the *distribution* of the ponderomotive forces over the volume of a magnetic, whereas the resultant \mathbf{F} and the resultant moment \mathbf{N} of the striction forces \mathbf{f}'' applied to all the elements of volume of a body are either equal to zero (if the body is in a vacuum) or are balanced by the hydrostatic pressure appearing under the action of the magnetic field in the liquid or gaseous phase surrounding the body (see also Sec. 6.9).

6.9 Stress Tensor of a Magnetic Field

1. The ponderomotive forces of a magnetic field, in exactly the same way as the ponderomotive forces of an electric field, can be reduced to a system of stresses (tensions) equivalent to these forces that is characterized by a stress tensor \mathbf{T} .

The corresponding calculations are absolutely similar to those of Sec. 2.15. We shall represent the general expression for the ponderomotive forces (6.73) as the sum of two addends:

$$\left. \begin{aligned} \mathbf{f} &= \mathbf{f}' + \mathbf{f}'' \\ \mathbf{f}' &= \frac{1}{c} [\mathbf{jB}] - \frac{1}{8\pi} H^2 \text{grad } \mu \\ \mathbf{f}'' &= \frac{1}{8\pi} \text{grad} \left(H^2 \tau \frac{\partial \mu}{\partial \tau} \right) \end{aligned} \right\} \quad (6.77)$$

[cf. Eq. (6.75)]. We shall also consider that the stress tensor \mathbf{T} consists of the sum of two tensors \mathbf{T}' and \mathbf{T}'' :

$$\mathbf{T} = \mathbf{T}' + \mathbf{T}'' \quad (6.78)$$

so that \mathbf{T}' will correspond to \mathbf{f}' and \mathbf{T}'' to \mathbf{f}'' , i.e. so that the following equations and similar ones for f'_x, f''_x , etc. be satisfied [cf. Eq. (2.117)]:

$$\left. \begin{aligned} f'_x &= \frac{\partial T'_{xx}}{\partial x} + \frac{\partial T'_{xy}}{\partial y} + \frac{\partial T'_{xz}}{\partial z} \\ f''_x &= \frac{\partial T''_{xx}}{\partial x} + \frac{\partial T''_{xy}}{\partial y} + \frac{\partial T''_{xz}}{\partial z} \end{aligned} \right\} \quad (6.79)$$

2. Let us first consider the forces \mathbf{f}' and the tensor \mathbf{T}' . Expressing the density of the currents \mathbf{j} through curl \mathbf{H} [Eq. (5.24)], we can write

$$\frac{1}{c} [\mathbf{jB}] = \frac{1}{4\pi} [\text{curl } \mathbf{H} \cdot \mathbf{B}] = \frac{\mu}{4\pi} [\text{curl } \mathbf{H} \cdot \mathbf{H}]$$

Further, on the basis of Eq. (A.47), we have

$$[\text{curl } \mathbf{H} \cdot \mathbf{H}] = \mathbf{H} \nabla \cdot \mathbf{H} - \frac{1}{2} \text{grad } H^2$$

and, consequently,

$$\begin{aligned} \mathbf{f}' &= \frac{\mu}{4\pi} \mathbf{H} \nabla \cdot \mathbf{H} - \frac{\mu}{8\pi} \text{grad } H^2 - \frac{1}{8\pi} H^2 \text{grad } \mu = \\ &= \frac{1}{4\pi} \mathbf{B} \nabla \cdot \mathbf{H} - \frac{1}{8\pi} \text{grad } (\mu H^2) \end{aligned}$$

Let us consider the component of the force density \mathbf{f}' along the x -axis:

$$f'_x = \frac{1}{4\pi} \mathbf{B} \nabla \cdot H_x - \frac{1}{8\pi} \frac{\partial}{\partial x} (\mu H^2) \quad (6.80)$$

It is easy to show [see Eq. (A. 43₂), where φ in our case corresponds to H_x] that

$$\mathbf{B} \nabla \cdot H_x = \mathbf{B} \cdot \nabla H_x = \text{div} (\mathbf{B} H_x) - H_x \text{div } \mathbf{B}$$

or since $\text{div } \mathbf{B} = 0$,

$$\mathbf{B} \nabla \cdot H_x = \text{div} (\mathbf{B} H_x) = \frac{\partial}{\partial x} (B_x H_x) + \frac{\partial}{\partial y} (B_y H_x) + \frac{\partial}{\partial z} (B_z H_x)$$

Introducing this expression into Eq. (6.80), we see that it coincides in form with Eq. (6.79) if we assume that

$$T'_{xx} = \frac{1}{4\pi} \left(\mu H_x^2 - \frac{\mu H^2}{2} \right), \quad T'_{xy} = \frac{\mu}{4\pi} H_x H_y, \quad T'_{xz} = \frac{\mu}{4\pi} H_x H_z$$

The remaining components of the tensor \mathbf{T}' are determined similarly. Their totality can be written in a form similar to (2.116):

$$4\pi \mathbf{T}' = \left\{ \begin{array}{ccc} \mu \left(H_x^2 - \frac{H^2}{2} \right) & \mu H_x H_y & \mu H_x H_z \\ \mu H_y H_x & \mu \left(H_y^2 - \frac{H^2}{2} \right) & \mu H_y H_z \\ \mu H_z H_x & \mu H_z H_y & \mu \left(H_z^2 - \frac{H^2}{2} \right) \end{array} \right\} \quad (6.81)$$

Thus, the stress tensor of a magnetic field \mathbf{T}' can be obtained from the corresponding tensor \mathbf{T}' of an electric field [Eq. (2.122)] by simply

substituting \mathbf{H} for \mathbf{E} and μ for ϵ^* . The same relates to the stress tensor \mathbf{T}'' equivalent to the forces \mathbf{f}'' because the expression for the density of these forces in a magnetic field (6.77) is obtained from the expression for the force density \mathbf{f}'' in an electric field (2.121) by substituting \mathbf{H} for \mathbf{E} and μ for ϵ . Therefore, by analogy with Eq. (2.123), we have

$$T''_{xx} = T''_{yy} = T''_{zz} = \frac{1}{8\pi} H^2 \frac{\partial \mu}{\partial \tau} \quad (6.82)$$

while the non-diagonal components of the tensor \mathbf{T}'' (T''_{xy} , T''_{xz} , etc.) equal zero.

3. We have thus proved the equivalence of the volume forces (6.77) to the system of stresses (6.81) and (6.82) (it should be noted that both tensors \mathbf{T}' and \mathbf{T}'' are symmetrical).

Owing to the similarity of the stresses of a magnetic field and those of an electric field, all the results which we have obtained in Sec. 2.15 can also be directly applied to the stresses of a magnetic field. For example, the system of stresses in a magnetic field consists

in the pull $\frac{\mu + \tau \frac{\partial \mu}{\partial \tau}}{8\pi} H^2$ in the direction of the field \mathbf{H} and in the

pressure $\frac{\mu - \tau \frac{\partial \mu}{\partial \tau}}{8\pi} H^2$ in a direction perpendicular to \mathbf{H} [see Eqs. (2.126)

and (2.127)].

Further, we can show, as in Sec. 2.15, that the resultant \mathbf{F} and the resultant moment \mathbf{N} of the forces acting on a body in a magnetic field are completely determined by the *Maxwellian stress tensor* \mathbf{T}' and do not depend on the tensor \mathbf{T}'' not taken into consideration by Maxwell, so that the striction tensions \mathbf{T}'' affect only the *distribution* of the ponderomotive forces over the volume of a body and also cause the appearance in the liquid or gaseous phase surrounding the body of a hydrostatic pressure balancing them [see Eq. (2.124)]:

$$p = -\frac{1}{8\pi} H^2 \frac{\partial \mu}{\partial \tau} \quad (6.83)$$

* The same result is also obtained when substituting \mathbf{B} for \mathbf{E} and $1/\mu$ for ϵ in Eq. (2.122), for example $\epsilon E_x E_y \rightarrow \frac{1}{\mu} B_x B_y = \mu H_x H_y$. From the viewpoint of the modern notions of the field in magnetics, it is exactly the latter substitution that has a direct physical meaning (owing to the relationships $\mathbf{E} = \bar{\mathbf{E}}_{\text{micro}}$, $\mathbf{B} = \bar{\mathbf{H}}_{\text{micro}}$, $\mathbf{D} = \epsilon \bar{\mathbf{E}}_{\text{micro}}$, and $\mathbf{H} = \frac{1}{\mu} \bar{\mathbf{B}}_{\text{micro}}$).

4. Although the expressions obtained in Secs. 6.6-6.8 for the density of the energy and the density of the volume forces of a magnetic field, as we have repeatedly remarked, hold only for non-ferromagnetic media, the reduction of the volume forces to stresses allows us, as was shown in Sec. 5.16, to also determine the resultant \mathbf{F} and the resultant moment \mathbf{N} of the forces acting on ferromagnetics (but not the distribution of the forces over the volume of a ferromagnetic).

6.10 Vortices of an Electric Field

1. In Sec. 6.1, we derived the laws for currents induced in *moving* conductors on the basis of the fact that according to Sec. 4.4 electric charges (or electrons) are acted upon by the Lorentz force

$$\mathbf{F} = e \left\{ \mathbf{E} + \frac{1}{c} [\mathbf{v}\mathbf{H}] \right\}$$

The second term of this expression is proportional to the velocity of the charge and the intensity of the magnetic field \mathbf{H} . Next, on the basis of the principle of the relativity of motion, we showed that currents must also be induced in *stationary* conductors upon changes in the magnetic field, and

$$\mathcal{E}_{\text{ind}} = -\frac{1}{c} \frac{d\Psi}{dt} = -\frac{1}{c} \frac{d}{dt} \int B_n dS$$

where integration may be extended over any surface resting on the contour of a conductor. For *stationary** conductors, this surface may also be a stationary one, and in this case differentiation with respect to time can be performed inside the integral:

$$\mathcal{E}_{\text{ind}} = -\frac{1}{c} \int \frac{\partial B_n}{\partial t} dS \quad (6.84)$$

We have used the partial derivative instead of the total derivative with respect to time in order to show that $\partial B_n/\partial t$ is the rate of change with time of the quantity B_n at a *fixed* point of space.

We thus arrive at the conclusion that a change in a magnetic field should induce in stationary conductors forces acting on electric charges. The circulation of these forces around the contour of a conductor, which we have denoted by \mathcal{E}_{ind} , is determined by Eq. (6.84).

2. In Sec. 1.2, we defined the electric field intensity \mathbf{E} as the force acting on a unit positive test charge. In Sec. 4.4, however, we saw

* Stationary relative to the inertial frame in which the field is being measured (see Sec. 6.2).

that in the absence of an electric field a *moving* charge (electron) can be acted upon by the force .

$$\mathbf{F} = \frac{e}{c} [\mathbf{v}\mathbf{H}]$$

This makes it necessary to determine the definition of the electric field intensity \mathbf{E} more exactly in the sense that \mathbf{E} equals the force acting on a *stationary* unit positive charge. Indeed, it follows from Eq. (4.21) that when $v = 0$, we have

$$\mathbf{E} = \frac{1}{e} \mathbf{F}$$

(we assume that extraneous e.m.f.'s of a chemical and thermal origin are absent).

On the basis of this definition of an electric field, we must conclude from Eq. (6.84) relating to *stationary* conductors that *changes in the magnetic field in these conductors induce an electric field*. The circulation of its intensity around the contour of a conductor L is

$$\oint_L E_s ds = \mathcal{E}_{\text{ind}} = -\frac{1}{c} \int \frac{\partial B_n}{\partial t} dS = -\frac{1}{c} \frac{\partial \Psi}{\partial t} \quad (6.85)$$

Thus, the induction of currents in conductors *moving* in a constant *magnetic* field has been interpreted by us as the result of the action of a *magnetic* field (Lorentz force), whereas the induction in *stationary* conductors upon changes in the magnetic field has been interpreted in an absolutely different way—as the result of the action of an *electric* field induced by changes in the magnetic field. As we have seen in Sec. 6.1, however, there is no fundamental difference between these two kinds of induction because the concept of motion is relative. The first of A. Einstein's works devoted to the theory of relativity begins with an indication of the need to eliminate this difference of principle in the interpretation of two phenomena that objectively are indistinguishable from each other. The theory of relativity has solved this problem, and it will be briefly considered in Sec. 8.6. 3. Since according to Eq. (1.33) the circulation of the electric field vector of stationary charges equals zero, then Eq. (6.85) also holds if we agree throughout the following to understand by \mathbf{E} *the total intensity of an electric field regardless of whether the field is induced (completely or partly) by stationary electric charges (a Coulomb field) or by changes of a magnetic field*.

In deriving Eq. (6.85), we assumed that the integration contour L coincides with the contour of a linear conductor. It is natural to assume, however, that if changes in a magnetic field induce an electric field in conductors, then they also induce it outside of conductors.

In other words, it is natural to assume that Eq. (6.85) may be applied to *any* closed stationary integration contour *regardless* of whether this contour passes through conductors, dielectrics or through a vacuum, and that the distinction between a conducting contour and a non-conducting one tells in that only in conductors does the induction of a field lead to the appearance of a current.

Thus, we shall assume that Eq. (6.85) may be applied to any closed stationary integration contour L . Presuming that there are no points of discontinuity of the vector \mathbf{E} on the surface S resting on the contour L , we can transform the left-hand side of this equation with the aid of the Stokes theorem [Eq. (A. 27)]:

$$\oint_L \mathbf{E}_s \, ds = \int_S \text{curl}_n \mathbf{E} \, dS = - \frac{1}{c} \int_S \frac{\partial B_n}{\partial t} \, dS$$

This equation must hold with any choice of the contour L and the integration surface S , which can take place only when

$$\text{curl } \mathbf{E} = - \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \quad (6.86)$$

Thus, the phenomena of induction lead us to the conclusion that *an electric field can be induced not only by electric charges, but also by changes in the magnetic field* (a magnetic field, in turn, is induced by the motion of electric charges).

Equation (6.86) relating the value of the curl of the electric vector to the time derivative of the magnetic induction is one of the *fundamental* equations of the electromagnetic field. It is obvious that by repeating our reasoning in the reverse sequence, we can derive the initial integral relationship (6.85) from the differential equation (6.86).

4. Since the divergence of the curl equals zero [Eq. (A. 42₂)], it follows from Eq. (6.86) that

$$\text{div } \frac{\partial \mathbf{B}}{\partial t} = \frac{\partial}{\partial t} \text{div } \mathbf{B} = 0 \quad (6.87)$$

Hence, it follows from Eq. (6.86) that at every point of space $\text{div } \mathbf{B}$ must have a constant value that cannot change in any physical processes. It is sufficient to assume that in the absence of currents and magnetics, the magnetic induction \mathbf{B} (and, consequently, also $\text{div } \mathbf{B}$) vanishes throughout the entire space in order to obtain from Eq. (6.87), in turn, one of the fundamental equations of an electromagnetic field, namely, Eq. (5.25):

$$\text{div } \mathbf{B} = 0$$

6.11 Dependence of Electric Voltage on Integration Path. Voltage of Alternating Current

1. We saw in Sec. 4.7 that for a vector to have a (single-valued) scalar potential, it is necessary and sufficient that its curl equal zero at all points of space or, which is the same, that its circulation around an arbitrary contour equal zero [see also Sec. 1.7, in particular Eq. (1.36)]. It follows from Eqs. (6.85) and (6.86) that this condition is complied with for the electric vector \mathbf{E} only in stationary fields ($\partial\mathbf{B}/\partial t = 0$) and that, consequently, *the electric vector \mathbf{E} of a varying electromagnetic field ($\partial\mathbf{B}/\partial t \neq 0$) does not have a (single-valued) scalar potential φ .*

2. In this connection, a considerable number of concepts which we introduced in studying a stationary electric field having a potential lose their direct physical meaning in a varying field. For example, in Sec. 3.1, we introduced the concept of electric voltage \mathcal{E}_{12} existing between two arbitrary points of a field 1 and 2 and defined it as the line integral of the field intensity E over an *arbitrary path* connecting the points 1 and 2 [Eq. (3.3)]:

$$\mathcal{E}_{12} = \int_1^2 E_s ds$$

For a stationary field having a potential, by Eq. (3.2), we have

$$\mathcal{E}_{12} = \int_1^2 E_s ds = \varphi_2 - \varphi_1$$

so that the voltage \mathcal{E}_{12} equals the difference between the potentials of the points 1 and 2 and is *unambiguously* determined by the position of these points. For a varying field deprived of a potential, however,

the value of the integral $\int_1^2 E_s ds$ depends appreciably on the selec-

tion of the integration path. Hence, we can speak only of the *voltage \mathcal{E}_{12} existing between the given points 1 and 2 along a given path.*

3. Insufficient attention to this exceedingly important distinction of a varying field from a stationary field may result in very grave errors.

Let, for instance, 1 and 2 be two arbitrary points of a closed conductor L to which a galvanometer G is connected parallel to L

(Fig. 72). If R_0 is the total resistance of the galvanometer and the wires connecting it to the points 1 and 2, then the current I_0 in the galvanometer circuit, according to Ohm's law [Eq. (3.4) or (3.26)] will be

$$I_0 = \frac{1}{R_0} \mathcal{E}_{12} = \frac{1}{R_0} \int_1^2 E_s ds$$

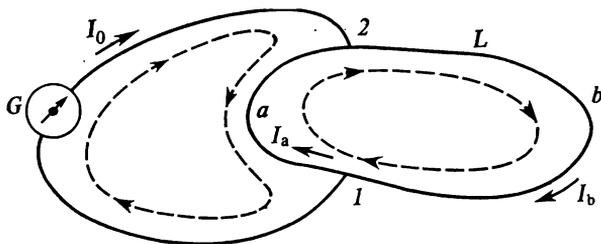


Fig. 72

In a stationary electromagnetic field, the value of the last integral is independent of the integration path. In a varying field, however, formula (3.26) expressing Ohm's law *remains correct*, as follows

from its derivation given in Sec. 3.4, *only if* by $\int_1^2 E_s ds$ we understand

the *line integral of the vector E* from the point 1 to the point 2 taken *over the conductor through which the current I_0 is flowing, and only over this conductor*. Since the value of this integral in a varying field noticeably depends on the position and the shape of the integration path, then, therefore, the readings I_0 of the galvanometer will depend noticeably on the arrangement of the connecting wires.

Let us assume that extraneous e.m.f.'s are absent in the conductors we are considering. Let R_a and R_b be the resistances, and I_a and I_b the currents in portions $1a2$ and $2b1$ of the contour L , respectively. Let, further, Ψ be the magnetic induction flux through the contour L , and Ψ_0 its flux through the contour formed by the galvanometer circuit $1G2$ and the portion $2a1$ of the contour L (Fig. 72). Finally, let us choose in a definite way the positive direction around these contours, for example as shown by the dash line in the figure. Applying Kirchhoff's second law to the contour $1G2a1$, on the basis of Eqs. (3.26) and (6.85) we can write that

$$I_0 R_0 - I_a R_a = \oint_{1G2a1} E_s ds = -\frac{1}{c} \frac{\partial \Psi_0}{\partial t}$$

(The minus sign has been used before I_a because in accordance with the figure, the positive direction around the contour is opposite to

the direction of the current I_a .) Similarly, for the contour L , we get

$$I_a R_a + I_b R_b = \oint_{la2bl} E_s ds = -\frac{1}{c} \frac{\partial \Psi}{\partial t}$$

Finally, applying Kirchhoff's first law to the circuit branch-points 1 and 2, we get

$$I_b = I_0 + I_a$$

Eliminating I_a and I_b from these equations, we have

$$\begin{aligned} I_0 \{R_0 (R_a + R_b) + R_a R_b\} &= \\ &= -\frac{1}{c} \left\{ R_a \frac{\partial \Psi}{\partial t} + (R_a + R_b) \frac{\partial \Psi_0}{\partial t} \right\} \end{aligned}$$

Thus, the readings I_0 of the galvanometer do indeed appreciably depend on the rate of change in the flux Ψ_0 through the contour $IG2aI$, the latter, in turn, depending on the arrangement of the galvanometer circuit.

4. Let us assume, for example, that we may disregard the proper magnetic field of the currents I_a , I_b , and I_0 in comparison with the "external" field of an alternating current I' having a preset intensity and period that is circulating through the solenoid AB (Fig. 73) encircled by the contour L .

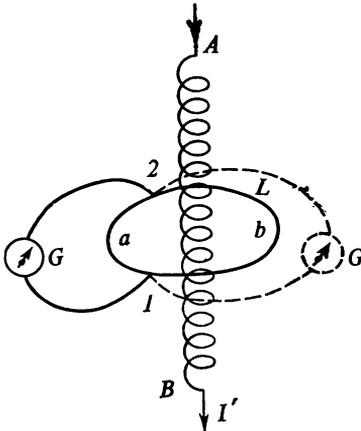


Fig. 73

Let us assume, further, for simplification that the magnetic field of the solenoid can with sufficient accuracy be considered as concentrated inside the solenoid, i.e. that outside of the solenoid $\mathbf{B} = \mathbf{H} = 0$ (see the example in Sec. 4.8). Let, finally, $R_a = R_b$. For these conditions, it is a simple matter to show (we shall let our reader do this

as an exercise) that in the two different positions of the galvanometer circuit shown in the figure by a solid and a dash lines, respectively, the current in the galvanometer circuit $IG2$ will have the same value, but an *opposite direction* (for instance $IG2$ in the first and $2G1$ in the second case). Hence, if we use the term "voltage" between the points 1 and 2 without indicating the integration path, then we must say that the transfer of the galvanometer circuit from the left to the right involves a change in the sign of the voltage \mathcal{E}_{12} applied to the terminal points 1 and 2 of this circuit.

5. It is sometimes convenient to use the concept of voltage, however, for alternating currents too in certain conditions and when care is taken.

Let us consider as an example a very simple scheme of the circuit of a central alternating current electric power plant consisting of two almost closed contours I and II whose ends are connected by means of two conductors ac and bd in close proximity to each other

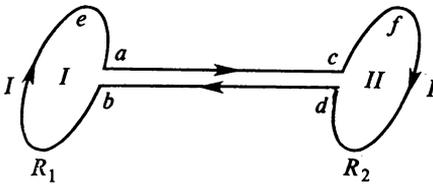


Fig. 74

(Fig. 74). Circuit I includes the generator installation of the power plant, and circuit II the current consumers.

Disregarding the induction flux through the strip confined between conductors ac and bd , we can consider that the induction flux Ψ through the contour of the entire circuit equals the sum of the fluxes Ψ_1 and Ψ_2 through the loops I and II . When calculating the fluxes Ψ_1 and Ψ_2 , we shall agree to consider these loops completed to a closed state by lengths ab and cd . In this case, according to Eq. (6.14), we have

$$I(R_1 + R_2) = \mathcal{E}_{\text{ext}, 1} + \mathcal{E}_{\text{ext}, 2} - \frac{1}{c} \frac{d\Psi_1}{dt} - \frac{1}{c} \frac{d\Psi_2}{dt}$$

where

R_1 = total resistance of the circuit I up to the points c and d

R_2 = resistance of the circuit II

$\mathcal{E}_{\text{ext}, 1}$ and $\mathcal{E}_{\text{ext}, 2}$ = extraneous e.m.f.'s in these circuits.

Let us write the last equation in the form

$$IR_2 - \mathcal{E}_{\text{ext}, 2} + \frac{1}{c} \frac{d\Psi_2}{dt} = - \left(IR_1 - \mathcal{E}_{\text{ext}, 1} + \frac{1}{c} \frac{d\Psi_1}{dt} \right) = \mathcal{E}_{\text{cd}} \quad (6.88)$$

where \mathcal{E}_{cd} stands for the value of each of the terms of the equation.

If loops *I* and *II* are sufficiently far from each other, then their mutual induction may be ignored, and we can assume that

$$\Psi_1 = \frac{1}{c} L_{11} I \text{ and } \Psi_2 = \frac{1}{c} L_{22} I$$

where L_{11} and L_{22} are the self-inductances of the loops *I* and *II*. It is significant that to find the value of \mathcal{E}_{cd} in this case it is enough to know the values of the quantities I , R_1 , $\mathcal{E}_{\text{ext}, 1}$, and L_{11} relating only to the “generating” section *I*. Conversely, if \mathcal{E}_{cd} is known, then the current in the circuit can be determined depending on the value of the quantities R_2 , $\mathcal{E}_{\text{ext}, 2}$, and L_{22} relating only to the “consuming” section *II*. Here \mathcal{E}_{cd} plays the part of an additional extraneous e.m.f. applied to the almost closed contour *II*.

It is exactly this quantity \mathcal{E}_{cd} that is called the *voltage applied to the “consuming” portion of the circuit II and induced by the “generating” portion of the circuit I*.

6. Let us apply Ohm’s law to section *II*:

$$IR_2 = \mathcal{E}_{\text{ext}, 2} + \int_{cfd} E_s ds$$

Since, on the other hand,

$$- \frac{1}{c} \frac{d\Psi_2}{dt} = \oint_{cfdc} E_s ds$$

then

$$\begin{aligned} \mathcal{E}_{cd} = IR_2 - \mathcal{E}_{\text{ext}, 2} + \frac{1}{c} \frac{d\Psi_2}{dt} &= \int_{cfd} E_s ds - \\ &- \oint_{cfdc} E_s ds = \int_c^d E_s ds \end{aligned}$$

where the last integral, according to the above definition of the quantity Ψ_2 , must be taken over the shortest path connecting the points *c* and *d*. Thus, the *voltage* \mathcal{E}_{cd} equals the line integral of the electric field intensity **E** over this path:

$$\mathcal{E}_{cd} = \int_c^d E_s ds \tag{6.89}$$

and, consequently, *can be measured according to the current in a galvanometer connected between the points c and d*. The circuit of the galvanometer may deviate from the straight line *cd* only within such limits (quite broad in practice, however) that the deviation will have

no appreciable influence on the value of the integral $\int_c^d E_s ds$.

7. Assuming for simplicity that the current contour does not deform so that the self-inductances L_{11} and L_{22} of the loops I and II are constant in time, on the basis of the first of Eqs. (6.53) we get

$$\frac{1}{c} I \frac{d\Psi_1}{dt} = \frac{1}{c^2} I L_{11} \frac{dI}{dt} = \frac{d}{dt} \left(\frac{1}{2c^2} L_{11} I^2 \right) = \frac{dU_1}{dt}$$

where U_1 is the magnetic energy of the field of the current in the portion I of the circuit. A similar relationship holds for the portion II . Hence, multiplying Eq. (6.88) by I , we can write

$$\begin{aligned} \left(I^2 R_2 - I \mathcal{E}_{\text{ext}, 2} + \frac{dU_2}{dt} \right) &= \\ = - \left(I^2 R_1 - I \mathcal{E}_{\text{ext}, 1} + \frac{dU_1}{dt} \right) &= I \mathcal{E}_{\text{cd}} \end{aligned} \quad (6.90)$$

Since $I^2 R$ equals the heat liberated by the current, and $I \mathcal{E}_{\text{ext}}$ equals the work of the extraneous e.m.f.'s (in a unit time), then, consequently, $I \mathcal{E}_{\text{cd}}$ equals the total reduction in the energy of the section I of the circuit and at the same time equals the growth in the energy of the section II . In other words, $I \mathcal{E}_{\text{cd}}$ equals the energy transmitted by the generating section I to the consuming section II in a unit time.

Thus, provided that the wires ac and bd are sufficiently close to each other and the sections I and II are sufficiently remote from each other, it is indeed very good practice to introduce into consideration the quantity \mathcal{E}_{cd} determined by Eqs. (6.88) and (6.89). In engineering, it is called the *alternating current voltage*.

6.12 Equation of Continuity

1. Up to now, we assumed that varying or alternating currents, similar to steady ones, are *closed*. As we have seen in Sec. 3.3, a necessary condition for closure of the current lines is the requirement that at each point of a conductor Eq. (3.19) be satisfied, i.e.

$$\text{div } \mathbf{j} = 0$$

It follows from Eq. (3.19), in particular, that in an unbranched conductor the current at each given moment is identical in all its cross sections, and that Kirchhoff's first law (3.16) is complied with at the branch-points of the conductors.

Equation (3.19) and the conclusions following from it, however, cannot be applied to varying currents or, more exactly, may be applied with a certain approximation only to a definite class of varying currents (closed quasistationary currents, see below). In general,

varying currents can flow through open contours (a circuit with a capacitor having a dielectric between its plates, currents in an aerial, etc.), their intensity may differ in different cross sections of the conductor, etc. We remind our reader that Eq. (3.19) itself was obtained in Sec. 3.3 from the more general "continuity equation" (3.17):

$$\oint_S j_n dS = -\frac{\partial q}{\partial t}$$

(q is the charge in the volume V confined by the surface S) on the grounds that the distribution of the electric charges in the field of a steady current must remain constant. For varying currents, this condition, generally speaking, is not observed, and the equation of continuity (3.17) does not lead to Eq. (3.19).

2. For our further discussion, it will be convenient to transform the equation of continuity (3.17) as follows. If in the volume V enclosed by the surface S there are neither surface charges nor discontinuities of the current density \mathbf{j} (which may occur only on the interfaces between different media), then

$$q = \int_V \rho dV$$

where ρ is the volume density of the electric charges, and

$$\oint_S j_n dS = \int_V \operatorname{div} \mathbf{j} dV$$

[according to Gauss's theorem, Eq. (A.17)]. Consequently, Eq. (3.17) becomes

$$\int_V \operatorname{div} \mathbf{j} dV = -\frac{\partial}{\partial t} \int_V \rho dV = -\int_V \frac{\partial \rho}{\partial t} dV$$

where a change in the sequence of integration over V and differentiation with respect to t is possible provided that the volume V we are considering is stationary (this is why the sign of the *partial* derivative with respect to time has been used here, see p. 424). Owing to the arbitrary nature of the volume V , it follows from the last equation that

$$\operatorname{div} \mathbf{j} = -\frac{\partial \rho}{\partial t} \quad (6.91)$$

3. It is exactly Eq. (6.91) that is the differential form of the *equation of continuity*. On surfaces of discontinuity of the vector \mathbf{j} , it must naturally be replaced with the equation

$$\operatorname{div} \mathbf{j} = j_{2n} - j_{1n} = - \frac{\partial \sigma}{\partial t} \quad (6.92)$$

which can be seen either directly from Eq. (3.17) or on the basis of Eq. (1.30). Particularly, on the interface between a conductor and a non-conducting medium, we have $j_{2n} = 0$ (if the normal \mathbf{n} is an outward one), and, consequently, the following equation holds:

$$j_n = \frac{\partial \sigma}{\partial t} \quad (6.93)$$

It states that the quantity of electricity j_n flowing in a unit time to a unit surface of a conductor equals the increment of the charge σ of this section of surface (in a unit time).

Equations (6.91) and (6.92) make it possible, knowing the density of currents, to determine the change in charge distribution due to these currents.

6.13 Displacement Currents

1. We shall now give our attention to the exceedingly significant circumstance that when $\partial \rho / \partial t \neq 0$ equation of continuity (6.91) is *incompatible* with the equation for the magnetic field of currents (5.24):

$$\operatorname{curl} \mathbf{H} = \frac{4\pi \mathbf{j}}{c}$$

Indeed, according to the last equation, \mathbf{j} is proportional to the curl of the vector \mathbf{H} . But the divergence of a curl always equals zero [Eq. (A.42₂)], whereas according to Eq. (6.91), $\operatorname{div} \mathbf{j}$, generally speaking, does not equal zero.

Similarly, Eq. (4.40)

$$\oint_L H_s ds = \frac{4\pi I}{c} = \frac{4\pi}{c} \int_S j_n dS$$

which is equivalent to Eqs. (5.24) or (4.38), cannot be applied to varying currents if only for the reason that the intensity of *unclosed* currents flowing through the surface S resting on the contour L depends essentially not only on the contour L , but also on the shape and arrangement of the surface S . Particularly, with unclosed currents,

it is quite often possible to conduct the surface S so that it will not intersect the conductors carrying a current at all (Fig. 75, showing a varying current circuit including a capacitor C).

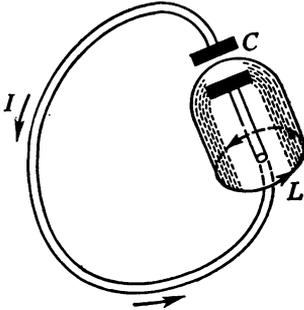


Fig. 75

Thus, the equations for the magnetic field of steady currents obtained in Chapter 4, generally speaking, cannot be applied to varying currents and need to be modified and supplemented.

2. Proceeding from the correctness of the equation of continuity (6.91), we can try, using purely formal mathematical reasoning, to determine the simplest kind of correction whose introduction into Eq. (5.24) will eliminate the above contradiction between it and Eq. (6.91).

Up to now, we proceeded from the notion that electric currents are the transfer of electric charges through conductors and that their density, according to Eq. (3.23) is determined by the conductivity of the conductor κ :

$$\mathbf{j} = \kappa (\mathbf{E} + \mathbf{E}_{ext})$$

From now on, we shall call these currents *conduction currents* and assume that apart from them* there may also be currents of a somewhat different kind which we shall call *displacement currents* (the meaning of this term will be explained on a following page). The density of the total current \mathbf{j}_{tot} will therefore equal the sum of the densities of the conduction current \mathbf{j} and the displacement current \mathbf{j}^{dis} :

$$\mathbf{j}_{tot} = \mathbf{j} + \mathbf{j}^{dis} \tag{6.94}$$

We shall assume here that the true dependence of the vector \mathbf{H} on \mathbf{j} differs from Eq. (5.24) only in the respect that curl \mathbf{H} is proportional

* And apart from the molecular currents in magnetics. The molecular currents are not included among the macroscopic currents which we shall treat here, but are taken into consideration by the magnetization vector \mathbf{M} [see Eqs. (5.13) and (5.14)] which, in turn, is unambiguously determined by the values of the vectors \mathbf{B} and \mathbf{H} [see Eq. (5.26)].

to the density of the *total* current \mathbf{j}_{tot} instead of to \mathbf{j} :

$$\text{curl } \mathbf{H} = \frac{4\pi}{c} \mathbf{j}_{\text{tot}} = \frac{4\pi}{c} (\mathbf{j} + \mathbf{j}_{\text{dis}}) \quad (6.95)$$

In other words, we shall assume that in the magnetic respect *the displacement currents* are equivalent to the conduction currents, i.e. *induce a magnetic field according to the same laws as the conduction currents do.*

Taking the divergence of both sides of Eq. (6.94), on the basis of Eq. (A.42₂) we get

$$\text{div } \mathbf{j}_{\text{tot}} = \text{div } \mathbf{j} + \text{div } \mathbf{j}_{\text{dis}} = 0 \quad (6.96)$$

Thus, the field of the total current must be deprived of sources, i.e. *the lines of a total current* cannot begin nor terminate anywhere and *must be closed or pass to infinity* (or, finally, must tightly fill a certain surface everywhere, see Sec. 4.12). Consequently, at the place where conduction current lines terminate, the lines of the displacement current continuing them must directly join them.

Further, from Eqs. (6.96) and (6.91), it follows that

$$\text{div } \mathbf{j}_{\text{dis}} = - \text{div } \mathbf{j} = \frac{\partial \rho}{\partial t}$$

But by Eq. (2.18) we have

$$\text{div } \mathbf{D} = 4\pi\rho$$

Hence,

$$\text{div } \mathbf{j}_{\text{dis}} = \frac{1}{4\pi} \frac{\partial}{\partial t} \text{div } \mathbf{D} = \text{div} \left(\frac{1}{4\pi} \frac{\partial \mathbf{D}}{\partial t} \right)$$

3. The simplest, although, naturally, not the only way to satisfy this equation is to assume that

$$\mathbf{j}_{\text{dis}} = \frac{1}{4\pi} \frac{\partial \mathbf{D}}{\partial t} \quad (6.97)$$

i.e. to assume that the density of the displacement current at every point of the field is proportional to the rate of change of the electric displacement vector \mathbf{D} . According to this definition, a displacement current may be encountered not only in conductors, but also in dielectrics and even in a vacuum. In a stationary field, however, this current will always vanish. Equation (6.95) will acquire the form

$$\text{curl } \mathbf{H} = \frac{4\pi}{c} \mathbf{j} + \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} = \frac{4\pi}{c} \kappa (\mathbf{E} + \mathbf{E}_{\text{ext}}) + \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} \quad (6.98)$$

When $\partial \mathbf{D} / \partial t = 0$, Eq. (6.98) coincides with Eq. (5.24), so that all the results we have obtained previously from Eqs. (5.24) and (4.38) hold for stationary fields.

Thus, by introducing a hypothesis of the existence of displacement currents in a purely formal way, we can eliminate the contradiction between Eqs. (6.91) and (5.24) without introducing any modification into the laws of a stationary electric field.

Experiments completely confirm the correctness of both this hypothesis and of Eq. (6.98), which is one of the fundamental equations of electrodynamics. We shall see below that the very fact of the propagation of electromagnetic waves is the most convincing proof of this equation.

4. To understand the physical meaning of Eqs. (6.97) and (6.98), let us consider the case when the total current consists only of a displacement current, i.e. when the conduction current is absent (a non-conducting medium, $\kappa = 0$).

In this case, Eq. (6.98) becomes

$$\operatorname{curl} \mathbf{H} = \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} \quad (6.99)$$

This equation is quite similar to Eq. (6.86):

$$\operatorname{curl} \mathbf{E} = - \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}$$

It follows from Eq. (6.99) that in the same way as an electric field can be induced not only by electric charges, but also by changes in a magnetic field (i.e. changes in the magnetic induction vector \mathbf{B}), so can a magnetic field in turn be induced not only by movements of charges (conduction currents), but also by changes in an electric field (the electric displacement vector \mathbf{D})*.

On the basis of this analogy between Eqs. (6.86) and (6.99), the quantity $\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}$ could be called the density of the *magnetic displacement current*.

It must be noted that the right-hand sides of Eqs. (6.99) and (6.86) have different signs. This difference is due to the fact that the lines of force of the magnetic field \mathbf{H} induced by the displacement currents $\mathbf{j}_{\text{dis}} = \frac{1}{4\pi} \frac{\partial \mathbf{D}}{\partial t}$ form a right-handed system with the direction of these currents**, whereas the directions of the vectors \mathbf{E} and $\partial \mathbf{B} / \partial t$ are in the inverse relationship (Fig. 76, cf. Fig. 70 on p. 386).

* Naturally, a change in the electric field *in the long run* is due to the same movement of the charges. Moreover, from the electron viewpoint, a part (*but only a part*) of the density of the displacement currents in dielectrics (but not in a vacuum) consists *directly* in the motion of "bound" charges of the medium (see below).

** This follows from the equivalence of displacement currents to conduction currents and from the results of Sec. 4.12.

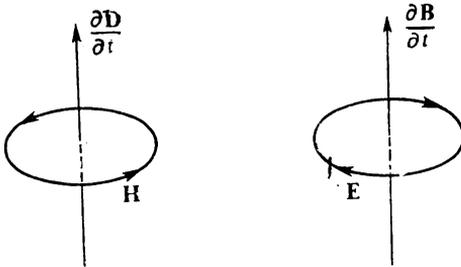


Fig. 76

5. We shall note in conclusion that from the modern viewpoint* *conduction currents* on one hand and *displacement currents* in a vacuum on the other, notwithstanding the similarity of their names, are in essence *absolutely different physical concepts*. Their *only* common feature is that they induce a magnetic field in an identical manner, i.e. enter the right-hand side of Eq. (6.98) in the same way. In all other respects, these currents sharply differ from each other.

The most significant distinction is that conduction currents correspond to the motion of electric charges whereas a “pure” displacement current—a displacement current in a vacuum—corresponds only to a change in the intensity of the electric field and is not attended by any motion of electric charges or other particles of a substance. Indeed, in a vacuum, $\mathbf{D} = \mathbf{E}$, and Eq. (6.97) becomes

$$\mathbf{j}_{\text{dis}} = \frac{1}{4\pi} \frac{\partial \mathbf{E}}{\partial t}$$

In the presence of a dielectric, Eq. (6.97) on the basis of Eqs. (2.6) and (2.17) becomes

$$\begin{aligned} \mathbf{j}_{\text{dis}} &= \frac{1}{4\pi} \frac{\partial \mathbf{D}}{\partial t} = \frac{1}{4\pi} \left(\frac{\partial \mathbf{E}}{\partial t} + 4\pi \frac{\partial \mathbf{P}}{\partial t} \right) = \\ &= \frac{1}{4\pi} \frac{\partial \mathbf{E}}{\partial t} + \sum q_i \frac{\partial \mathbf{R}_i}{\partial t} \end{aligned} \quad (6.100)$$

Thus, the displacement current in a dielectric consists of the “pure” displacement current $\frac{1}{4\pi} \frac{\partial \mathbf{E}}{\partial t}$ not connected with the motion of charges and of the term $\sum q_i \frac{\partial \mathbf{R}_i}{\partial t} = \sum q_i \mathbf{v}_i$ taking into account the motion of charges connected with the molecules of the dielectric. From the microscopic viewpoint, the latter term, in essence, is a part of the conduction current [cf., for example, Eq. (3.39)]; in the last of the above equations \mathbf{v}_i stands for the velocity of the charge q_i .

* Unlike the original notions of J. Maxwell who was the first to establish the existence of displacement currents and who gave them this name.

We shall also mention that *displacement currents*, unlike their conduction counterparts, *are not attended by the liberation of Joule heat*. This is self-obvious for displacement currents in a vacuum. We shall prove in Sec. 7.2 that this statement also strictly holds when applied to displacement currents in dielectrics whose permittivity does not depend on the temperature (quasi-elastic dipoles). As regards dielectrics with constant dipoles, then as we mentioned in Sec. 2.12, a change in the polarization of dielectrics of this class is attended by a certain liberation or absorption of heat. Hence, the displacement currents in them are also attended by heat effects. In fast-varying fields, this causes the liberation of appreciable quantities of heat in the dielectric. This phenomenon, however, obeys absolutely different laws than the liberation of Joule heat in conductors.

6. Let us now consider the extent to which the existence of displacement currents violates the correctness of the conclusions which we arrived at in the preceding sections of the present chapter. It must be noted, first of all, that inside of conductors, particularly inside of metals, the density of displacement currents is usually so small in comparison with that of conduction currents,

$$j_{\text{dis}} \ll j$$

that without any harm for the accuracy of our calculations it may in general be disregarded.

Assume, for example, that we have to do with a periodic current having the frequency ν so that

$$E = E_0 \sin 2\pi\nu t \quad \text{and} \quad j_{\text{dis}} = \frac{\epsilon}{4\pi} \frac{\partial E}{\partial t} = \frac{\epsilon\nu}{2} E_0 \cos 2\pi\nu t$$

In this case the condition $j_{\text{dis}} \ll j = \kappa E$ is equivalent to the requirement

$$\epsilon\nu \ll \kappa \tag{6.101}$$

For mercury, $\kappa = 9.57 \times 10^{15}$ abs. units (s^{-1}), the conductivity of pure solid metals is from 10 to 50 times greater: for them $\kappa \approx 10^{17}$ abs. units.

Because of such a great value of κ , the condition (6.101) is observed in metals for all the frequencies employed in engineering (including radio engineering), and is violated only in the region of frequencies corresponding to the infrared part of the spectrum. At these frequencies, however, the dependence of the material constants ϵ and κ on the frequency of the field (dispersion) begins to play an appreciable

part, so that Maxwell's equations, which do not take into account the possibility of such dispersion, stop being applicable*.

Of an absolutely different nature is the question of whether it is possible to ignore the displacement currents that appear in the non-conducting medium surrounding a conductor in comparison with the conduction currents. In the majority of technically interesting cases, this is permissible provided that the conductors form a closed circuit. If, however, there are discontinuities in a circuit conducting a current, for instance if it includes a capacitor (see Fig. 75), then it is impossible to disregard the displacement current in the space between the capacitor plates. Indeed, the total current is always deprived of sources and sinks [Eq. (6.96)], therefore the total displacement current flowing through the capacitor equals the current in the wires leading to it. We must note, however, that for a capacitor connected to a circuit, the electric field of the current and, consequently, the displacement current are concentrated between the capacitor plates. Hence, with a certain approximation, a circuit with a capacitor can be considered similar to a *closed conducting circuit*. In other words, the self-inductance of the circuit, the magnetic field of the current, etc. can be calculated in the same way as if a conduction current circulates between the plates of the capacitor that has the same intensity as the current in the wires leading to it (because from a magnetic stand-point a displacement current is equivalent to a conduction current). Thus, the presence of a capacitor in the circuit of a varying (alternating) current tells directly only on the electric, but not on the magnetic field of the current (see Sec. 6.14).

Matters are different if we have an absolutely open conducting circuit, for example a straight conductor of a finite length (an aerial). In this case, the displacement currents will be distributed over the entire space surrounding the conductor, and we cannot speak of a closed circuit. Finally, with sufficiently rapid electric oscillations and a sufficient length of the conductor (comparable with the length of an electric wave), the conduction current may differ in various cross sections of the conductor (even if the conductor is closed and unbranched). If, notwithstanding the above, in certain conditions it does become possible to disregard the field of the displacement currents when determining the magnetic field, nevertheless such conduction currents cannot be considered as similar to steady currents and do not comply with the conditions for quasistationary currents.

* It must be noted that the concept of the permittivity of metals can be defined only as applied to varying fields because in static fields the polarization of the conductors, if it does occur, is completely veiled by conduction phenomena (currents). The values of ϵ for metals in fast-varying fields can be determined by studying the reflection and refraction of electromagnetic waves in metals. For example, we shall show in Sec. 7.12 that the coefficient of reflection of waves from a metal surface depends, particularly, on the permittivity of the metal [cf. Eqs. (7.110), (7.111), and (7.116)].

At any rate, a sufficiently slow change of the field is a necessary condition to ignore the displacement currents (which are proportional to the rate of field changes) and consider conduction currents to be closed.

6.14 A Capacitor in the Circuit of a Quasistationary Current. Electric Oscillations

1. Up to now, we considered only quasistationary currents in closed conducting contours; now we shall consider in somewhat greater detail a circuit containing a capacitor as an example.

Let C be the capacitance of the capacitor, R the resistance of the circuit connecting its plates, and L the self-inductance of the closed current circuit that is obtained from the given one by short-circuiting the capacitor plates with a piece of wire placed between them*.

Let us assume that the current I is identical in all the cross sections of the circuit (in the conductors it is a conduction current, between the capacitor plates a displacement current) and choose a definite positive direction of the current (the arrow in Fig. 77). Let q_1 and q_2 be the charges of the capacitor plates, and $q_1 = -q_2$. We shall assume that the capacitance of the capacitor is so much greater than that of the remaining circuit that the latter may in general be disregarded, and that the total energy of the electric field is virtually equal to the energy of the portion of the field confined between the capacitor plates. If the field corresponds to quasistationary conditions, then at each given moment the value of this energy should depend in the

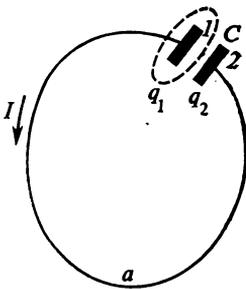


Fig. 77

same way on the instantaneous value of the capacitor charge as in the static case. Matters will be similar for the dependence of the

* If the distance between these plates is sufficiently small, then within certain limits the cross section of the wire closing them will not affect the value of the self-inductance L .

magnetic energy on the current in the circuit. Hence, the total electromagnetic energy of the system should be [Eqs. (1.104) and (6.26)]

$$U = \frac{1}{2} \frac{q_1^2}{C} + \frac{1}{2c^2} LI^2 \quad (6.102)$$

To obtain a differential equation determining the current upon the discharge of the capacitor, we shall take into account that the quantity of Joule heat liberated in the circuit must equal the reduction in the electromagnetic energy of the system:

$$I^2 R = - \frac{dU}{dt} = - \left(\frac{q_1}{C} \frac{dq_1}{dt} + \frac{1}{c^2} LI \frac{dI}{dt} \right) \quad (6.103)$$

On the other hand, by applying the continuity equation (3.17) to the closed surface enveloping one of the capacitor plates (the dash line in Fig. 77), it is easy to see that with the positive direction of the current that we have chosen the value of this current I should equal the decrease in the charge q_1 of the first capacitor plate and the growth in the charge q_2 of its second plate:

$$I = - \frac{dq_1}{dt} = \frac{dq_2}{dt} \quad (6.104)$$

Using this equation in Eq. (6.103) and then cancelling I , we get

$$RI = \frac{q_1}{C} - \frac{1}{c^2} L \frac{dI}{dt} \quad (6.105)$$

Differentiating this equation with respect to t and again using Eq. (6.104), we finally get

$$\frac{1}{c^2} L \frac{d^2 I}{dt^2} + R \frac{dI}{dt} + \frac{I}{C} = 0 \quad (6.106)$$

If we express the values of the electromagnetic quantities in practical units instead of absolute ones, the constant $1/c^2$ will vanish (cf. Sec. 6.5), and we get

$$L' \frac{d^2 I'}{dt^2} + R' \frac{dI'}{dt} + \frac{I'}{C} = 0 \quad (6.107)$$

2. Equation (6.107) has the well-known form of the equation of damped harmonic oscillations. Its general solution is expressed by the formula

$$I' = Ae^{k_1 t} + Be^{k_2 t}$$

where A and B are arbitrary integration constants, and k_1 and k_2 are the roots of the quadratic equation

$$L'k^2 + R'k + \frac{1}{C} = 0$$

i.e.

$$k = -\frac{R'}{2L'} \pm \sqrt{\left(\frac{R'}{2L'}\right)^2 - \frac{1}{L'C'}}$$

If the radicand is not negative, then both roots k_1 and k_2 are real and *negative* (because the absolute value of a root is less than the value of the first term), so that the discharge of the capacitor will be aperiodic. If the radicand is negative, then k_1 and k_2 are complex numbers. Introducing the notation

$$\omega_0^2 = \frac{1}{L'C'} - \left(\frac{R'}{2L'}\right)^2 \tag{6.108}$$

we get in this case

$$k = \frac{R'}{2L'} \pm i\omega_0$$

and

$$I' = A \exp(k_1 t) + B \exp(k_2 t) = \exp\left(-\frac{R'}{2L'} t\right) \{A \exp(i\omega_0 t) + B \exp(-i\omega_0 t)\}$$

Expressing $\exp(\pm i\omega_0 t)$ in trigonometric functions, it is easy to reduce the real part of this expression to the form

$$I' = a \exp\left(-\frac{R'}{2L'} t\right) \sin(\omega_0 t - \varphi)$$

where a and φ are new constants whose values depend on A and B . Hence, in this case, a damped oscillatory current having the cyclic frequency ω_0 flows through the circuit. The period of oscillations of this current, i.e. the time interval between consecutive moments of passage of the current intensity through zero in the *same* direction (i.e., for example, from negative values of I' to positive ones), equals

$$T = \frac{2\pi}{\omega_0}$$

If we have

$$R'^2 \ll \frac{4L'}{C'}$$

we may disregard the second term of Eq. (6.108) so that the last equation is transformed into the well-known Thomson formula

$$T = 2\pi \sqrt{L'C'} \tag{6.109}$$

The rate of attenuation of the current is determined by the value of the ratio $R'/2L'$: during one complete period the current diminishes $\exp\left(\frac{R'}{2L'} T\right)$ times. The logarithm of this factor

$$\delta = \frac{R'}{2L'} T = \frac{R'\pi}{L'\omega_0} \quad (6.110)$$

is called the *logarithmic decrement* (or the *damping decrement*).

3. Electric oscillations in a circuit with a capacitor are quite similar to the vibrations of a body retained by elastic forces near the position of equilibrium. In charging a capacitor or in bringing a body out of its position of equilibrium, we impart a certain store of potential energy to the system (electric or elastic energy). Upon discharge of our capacitor and upon vibrations of the body, this potential energy transforms respectively into the "electrokinetic" energy of the magnetic field and the kinetic energy of the moving body. When the charge of the capacitor becomes equal to zero, however, (the body passes through its equilibrium position), the current does not immediately vanish (the body does not stop), but owing to its electromagnetic inertia, as it were (a measure of which is the self-inductance, cf. p. 397), it continues to flow in the previous direction for a certain time. As a result, the capacitor is recharged (the body moves away from its equilibrium position), after which the process of discharging repeats in the opposite direction. The resistance of the circuit, which the attenuation of the electric oscillations is due to, is naturally quite similar to friction.

The science of electric oscillations plays an exceedingly important part in a considerable number of branches of electrical engineering and in this connection has developed to such an extent that it is, in essence, a special branch of the science of electricity. Not having the possibility to treat these matters in detail, we are forced to refer our reader to special publications.

4. In conclusion, we shall show that Eq. (6.106), which we have derived on the basis of energy considerations, can also be obtained from Ohm's law [Eq. (3.4)]. Applying Eq. (3.4) to the conductor $1a2$ connecting the capacitor plates (Fig. 77) and through which the conduction current I flows, we get

$$IR = \int_{1a2} E_s ds.$$

But

$$\int_{1a2} E_s ds = \oint_{1a2b1} E_s ds - \int_{2b1} E_s ds = \mathcal{E}_{\text{ind}} + \int_{1b2} E_s ds$$

because the circulation of the electric vector around a closed contour equals the induced e.m.f. Here $2b1$ signifies the length that supplements the contour $1a2$ to a closed state. As regards the last integral, in an *electrostatic field* it would equal the difference of potentials of the capacitor plates $\varphi_1 - \varphi_2$. In a varying field, as we have seen in Sec. 6.11, the concept of potential loses its meaning because the vortex field of induction is added to the potential Coulomb field of the electric charges.

The assumption of the quasistationary nature of the field of the current circulating in a capacitor circuit, however, includes in particular the assumption that the electric field near the capacitor is in the same relationship with the instantaneous value of its charge as in the static case. In other words, it is assumed that at least *between the plates* of a capacitor the intensity of the vortex field induced by the changes in the magnetic field is vanishingly small in comparison with the intensity of the Coulomb field of the capacitor charges. On the other hand, it follows from the meaning of the last equation that the integral $\int_{1b2} E_s dS$ in it must be taken over the shortest path

connecting the capacitor plates, i.e. over the path *between* the plates. Consequently, with a quasistationary current, this integral may be assumed to equal the potential difference $\varphi_1 - \varphi_2$ that would exist between the capacitor plates in the static case. Hence,

$$IR = \mathcal{E}_{\text{ind}} + \varphi_1 - \varphi_2 \tag{6.111}$$

Expressing $\varphi_1 - \varphi_2$ in this equation through the charge and the capacitance of the capacitor [Eq. (1.52)], and \mathcal{E}_{ind} through L and dI/dt , we again arrive at Eq. (6.106).

It should be noted that Eq. (6.111) is quite similar to Ohm's law for steady currents [Eq. (3.27)], \mathcal{E}_{ind} playing the part of the "extraneous" e.m.f.'s.

Problem 36. A given alternating voltage $\mathcal{E}'_{cd} = \mathcal{E}'_0 \cos \omega t$ (we are using the practical system of units) is applied from outside to the ends c and d of an almost closed "consuming" portion of the circuit II (Fig. 74, p. 430). There are no extraneous e.m.f.'s in this portion of the circuit, but a capacitor having the capacitance C' is connected to it. Show that the current I' in the circuit is

$$I' = I'_0 \cos(\omega t - \varphi)$$

the amplitude of the current I'_0 being

$$I'_0 = \frac{\mathcal{E}'_0}{\sqrt{R^2 + \left(\omega L' - \frac{1}{\omega C'}\right)^2}}$$

while the phase shift of the current with respect to the voltage is determined by the relationship

$$\tan \varphi = \frac{\omega L' - \frac{1}{\omega C'}}{R'}$$

Note 1. This relationship also holds for a closed circuit, the current in which is induced by periodic changes in the magnetic induction flux of the external magnetic field. To pass over to this case, it is sufficient to replace \mathcal{E}_{cd} with $\mathcal{E}'_{ext, e}$:

$$\mathcal{E}'_{ext, e} = - \frac{d\psi'_e}{dt}$$

Note 2. It follows from the expression for I'_0 that for a given value of \mathcal{E}'_0 the amplitude of the current reaches a maximum when $\omega L' = 1/\omega C'$, i.e. when

$$\omega^2 = \frac{1}{L'C'}$$

If the logarithmic decrement of the natural oscillations of the circuit is small, then this condition corresponds to the approximate equality of the (cyclic) frequency ω of the external exciting forces and the frequency of the natural oscillations of the circuit ω_0 [Eqs. (6.108) and (6.109)]. Thus, a circuit containing a capacitance and inductance, like any system with a natural period of oscillations, displays resonance when it is excited by external periodic forces. It is known that such *resonance* phenomena are widely used in electrical engineering in general and in radio engineering in particular.

6.15 The Skin Effect

We did not at all consider above the distribution of varying currents over the cross section of conductors. This question, however, is important not only from the theoretical viewpoint, but also from an engineering one. We shall now show that even in a homogeneous quasilinear conductor (Sec. 3.4) a varying current, unlike a steady one, is not distributed uniformly over the cross section of the conductor, but is concentrated on its surface*. This phenomenon, which has been called the *skin effect* (the current is concentrated on the "skin" of the conductor), leads, in turn, to a change in the effective resistance and self-inductance of the conductor.

* The reverse phenomenon, i.e. axial concentration of the current, is sometimes possible; see Z. Libin, *Zh. prikl. fiz.*, 4, vyp. 3, p. 45 (1927).

1. In treating the theory of the skin effect of conductors, we shall proceed from the basic equations of an electromagnetic field (6.86) and (6.98):

$$\operatorname{curl} \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = -\frac{\mu}{c} \frac{\partial \mathbf{H}}{\partial t}$$

$$\operatorname{curl} \mathbf{H} = \frac{4\pi\kappa}{c} \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t}$$

(we assume that $\mathbf{E}_{\text{ext}} = 0$ and that the permeability μ is independent of time).

As we have indicated in Sec. 6.13, the density of the displacement currents in conductors or at least in metals is vanishingly small in comparison with the density of the conduction currents. Therefore, in the last equation we may ignore the second term and assume [cf. Eq. (4.38)] that

$$\operatorname{curl} \mathbf{H} = \frac{4\pi\kappa}{c} \mathbf{E}$$

Let us consider a homogeneous conductor over whose length the quantities ε , κ and μ are constant. In this case, forming a curl of Eq. (6.86) and introducing the above value of $\operatorname{curl} \mathbf{H}$ into it, we get

$$\operatorname{curl} \operatorname{curl} \mathbf{E} = -\frac{\mu}{c} \frac{\partial}{\partial t} (\operatorname{curl} \mathbf{H}) = -\frac{4\pi\mu\kappa}{c^2} \frac{\partial \mathbf{E}}{\partial t}$$

On the other hand, on the basis of Eq. (A.42₃),

$$\operatorname{curl} \operatorname{curl} \mathbf{E} = \operatorname{grad} \operatorname{div} \mathbf{E} - \nabla^2 \mathbf{E}$$

and if there are no space charges inside the conductor ($\rho = 0$), then owing to the constancy of ε it follows from Eq. (2.20) that

$$\operatorname{div} \mathbf{D} = \operatorname{div} \varepsilon \mathbf{E} = \varepsilon \operatorname{div} \mathbf{E} = 0$$

Hence, the differential equation for an electric field inside a homogeneous conductor can be written as follows:

$$\nabla^2 \mathbf{E} = \frac{4\pi\mu\kappa}{c^2} \frac{\partial \mathbf{E}}{\partial t} \tag{6.112}$$

In a similar way, it is easy to obtain an equation like this for the magnetic vector \mathbf{H} :

$$\nabla^2 \mathbf{H} = \frac{4\pi\mu\kappa}{c^2} \frac{\partial \mathbf{H}}{\partial t} \tag{6.113}$$

2. We shall limit ourselves to a consideration of varying fields whose intensity is a *sinusoidal* function of time, and shall express their intensity in the *complex* form (cf. Sec. 6.5, p. 398):

$$\mathbf{E} = \mathbf{E}_0(x, y, z) e^{i\omega t}, \quad \mathbf{H} = \mathbf{H}_0(x, y, z) e^{i\omega t} \quad (6.114)$$

where the amplitudes \mathbf{E}_0 and \mathbf{H}_0 may be complex vectors*, but do not depend on the time.

Inserting Eq. (6.114) into Eq. (6.112) and then cancelling the time factor $e^{i\omega t}$, we get the following equation for the amplitude of the electric vector:

$$\nabla^2 \mathbf{E}_0 = \frac{4\pi\mu\kappa\omega}{c^2} \mathbf{E}_0 = 2ip^2 \mathbf{E}_0 \quad (6.115)$$

where we have introduced the notation

$$p^2 = \frac{2\pi\mu\kappa\omega}{c^2} \quad (6.116)$$

3. Let us first consider the following very simple case. Let an infinite homogeneous conductor occupy the half-space $z > 0$ so that its surface coincides with the plane $z = 0$. We shall assume that the electric field and, consequently, the current are directed along the x -axis parallel to the boundary surface ($E_y = E_z = 0$) (Fig. 78). Let us further assume that the field intensity depends only on the distance z to the point of the conductor being considered from its surface, but does not depend on x and y .

For these conditions, Eq. (6.115) of the field inside the conductor (i.e. with $z > 0$) acquires the form

$$\nabla^2 E_{0x} = \frac{\partial^2 E_{0x}}{\partial z^2} = 2ip^2 E_{0x}$$

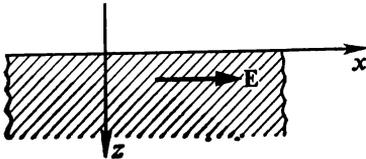


Fig. 78

The general solution of this linear equation is known to have the form

$$E_{0x} = Ae^{kz} + Be^{-kz}$$

* Do not confuse complex vectors with complex numbers depicted by vectors on a plane. Any complex vector \mathbf{a} can be represented in the form of the sum $\mathbf{b} + i\mathbf{c}$, where $i = \sqrt{-1}$, while \mathbf{b} and \mathbf{c} are ordinary real vectors in space.

where A and B are integration constants, and k is the root of the equation

$$k^2 = 2ip^2$$

i.e.

$$k = p\sqrt{2i} = p(1 + i)$$

Thus,

$$E_{0x} = Ae^{pz} e^{ipt} + Be^{-pz} e^{-ipt}$$

where by Eq. (6.116) p is a real quantity. Since the field in the conductor is bounded by a thin surface layer, the integration constant A should be assumed equal to zero. Otherwise when z grows, i.e. upon penetrating from the surface into the conductor, E_{0x} would increase to infinity. Passing over, in addition, from the amplitude of the electric vector to its complete complex expression, we get

$$E_x = E_{0x} e^{i\omega t} = Be^{-pz} e^{i(\omega t - pz)} \tag{6.117}$$

Discarding the imaginary part, we finally have

$$E_x = Be^{-pz} \cos(\omega t - pz) \tag{6.118}$$

The current density, accordingly, is expressed by the equation

$$j_x = \kappa E_x = j_0 e^{-pz} \cos(\omega t - pz) \tag{6.119}$$

where $j_0 = \kappa B$ stands for the amplitude of the current density on the surface of the conductor.

Thus, as we penetrate into the conductor, the phase of the electric vector and the current density changes linearly, while their amplitudes Be^{-pz} and κBe^{-pz} diminish according to an exponential law. The main part of the current can be considered concentrated in a surface layer having a thickness of $1/p$ cm because at this depth the current density is already $1/e$ (i.e. about $1/2.7$) of the current density at the surface of the conductor.

To assess the thickness of this field, let us consider a concrete example. We can assume for copper that $\mu = 1$, $\kappa = 6 \times 10^{-4}$ emu = $c^2 \times 6 \times 10^{-4}$ abs. units (esu). The cyclic frequency is

$$\omega = \frac{2\pi}{T} = 2\pi \nu$$

where ν is the number of periods a second (Hz). When $\nu = 1000$ Hz it follows from Eq. (6.116) that

$$p = \sqrt{\frac{2\pi\mu\kappa\omega}{c^2}} \approx 5 \text{ cm}^{-1}$$

When $\nu = 10^5$ Hz, which corresponds to comparatively slow radio telegraph oscillations (a wavelength of 3000 m), we get

$$p \approx 50 \text{ cm}^{-1}$$

Thus, in the first case, the current is virtually concentrated in a layer having a thickness of $1/5 \text{ cm} = 2 \text{ mm}$, and in the second in a layer only $1/50 \text{ cm} = 0.2 \text{ mm}$ thick.

For steady currents, $\omega = 0$ and, therefore, $p = 0$ so that the field intensity and current density retain a constant value over the entire thickness of a conductor.

4. The results obtained when considering an infinite conductor remain qualitatively applicable to the practically most interesting case of *cylindrical conductors*. In this case too, an alternating current is concentrated on the surface of a conductor to an extent that grows with the current frequency. The concentration of the current at the surface causes a change in the resistance and self-inductance of the conductor. Thus, for alternating currents, these quantities are no longer constant, but depend on the current frequency. For instance, if the entire current is concentrated in the surface layer of a cylindrical conductor, then the resistance of the conductor should become equal to that of a *hollow* cylinder having walls of the appropriate thickness. An increase in the frequency is attended by diminishing of the thickness of the conducting layer, and the resistance of the conductor should grow.

Let us introduce a cylindrical system of coordinates z, r , and α whose axis coincides with that of a cylindrical conductor and assume that in the portion of the conductor being considered the field \mathbf{E} is directed along the axis of the conductor, its intensity $E = |E_z|$ depends only on the coordinate r and does not depend on z or on α . In these conditions, Eq. (6.115) becomes*

$$\frac{d^2 E_0}{dr^2} + \frac{1}{r} \frac{dE_0}{dr} - 2ip^2 E_0 = 0 \quad (6.120)$$

* Indeed, since the vector \mathbf{E} is directed along the z -axis, Eq. (6.115) is equivalent to the equation

$$\nabla^2 E_{0z} - 2ip^2 E_{0z} = 0$$

or, since

$$|E_{0z}| = E_0$$

to the equation

$$\nabla^2 E_0 - 2ip^2 E_0 = 0.$$

The absolute value E_0 of the vector \mathbf{E}_0 is a scalar that in accordance with our condition depends only on r . Hence,

$$\text{grad}_r E_0 = \frac{dE_0}{dr}, \quad \text{grad}_z E_0 = \text{grad}_\alpha E_0 = 0$$

Further

$$\nabla^2 E_0 = \text{div grad } E_0.$$

Therefore, equating the vector \mathbf{a} to the vector $\text{grad } E_0$ in Eq. (A.22), we get

$$\nabla^2 E_0 = \frac{d^2 E_0}{dr^2} + \frac{1}{r} \frac{dE_0}{dr}$$

Equations of this kind are called Bessel equations. The coefficient $1/\overline{r}$ of dE_0/dr in this equation becomes equal to infinity when $r = 0$ (i.e. on the axis of the conductor). Therefore the solutions of the Bessel equation when $r = 0$ also become equal to infinity. There is only one quite definite (with an accuracy to an arbitrary constant factor) solution of Eq. (6.120) that remains finite on the axis of the conductor. This solution is called the zero-order Bessel function of the argument $pr\sqrt{-2i}$ and is designated by the symbol $J_0(pr\sqrt{-2i})$. Since E_0 must have a finite value throughout the entire thickness of the conductor, then it follows from Eq. (6.120) that

$$E_0 = \text{const} \cdot J_0(pr\sqrt{-2i}) \tag{6.121}$$

The following two formulas are proved in the theory of Bessel functions:

(1) $pr \ll 1$:

$$J_0(pr\sqrt{-2i}) = 1 + \frac{i(pr)^2}{2} - \frac{(pr)^4}{16} + \dots \tag{6.122}$$

(2) $pr \gg 1$:

$$J_0(pr\sqrt{-2i}) = \frac{\exp\left[pr - i\left(pr - \frac{\pi}{8}\right)\right]}{2^{1/4}\sqrt{2\pi pr}} \tag{6.123}$$

We shall consider only two limiting cases: $pr_0 \ll 1$ and $pr_0 \gg 1$, where r_0 is the radius of the conductor. In the first case (a low frequency ω and a thin conductor), we may limit ourselves to the first terms of expansion (6.122) for the entire cross section of the conductor. The field intensity E and the current density j will change only insignificantly over the cross sections in this case: it is simple to show with the aid of Eq. (6.122) that the amplitude of the real part of the current density will grow with an increasing distance from the axis of the current approximately proportional to $1 + (pr)^4/16$.

With great frequencies ω and thick conductors, we have $pr_0 \gg 1$ and for the greater part of the conductor cross section $I_0(pr_0\sqrt{-2i})$ can be replaced by the approximate expression (6.123), so that

$$E_0 = \frac{\text{const}}{\sqrt{r}} \exp\left[pr - i\left(pr - \frac{\pi}{8}\right)\right]$$

Passing over from the amplitude E_0 of the field intensity to its complete complex expression $E_0e^{i\omega t}$ and then discarding the imaginary

part, we get

$$\left. \begin{aligned} E_z &= \frac{a}{\sqrt{r}} e^{pr} \cos\left(\omega t + \frac{\pi}{8} - pr\right) \\ j_z &= \kappa E_z = \frac{\kappa a}{\sqrt{r}} e^{pr} \cos\left(\omega t + \frac{\pi}{8} - pr\right) = \\ &= \frac{be^{-p(r_0-r)}}{\sqrt{r}} \cos\left(\omega t + \frac{\pi}{8} - pr\right) \end{aligned} \right\} \quad (6.124)$$

where a and $b = \kappa a e^{pr_0}$ are constants.

Thus, in this case, the current density exponentially diminishes as we move from the surface $r = r_0$ into the conductor. The main part of the current is concentrated, as in the preceding case of an infinite conductor, in the surface layer having a thickness of $1/p$ cm.

Consequently, when determining, for instance, the total current in a conductor, we can integrate Eq. (6.124) for j_z over the entire cross section of the conductor. Although inside the conductor, with $pr \ll 1$, this expression stops being correct and even becomes equal to infinity on the axis of the conductor, nevertheless the internal regions of the conductor (provided that $pr_0 \gg 1$) add virtually nothing to the total current.

Thus, the total current in the conductor when $pr_0 \gg 1$ is

$$I = \int_0^{r_0} j_z 2\pi r dr = 2\pi b \int_0^{r_0} \sqrt{r} e^{-p(r_0-r)} \cos\left(\omega t + \frac{\pi}{8} - pr\right) dr$$

Since the integrand diminishes very rapidly with an increasing distance from the surface of the conductor, we may approximately replace \sqrt{r} with its value at the surface:

$$I = 2\pi b \sqrt{r_0} \int_0^{r_0} e^{-p(r_0-r)} \cos\left(\omega t + \frac{\pi}{8} - pr\right) dr$$

After integration, and disregarding the term of the order of magnitude of e^{-pr_0} , we get

$$I = \frac{\pi b \sqrt{r_0}}{p} \left\{ \sin\left(\omega t + \frac{\pi}{8}\right) - \cos\left(\omega t + \frac{\pi}{8}\right) \right\}$$

Finally, the mean value of the square of the current during a period is

$$\bar{I}^2 = \frac{\pi^2 b^2 r_0}{p^2} \quad (6.125)$$

Let us determine, finally, the quantity of Joule heat liberated in a unit length of the conductor in a unit time:

$$Q = \int_{\frac{j_z^2}{\kappa}}^{\overline{j_z^2}} 2\pi r \, dr = \frac{2\pi b^2}{\kappa} \int_0^{r_0} e^{-2p(r_0-r)} \overline{\cos^2 \left(\omega t + \frac{\pi}{8} - pr \right)} dr$$

Since the mean value of the square of the cosine during a period is $1/2$, then

$$Q = \frac{\pi b^2}{\kappa} \int_0^{r_0} e^{-2p(r_0-r)} \, dr = \frac{\pi b^2}{2p\kappa} (1 - e^{-2pr_0})$$

or, ignoring the term e^{-2pr_0} in comparison with unity, we get

$$Q = \frac{\pi b^2}{2p\kappa} \tag{6.126}$$

5. If Q is the mean quantity of Joule heat liberated in the conductor in a unit time by an alternating current I , then it is expedient to understand by the resistance of a conductor to an alternating current the ratio of the quantity Q to the mean value of the square of the alternating current $\overline{I^2}$ during a period:

$$R = \frac{Q}{\overline{I^2}} \tag{6.127}$$

In engineering, this value of R is customarily called the *ohmic* or *real resistance* of the conductor.

For a steady (direct) current $\overline{I^2} = I^2$ so that the above equation coincides with Joule's law [Eq. (3.7)].

For the quasistationary varying current in a circuit containing self-inductance considered in Sec. 6.5, the value of the ohmic resistance determined by this equation coincides with the direct-current resistance of the circuit. Indeed, by introducing the value of the mean square of the current from Eq. (6.36) into Eq. (6.39) expressing the power used in a circuit (which obviously equals the quantity of heat liberated in the circuit), we obtain

$$Q = \frac{1}{2} \frac{\mathcal{E}_0'^2 \cos \varphi}{\sqrt{R'^2 + \omega^2 L'^2}} = \overline{I^2} \cos \varphi \sqrt{R'^2 + \omega^2 L'^2} = R \overline{I^2}$$

For fast-varying currents when the current concentrates in a thin surface layer of the conductor ($pr \gg 1$), it follows from Eqs. (6.125) and (6.126) that the ohmic resistance of a unit length of the conductor is

$$R = \frac{Q}{\overline{I^2}} = \frac{p}{2\pi\kappa r_0} \tag{6.128}$$

Thus, a conductor has the same resistance to an alternating current with the cyclic frequency ω as it would have to a steady current if the latter were concentrated in a surface layer of the conductor having the cross section $2\pi r_0/p$, i.e. the thickness*

$$\delta = \frac{1}{p} = \sqrt{\frac{c^2}{2\pi\mu\kappa\omega}} \quad (6.129)$$

6. Unlike the resistance, the self-inductance of a conductor diminishes with a growth in the current frequency. Indeed, according to definition, the self-inductance of a conductor is proportional to the energy of the magnetic field of the current flowing through this conductor [Eq. (6.53)]. On the other hand, we know that if a current is concentrated, for instance, on the surface of a cylindrical conductor, then the magnetic field inside the latter equals zero (see Problem 29 on p. 229). The field outside the cylinder, however, does not depend on the distribution of the current over its cross section (since this distribution retains axial symmetry). Hence, the increase in concentration of the current on the surface of the conductor is attended by a reduction in the energy of its field and, consequently, in the self-inductance of the conductor. The latter tends to the limit L' equal to the external self-inductance of the conductor (see p. 409 f).

We can arrive at the same conclusion if we take into consideration that the magnetic field in conductors is determined by the differential equation (6.113) of the same kind as that for their electric field, and that, consequently, both the electric and the magnetic fields of fast-varying currents do not penetrate into the conductors.

* Equation (6.129) holds only for the so-called normal skin effect that occurs when the depth of the skin layer δ is considerably greater than the mean free path of the electrons λ . Indeed, only in this case may the differential form of Ohm's law (3.13) be used for each point of the skin layer. With a growth in the frequency ω , the depth of penetration of the field δ in the normal skin effect diminishes [see Eq. (6.129)], and an anomalous skin effect sets in when $\delta \lesssim \lambda$. In the region of a clearly expressed anomalous skin effect when $\delta \ll \lambda$, the thickness of the skin layer is $\delta_{\text{an}} \approx (c^2\lambda/2\pi\mu\kappa\omega)^{1/3}$. This expression is obtained from Eq. (6.129) if we introduce the effective conductivity equal to $\kappa(\lambda/\delta_{\text{an}})$. For the anomalous skin effect see, for example, J. M. Ziman. *Principles of the Theory of Solids*. Cambridge University Press (1964), Chap. 8, Sec. 7.

7

Varying Electromagnetic Field in a Stationary Medium and Its Propagation. Electromagnetic Waves

7.1 System of Maxwell's Equations for Macroscopic Electromagnetic Field

1. In the preceding chapters, our treatment was of an inductive nature: step by step we summarized empirically discovered regularities and formulated them as separate laws. Now we can consider that the task of finding the fundamental laws of the electromagnetic field (at least in the meaning in which these laws are understood in the classical theory of the macroscopic field) has been coped with, and the results obtained can be brought together into a *complete* system of equations of an electromagnetic field. If this system of equations is correct and is indeed *complete*, then *all* the properties of a field should *unambiguously* follow from it—both those we have studied and those we have not studied.

Thus, the system of fundamental equations is in essence a mathematical formulation of the fundamental postulates or “axioms” of classical electrodynamics. They play the same part in it as Newton’s axioms do in classical mechanics. The further task of theory is to disclose the content of these equations, apply them to separate problems and to compare the corollaries following from them with experimental data.

In the present chapter, we shall restrict ourselves to establishing a system of fundamental equations for a *macroscopic* electromagnetic field with the following simplifying assumptions made: (1) all the material bodies in the field are *stationary*; (2) at each point of the field the values of the quantities ϵ , μ , and κ characterizing the properties of the medium remain *constant*, i.e. do not change with time, do not depend on the field intensity, and are considered as preset quantities; and (3) the field contains no permanent magnets and ferromagnetics. Only in Sec. 7.18 shall we consider some questions relating to the theory of ferromagnetics.

It should be noted that the second restriction means, particularly, that we are either disregarding the temperature dependence of ϵ and μ or are limiting ourselves to a consideration of isothermal processes. We shall consider in the following that ϵ and μ are independent of the temperature, which will allow us to ignore also the

liberation and absorption of heat in the polarization and magnetization of the medium. At the same time, this approximation eliminates the difference between the free and "internal" energy of an electromagnetic field (Secs. 2.12 and 6.7). We have already noted that the effects connected with the liberation of heat in dielectrics and magnetics play an absolutely secondary part in the majority of cases so that they indeed may be left out of consideration.

2. The system of differential equations of classical electrodynamics whose treatment we are passing over to is called *Maxwell's equations*. J. Maxwell first formulated these equations in the sixties of the nineteenth century (particularly, he first introduced the concept of the displacement current) and disclosed their physical meaning. The final formulation of the equations of electrodynamics generally adopted at present, however, belongs to H. Hertz.

Maxwell's fundamental equations include first of all Eq. (6.98) determining the dependence of the curl of a magnetic field on the density of the conduction and the displacement currents:

$$\operatorname{curl} \mathbf{H} = \frac{4\pi}{c} \mathbf{j} + \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} \quad (\text{I})$$

and Eq. (6.86) expressing the law of induction of an electric field when the magnetic field changes:

$$-\operatorname{curl} \mathbf{E} = \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \quad (\text{II})$$

Two other equations generally listed among the fundamental equations of a field *follow* from these equations when certain additional assumptions are made. Thus, it was already shown in Sec. 6.10 that the time independence of the divergence of the vector \mathbf{B} [Eq. (6.87)] follows from Eq. (II). In other respects, the form of the function $\operatorname{div} \mathbf{B} = f(x, y, z)$ in solving the system of equations (I) and (II) remains indefinite so that this function plays the part of the *initial* conditions of integration. Assuming that this function equals zero at all points of space, we get Maxwell's third fundamental equation

$$\operatorname{div} \mathbf{B} = 0 \quad (\text{III})$$

that coincides with Eq. (5.25).

Taking a divergence from both sides of Eq. (I) and taking into account that the divergence of the curl equals zero [Eq. (A.42₂)], we get (after changing the sequence of differentiation with respect to the spatial coordinates and to time)

$$4\pi \operatorname{div} \mathbf{j} + \frac{\partial}{\partial t} \operatorname{div} \mathbf{D} = 0$$

Using the notation

$$\operatorname{div} \mathbf{D} = 4\pi\rho \quad (\text{IV})$$

the preceding equation becomes

$$\operatorname{div} \mathbf{j} = - \frac{\partial \rho}{\partial t} \quad (\text{IVa})$$

which coincides in form with the continuity equation (6.91) expressing the law of conservation of the quantity of electricity. Thus, the quantity ρ defined by Eq. (IV) can be interpreted as the density of electric charges.

In Faraday's and Maxwell's theory, the quantity ρ did indeed have the nature of an auxiliary *symbol*, while the concept of charge had the nature of an auxiliary term because from the viewpoint of Faraday's and Maxwell's concept of a field, electric charges are not a special kind of substance, but are only "nodes" of the lines of force of a field characterizing the deformation of elastic ether. Hence, the term "electric charge" is only a conditional name of the sources of the vector \mathbf{D} , i.e. of the portions of a field in which $\operatorname{div} \mathbf{D} \neq 0$. The consideration of this auxiliary quantity is justified by the fact that the magnitude of the charge inside a closed surface in a non-conducting medium does not change with time, i.e. is the first integral of equations of the field.

The expressions relating the values of the basic vectors of an electromagnetic field, i.e.

$$\mathbf{D} = \varepsilon\mathbf{E}, \quad \mathbf{B} = \mu\mathbf{H}, \quad \text{and } \mathbf{j} = \kappa(\mathbf{E} + \mathbf{E}_{\text{ext}}) \quad (\text{V})$$

must also be listed among Maxwell's fundamental equations.

The quantities ε , μ , and κ characterizing the properties of a medium are considered here to be preset position functions independent of the time, while the extraneous e.m.f.'s \mathbf{E}_{ext} are considered to be preset functions of position and of time. The existence of these forces directly affects only the density of the conduction current, and indirectly it also affects the distribution of the electric charges. Particularly, to establish electrostatic equilibrium ($\mathbf{j} = 0$), it is essential that \mathbf{E}_{ext} at every point of conductors be balanced by the intensity \mathbf{E} of the electrostatic field of the charges [see Eq. (3.31)].

We must note that some authors assume that not only \mathbf{j} , but also \mathbf{D} is proportional to $\mathbf{E} + \mathbf{E}_{\text{ext}}$ instead of to \mathbf{E} , i.e. instead of Eq. (V) they assume that $\mathbf{D} = \varepsilon(\mathbf{E} + \mathbf{E}_{\text{ext}})$. The question of whether or not an assumption is correct can be answered in each separate case by determining the physical nature of the extraneous e.m.f.'s. Sometimes \mathbf{D} is undoubtedly independent of \mathbf{E}_{ext} (for example in solutions of electrolytes \mathbf{D} does not depend on extraneous e.m.f.'s of an osmotic origin, see Abraham, *Theorie d. Elektr.*, 3rd ed., 1907, Vol. I, p. 257). The practical significance of whether or not the vector \mathbf{D} depends on \mathbf{E}_{ext} , however, is quite negligible.

3. The system of equations of an electromagnetic field (I)-(V) will acquire a definite physical content only if it will be exactly indicated in what phenomena available for observation and experimental studying, and how, the existence of an electromagnetic field manifests itself because humans are deprived of the ability to directly perceive this field (except for special cases, for example the field of a light wave). We can learn that an electric current is flowing through a given conductor only according to the thermal (heating of the conductor), mechanical (deviation of a galvanometer pointer), and similar actions of the current. We can learn that a given body is charged only because it attracts a pith ball or gives birth to a spark when it is brought close to another body, and so on. In other words, we can conclude that an electromagnetic field exists only according to the appearance or vanishing of energy forms that we are able to perceive (for example thermal or mechanical energy) and that we observe in certain conditions. Proceeding from the law of conservation of energy, we conclude that this appearance or vanishing of forms of energy known to us must occur at the expense of the transformation of some other form of energy which we call the energy of an electromagnetic field U .

Thus, only when we postulate a definite dependence of this energy U on the field intensity—in Maxwell's theory it is assumed that [cf. Eqs. (2.83) and (6.48)]

$$U = \frac{1}{8\pi} \int \mathbf{DE} \, dV + \frac{1}{8\pi} \int \mathbf{BH} \, dV \quad (\text{VI})$$

—only in this case will the *system* of Eqs. (I)-(V) and (VI) (but not each of them separately) become available for experimental verification, i.e. acquire a definite physical meaning. The reason is that Eqs. (I)-(V) determine the changes in an electromagnetic field in time, while Eq. (VI) makes it possible to determine the transformations of energy in which these changes of the field manifest themselves (see, for example, Sec. 7.2).

4. Let us now turn to a consideration of the discontinuity surfaces of electromagnetic field vectors. The theory of the field is based on the assumption that outside an interface between different media and outside surface electric charges all the electromagnetic vectors, and also the constants of the medium ϵ , μ , and κ are everywhere finite, continuous, and differentiable. The interfaces between different media, however, should be *surfaces of discontinuity* of the electromagnetic vectors because these vectors are related to one another by expressions (V), and the latter include the quantities ϵ , μ , and κ that change in a jump on the interfaces. For the system of equations of a field to be *complete*, i.e. for it to make possible the unambiguous determination of the field intensity according to the initial conditions at the moment $t = 0$, this system must be supplemented with *bound-*

ary conditions which the components of the electromagnetic vectors on the discontinuity surfaces must comply with.

To establish these conditions, we shall first assume that adjacent media with different values of the quantities ϵ , μ , and κ are separated by a transition layer of a finite thickness in which the values of these quantities change *continuously*, and that the volume charge density ρ and the space current density \mathbf{j} remain finite everywhere. We shall then assume that the thickness of these transition layers d and also that of the charged layers through which currents flow tends to zero, and shall require, as we have repeatedly done before, that the equations of a field (I)-(VI) also remain correct for these layers in the limiting case when $d = 0$. When this requirement is introduced, the needed boundary conditions are determined unambiguously.

Indeed, on the basis of this requirement, we get two boundary conditions from Eqs. (III) and (IV) in accordance with Eq. (1.30):

$$\text{Div } \mathbf{B} = B_{2n} - B_{1n} = 0 \quad (\text{III}')$$

$$\text{Div } \mathbf{D} = D_{2n} - D_{1n} = 4\pi\sigma \quad (\text{IV}')$$

coinciding with Eqs. (5.29) and (2.23). Similarly, from the continuity equation (IVa) which is a corollary of Eqs. (I) and (IV), we get

$$\text{Div } \mathbf{j} = j_{2n} - j_{1n} = -\frac{\partial\sigma}{\partial t} \quad (\text{IV}'a)$$

which coincides with Eq. (6.92).

Further, we get two more boundary conditions on the basis of Eqs. (4.55) and (4.56) from Eqs. (I) and (II):

$$\text{Curl } \mathbf{H} = [\mathbf{n}(\mathbf{H}_2 - \mathbf{H}_1)] = \frac{4\pi}{c} \mathbf{i} \quad (\text{I}')$$

$$\text{Curl } \mathbf{E} = [\mathbf{n}(\mathbf{E}_2 - \mathbf{E}_1)] = 0 \quad (\text{II}')$$

that coincide with Eqs. (4.57) and (4.58)*. The last equation is equivalent to the equation

$$E_{2t} = E_{1t} \quad (\text{II}''')$$

where t signifies any direction tangent to the discontinuity surface [cf. Eqs. (4.52) and (4.53)]. Equation (I'), however, can be written in a similar way only when surface currents are absent ($i = 0$):

$$H_{2t} = H_{1t} \quad (\text{I}''')$$

* The right-hand side of Eq. (I) together with the space density of the conduction currents also includes the space density of the displacement currents. The surface density of the displacement currents, however, always equals zero (if only the time derivative of the displacement vector \mathbf{D} has a finite value), and is therefore absent in Eq. (I').

5. Apart from the above conditions on discontinuity surfaces, account must also be taken of the boundary conditions in the proper meaning of this word because the solution of differential equations of type (I) and (II) is unambiguously determined according to the initial conditions for $t = 0$ only if we are given (as a function of time) the values of some of the required position functions (in our case certain components of the field vectors) on the boundaries of the region of space being considered (Sec. 7.3). In each separate case, the form of these boundary conditions depends completely on the concrete conditions of the problem. Particularly, if the region being considered includes the entire infinite space, then the boundary conditions become conditions at infinity.

We shall see in Sec. 7.3 that the system of Maxwell's equations (I)-(V) together with the conditions on the discontinuity surfaces listed above and with the required conditions at infinity is a *complete* system, i.e. it allows us to *unambiguously* determine the electromagnetic field at any point of space and at any moment of time according to the initial values of \mathbf{E} and \mathbf{H} given for the moment $t=0$.

6. We shall show in conclusion what most general assumptions can be the basis for obtaining Maxwell's equations from the equations of the *microscopic* quantities of an electromagnetic field

$$\left. \begin{aligned} \operatorname{curl} \mathbf{H}_m - \frac{1}{c} \frac{\partial \mathbf{E}_m}{\partial t} &= \frac{4\pi}{c} \mathbf{j}_m, & \operatorname{div} \mathbf{E}_m &= 4\pi \rho_m \\ \operatorname{curl} \mathbf{E}_m + \frac{1}{c} \frac{\partial \mathbf{H}_m}{\partial t} &= 0, & \operatorname{div} \mathbf{H}_m &= 0 \end{aligned} \right\} \quad (7.1)$$

where the subscript m signifies the microscopic value of the corresponding quantity.

Taking the mean value of Eqs. (7.1) over infinitely small volumes as we did, for example, in Secs. 2.7 and 5.3, using relationships such as Eq. (2.45), and introducing the symbols [cf. Eq. (5.23)]

$$\bar{\mathbf{E}}_m = \mathbf{E} \text{ and } \bar{\mathbf{H}}_m = \mathbf{B} \quad (7.2)$$

we get

$$\left. \begin{aligned} \operatorname{curl} \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} &= \frac{4\pi}{c} \bar{\mathbf{j}}_m, & \operatorname{div} \mathbf{E} &= 4\pi \bar{\rho}_m \\ \operatorname{curl} \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} &= 0, & \operatorname{div} \mathbf{B} &= 0 \end{aligned} \right\} \quad (7.3)$$

The third of these equations coincides with Eq. (II). To obtain Eq. (I) of Maxwell's two fundamental equations, we must assume that the mean density of the microscopic currents $\bar{\mathbf{j}}_m$ in a *homogeneous and isotropic* medium is a *linear* function of the field vectors \mathbf{E} and \mathbf{B} and their *first* spatial and time derivatives. The most general expres-

sion for $\bar{\mathbf{j}}_m$ compatible with this assumption can be written in the form

$$\bar{\mathbf{j}}_m = \kappa \mathbf{E} + \frac{\varepsilon - 1}{4\pi} \frac{\partial \mathbf{E}}{\partial t} + \frac{c(\mu - 1)}{4\pi\mu} \text{curl } \mathbf{B} \quad (7.4)$$

where κ , ε , and μ are arbitrary scalars. Indeed, the quantities \mathbf{B} , $\partial \mathbf{B} / \partial t$, and $\text{curl } \mathbf{E}$ cannot be included in the expression for $\bar{\mathbf{j}}_m$ because they are *axial* vectors, whereas $\bar{\mathbf{j}}_m$ (like \mathbf{E} , $\partial \mathbf{E} / \partial t$, and $\text{curl } \mathbf{B}$) are *polar* vectors (see the footnote on p. 609). The vectors listed above exhaust all the linear vector functions of the vectors \mathbf{E} and \mathbf{B} and their first derivatives. The properties of a homogeneous and isotropic medium, on the other hand, can be characterized only by scalars*.

Introducing Eq. (7.4) into the first of Eqs. (7.3), taking into account the fact that the characteristics of the medium ε and μ are presumably independent of the coordinates and time, and regrouping terms, we have

$$\text{curl} \left(\frac{1}{\mu} \mathbf{B} \right) - \frac{1}{c} \frac{\partial(\varepsilon \mathbf{E})}{\partial t} = \frac{4\pi}{c} \kappa \mathbf{E}$$

Introducing, finally, the *notation*

$$\frac{1}{\mu} \mathbf{B} = \mathbf{H}, \quad \varepsilon \mathbf{E} = \mathbf{D}, \quad \text{and} \quad \kappa \mathbf{E} = \mathbf{j}$$

in accordance with Eqs. (V), we get Maxwell's equation (I).

As regards Eq. (IV), then as we have already indicated, it may be considered as a *definition* of the quantity $\bar{\rho}_m$.

Thus, the indicated assumptions are indeed sufficient (and necessary) to obtain Maxwell's system of differential equations and relationships (V) from the equations (7.1) for the microscopic quantities. If we discard the *linear* relationships (V) between \mathbf{E} , \mathbf{D} , \mathbf{B} , \mathbf{H} , and \mathbf{j} , then Maxwell's differential equations can be obtained from much more general assumptions (cf. Sec. 8.1).

7.2 Poynting's Theorem. Energy Flow

1. Having expressed the energy of an electromagnetic field in the form of the volume integral (VI), we have thus, as repeatedly indicated, obtained the possibility of *interpreting* this expression in the sense that the energy of a field is *localized* in space in a quite definite

* The assumption on the homogeneous and isotropic nature of the medium which we have made excludes the presence of extraneous e.m.f.'s \mathbf{E}_{ext} because the vector \mathbf{E}_{ext} characterizing the medium distinguishes a definite direction in space.

way, and the space density of the energy at an arbitrary point of the field is expressed by the equation

$$u = \frac{1}{8\pi} (\mathbf{DE} + \mathbf{BH}) = \frac{1}{8\pi} (\varepsilon E^2 + \mu H^2) \quad (\text{VIa})$$

We shall show in Sec. 7.19 that such an interpretation is not only possible, but is essential. Meanwhile, we shall take this statement on trust and consider the change in time of the amount of energy U inside the volume V confined by a certain fixed surface S :

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial t} \left\{ \frac{1}{8\pi} \int_V (\mathbf{DE} + \mathbf{BH}) dV \right\} = \int_V \frac{\partial u}{\partial t} dV \quad (7.5)$$

Since we have assumed that ε and μ are independent of time, we have

$$\frac{\partial u}{\partial t} = \frac{1}{4\pi} \left(\varepsilon \mathbf{E} \frac{\partial \mathbf{E}}{\partial t} + \mu \mathbf{H} \frac{\partial \mathbf{H}}{\partial t} \right) = \frac{1}{4\pi} \left(\mathbf{E} \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \frac{\partial \mathbf{B}}{\partial t} \right)$$

Introducing into this equation the values of $\partial \mathbf{D}/\partial t$ and $\partial \mathbf{B}/\partial t$ from Eqs. (I) and (II), we get with the aid of Eq. (A.44)

$$\begin{aligned} \frac{\partial u}{\partial t} &= -\mathbf{jE} + \frac{c}{4\pi} (\mathbf{E} \text{ curl } \mathbf{H} - \mathbf{H} \text{ curl } \mathbf{E}) = \\ &= -\mathbf{jE} - \frac{c}{4\pi} \text{div} [\mathbf{EH}] \end{aligned} \quad (7.6)$$

Using Eq. (7.6) in Eq. (7.5) and also Gauss's theorem (A.17), we finally get*

$$\frac{\partial U}{\partial t} = - \int_V \mathbf{jE} dV - \frac{c}{4\pi} \oint_S [\mathbf{EH}]_n dS \quad (7.7)$$

* If the volume V contains discontinuity surfaces S' of the vectors \mathbf{E} and \mathbf{H} , then the last integral in Eq. (7.7) must be extended not only over the external surface S of the volume V , but also over these discontinuity surfaces. Using the boundary conditions (I') and (II'), it is not difficult to show that by performing simple transformations the corresponding additional terms in Eqs. (7.7) and (7.8) can be presented in the form of the integral

$$- \int_{S'} \mathbf{iE} dS$$

Thus, these additional terms equal the work done by the electric field \mathbf{E} on the surface conduction currents \mathbf{i} taken with the reverse sign.

If the surface S confines a *total field*, the surface integral vanishes, and Eq. (7.7) becomes

$$\frac{\partial U}{\partial t} = - \int \mathbf{j} \mathbf{E} \cdot dV \quad (7.8)$$

This equation means that with all the material bodies in the field assumed to be stationary, the energy of the field U is used only for the work done by the electric field \mathbf{E} on the conduction currents \mathbf{j} and determined by the right-hand side of Eq. (7.8). Excluding the vector \mathbf{E} from this expression, by means of Eq. (V) we get

$$\frac{\partial U}{\partial t} = \int \mathbf{j} \mathbf{E}_{\text{ext}} \cdot dV - \int \frac{j^2}{\kappa} dV = A - Q \quad (7.9)$$

where

$$A = \int \mathbf{j} \mathbf{E}_{\text{ext}} \cdot dV \quad \text{and} \quad Q = \int \frac{j^2}{\kappa} dV \quad (7.10)$$

The quantity A obviously equals the work done by the extraneous e.m.f.'s in a unit time on the conduction currents \mathbf{j} [compare A with the second term of Eq. (3.32) expressing the work of the extraneous e.m.f.'s in a linear conductor], whereas Q coincides with the expression for the Joule heat liberated by the conduction currents in a unit time [because according to Eq. (3.14) the Joule heat liberated in a unit volume of a conductor in a unit time is $q = j^2/\kappa$].

Thus, Eq. (7.9) expresses the *law of conservation of energy*: the total increase in the electromagnetic energy (with the material bodies assumed to be stationary) equals the surplus work done by the extraneous e.m.f.'s (of a chemical, thermal and similar origin) over that needed for the liberation of the Joule heat (according to our initial assumption all the bodies are stationary so that the mechanical work equals zero). At the same time, we see that unlike conduction currents, *displacement currents liberate no heat*, and the extraneous e.m.f.'s when these currents flow do no work because the expressions for Q and A include only the density of the conduction currents, and not that of the displacement currents*. Having used Maxwell's equations to calculate the change in the electromagnetic energy U and the work A of the extraneous e.m.f.'s for a process, we can determine the Joule heat Q liberated in this process on the basis of formula (7.9). The value of Q can be directly measured, which allows us to verify the correctness of the theory.

* In this chapter, we take no account of the possibility of heat being liberated upon polarization and magnetization of the medium (if ϵ and μ depend on the temperature).

2. Let us now consider the case when the surface integral in Eq. (7.7) does not vanish, i.e. when the surface S does not envelop the total field.

Let us introduce the symbol

$$S = \frac{c}{4\pi} [\mathbf{E}\mathbf{H}] \quad (7.11)$$

and use the notation (7.10). Hence, Eq. (7.7) becomes

$$\frac{\partial U}{\partial t} = A - Q - \oint_S S_n dS \quad (7.12)$$

In the given case, the quantities U , A , and Q will evidently relate not to the entire field as previously, but only to its region V that is confined by the surface S . In this case, as follows from Eq. (7.12), the change in the electromagnetic energy in the volume V depends not only on the heat Q liberated in this volume and on the work A done by the extraneous e.m.f.'s in it, but also on the arrangement of the boundary surface S and on the value of the vector \mathbf{S} on it.

Proceeding from the notion of the localization of electromagnetic energy in space, we must conclude on its basis that *electromagnetic energy flows out through the surface S from the volume V being considered, its amount being $\oint S_n dS$ units of energy (ergs) a second.*

This statement is called *Poynting's theorem*, and the vector \mathbf{S} is called *Poynting's vector*.

Delving in more detail into this statement relating to *closed surfaces*, we can interpret it in the sense that *at every point of a field the flux of electromagnetic energy* (i.e. the amount of energy flowing in a unit time through a unit surface area perpendicular to the direction of the flux) *equals Poynting's vector \mathbf{S} in magnitude and direction.* This last assumption is not at all obligatory because attentive analysis of the possible physical experiments shows that direct experimental verification is possible only with respect to Poynting's theorem in its *integral* form applicable to *closed surfaces* *.

We shall nevertheless identify Poynting's vector with the energy flux at a given point of a field, first, because this interpretation of Poynting's vector leads to a number of quite simple and illustrative relationships [for example, to equality of the rate of energy flow and

* If we assume that the energy flux equals $\mathbf{S} + \text{curl } \mathbf{a}$, where \mathbf{a} is an arbitrary vector, then upon integration over any closed surface the integral of the second term will vanish [Eq. (A.28)] so that the total energy flux through this surface remains equal to $\oint_S S_n dS$.

the velocity of propagation of the carriers of this energy—electromagnetic waves (see Sec. 7.10, and also Secs. 7.13 and 7.14)], and, second, because it directly follows from the relativistic theory of the electromagnetic field.

We must note that the formulation of the law of conservation of energy with the aid of the concept of energy flux [an equation of the type of (7.12)] was first given in the general form by N. Umov back in 1874.

3. In the field of steady currents, the intensity of an electromagnetic field and, consequently, its energy remain constant so that the work of the extraneous e.m.f.'s completely transforms into heat [Eq. (3.37)].

This work is done, however, only in the portions of a circuit where \mathbf{E}_{ext} differs from zero, whereas the Joule heat is liberated in all the portions of the circuit. It is not difficult to see that the energy spent by the sources of extraneous e.m.f.'s flows to where it is consumed (i.e. liberated in the form of heat) as electromagnetic energy.

Let us consider for this purpose a portion of a cylindrical homogeneous conductor having the length l confined between two cross sections perpendicular to its axis (Fig. 79). Let r be the radius of the conductor, $\sigma = \pi r^2$ its cross-sectional area, $V = l\sigma$ the volume of the portion being considered, and, finally, I the current in the conductor. Let us assume that the magnetic field near our conductor coincides with sufficient accuracy with the field of an infinite straight

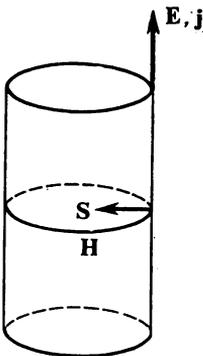


Fig. 79

current having the same intensity. Hence on the surface of the conductor (see Problem 30 on p. 240)

$$H = \frac{2I}{cr} = \frac{2j\sigma}{cr} = \frac{2\pi rj}{c}$$

and the magnetic lines of force are circles concentric to the current.

We shall first assume that $\mathbf{E}_{\text{ext}} = 0$ in the portion of the conductor we are considering. In this case, the electric vector \mathbf{E} has the same direction as the current and equals j/κ [Eq. (V)]. Consequently, on

the surface of the conductor since the vectors \mathbf{j} and \mathbf{H} are perpendicular, we have

$$|\mathbf{S}| = \left| \frac{c}{4\pi} [\mathbf{EH}] \right| = \frac{c}{4\pi\kappa} |[\mathbf{jH}]| = \frac{c}{4\pi\kappa} jH = \frac{j^2 r}{2\kappa}$$

and, according to the right-hand screw rule, \mathbf{S} is directed along an inward normal to the surface of the conductor (Fig. 79). Hence, in this case, energy *flows into* the conductor from the surrounding space through its outer surface in an amount of

$$\oint S_n dS = |\mathbf{S}| 2\pi r l = \frac{j^2}{\kappa} \pi r^2 l = \frac{j^2}{\kappa} V \text{ erg/s}$$

where V is the volume of the portion of the conductor being considered (no energy flows through the bases of the cylinder because \mathbf{S} is tangent to these bases). This amount of energy, as should be expected, equals the Joule heat Q liberated in this portion of the conductor in one second [Eq. (3.14)].

Thus, in the portions of a conductor where $\mathbf{E}_{\text{ext}} = 0$, the thermal energy liberated by a current flows into the conductor from its surroundings. It must evidently flow into this space from the portions of the conductor in which work of the extraneous e.m.f.'s is being done. Indeed, if $\mathbf{E}_{\text{ext}} \neq 0$, then according to Eq. (V)

$$\mathbf{E} = \frac{\mathbf{j}}{\kappa} - \mathbf{E}_{\text{ext}}$$

and

$$\mathbf{S} = \frac{c}{4\pi} [\mathbf{EH}] = \frac{c}{4\pi\kappa} [\mathbf{jH}] - \frac{c}{4\pi} [\mathbf{E}_{\text{ext}}\mathbf{H}]$$

The first term of the right-hand side is, as we have proved, the energy flux directed into the conductor; the second term has a minus sign and therefore has the *reverse* direction (because \mathbf{E}_{ext} , generally speaking, is parallel to \mathbf{j})*, i.e. is the energy flux *flowing out* of the conductor through its side surface. It is easy to see that with a steady current all this energy flowing out of the conductor returns to other portions of it so as to be liberated there in the form of heat.

4. Similar relationships are observed for quasistationary varying currents. For example, when a source of e.m.f. (an accumulator) is disconnected from a circuit, a reverse current continues to circulate in it for a time. As we have seen in Sec. 6.5 (Example 1), Joule heat

* Except for the portions of the conductor in which the extraneous e.m.f.'s do negative work at the expense of the positive work done by these forces in other portions of the conductor. At any rate, in a stationary field, the direction of circulation of the extraneous e.m.f.'s \mathcal{E}_{ext} around the current contour coincides with the direction of the current I .

is liberated by this current at the expense of a gradual reduction in the energy of the magnetic field of the current flowing into the conductor from the surrounding space; the amount of magnetic energy confined inside the conductor also diminishes.

In Secs. 6.4 and 6.5, we took no account of the change in the energy of the *electric* field surrounding a current. For quasistationary currents in closed conductors, this energy is in general so small in comparison with the magnetic energy of a current that it indeed may be ignored. If we include a capacitor in the field of a quasistationary current, for example, then the electric energy stored in its field is comparable with the magnetic energy of the current and it becomes impossible to ignore it (Sec. 6.14).

Generally speaking, we can say that in a conductor through which a current circulates electromagnetic energy is converted, in essence, into heat. This energy is mainly localized in the external space surrounding the conductor and enters the latter through its outer surface. This manifests itself especially clearly in fast-varying currents. When a field changes very rapidly its energy does not have time to reach the internal layers of the conductor and is converted into Joule heat only in the surface layers of the conductor in which the alternating currents are concentrated—the *skin effect* (Sec. 6.15).

5. We shall note in conclusion that the assumption on the energy flux of an electromagnetic field at every point of space being equal to Poynting's vector sometimes leads to corollaries that may seem to be deprived of a physical meaning. For example, in a constant field induced by a fixed electric charge and a stationary permanent magnet, Poynting's vector differs from zero, and the lines of this vector are closed (or fill a certain space). We thus arrive at the seemingly meaningless notion of the continuous circulation of energy along closed paths in a *static* electromagnetic field. The physical meaning of this notion will be revealed in Sec. 104.

7.3 Unambiguity of the Solutions of Maxwell's Equations

1. Having established in Sec. 7.1 a system of Maxwell's fundamental equations, we showed that this system is *complete*, i.e. that an electromagnetic field at each point of space and at each moment of time is *unambiguously* determined by this system if only the initial values of the vectors \mathbf{E} and \mathbf{H} are preset for all the points of space. This formulation of the "uniqueness theorem", however, is not quite accurate. We cannot determine the field intensity *throughout* infinite space—only a limited region of it is available for our observation. Therefore the uniqueness theorem acquires a direct physical meaning only if we limit ourselves to a certain finite region of space

and add definite *boundary conditions* on its boundaries to the conditions determining the solutions of Maxwell's equations.

2. We shall first prove the uniqueness theorem in the following wording: an electromagnetic field at any moment of time $t_1 > 0$ at any point of the volume V confined by an arbitrary closed surface S is unambiguously determined by Maxwell's equations if the initial values of the electromagnetic vectors \mathbf{E} and \mathbf{H} are given throughout this region of space for the moment $t = 0$ and if, in addition, *for one of these vectors (for example \mathbf{E}) we know the boundary values of its tangential components on the surface S during the entire interval of time from $t = 0$ up to $t = t_1^*$.*

Let us assume the opposite, i.e. that there are two different systems of solutions of Maxwell's equations \mathbf{E}', \mathbf{H}' and $\mathbf{E}'', \mathbf{H}''$ satisfying the same initial and boundary conditions. Owing to the linear nature of the equation of a field, the difference between these solutions $\mathbf{E}''' = \mathbf{E}' - \mathbf{E}''$ and $\mathbf{H}''' = \mathbf{H}' - \mathbf{H}''$ must also satisfy Maxwell's equations with the following additional conditions: (a) $\mathbf{E}_{\text{ext}} = 0$ **, (b) at the moment $t = 0$ at all points of the volume V we have $\mathbf{E}''' = \mathbf{H}''' = 0$ (because when $t = 0$ the quantities $\mathbf{E}', \mathbf{E}''$, and $\mathbf{H}', \mathbf{H}''$ have according to our assumption, identical given values), and (c) during the entire time interval from $t = 0$ to $t = t_1$ the tangential components of the vector \mathbf{E}''' or of the vector \mathbf{H}''' equal zero at all the points of the surface S (for the same reason).

Let us apply to this field $\mathbf{E}''', \mathbf{H}'''$ Poynting's theorem [Eq. (7.12)] following from Maxwell's equations, assuming in it in accordance with what has been said above that the work A of the extraneous forces equals zero. The surface integral in Eq. (7.12) will equal zero during the entire time interval from $t = 0$ to $t = t_1$ because it follows from condition (c) that on the surface S

$$S_n = [\mathbf{E}''' \mathbf{H}''']_n = 0$$

Hence, at any moment of this interval

$$\frac{\partial U'''}{\partial t} = -Q''' = -\int \frac{j'''^2}{\kappa} dV \quad (7.13)$$

Since the integrand is positive, then $\partial U''' / \partial t \leq 0$, i.e. the energy of the field U''' may either decrease or (when $j''' = 0$ everywhere) remain constant. But when $t = 0$, according to condition (b), the energy U''' of the field $\mathbf{E}''', \mathbf{H}'''$ was zero. It cannot have negative

* We must remind our reader here that in accordance with the main assumption underlying all the reasoning of this chapter, the values of the quantities ϵ, μ , and κ are considered to be preset position functions not depending on time, while \mathbf{E}_{ext} is a known function of position and time.

** Because from $\mathbf{I}' = \kappa(\mathbf{E}' + \mathbf{E}_{\text{ext}})$ and $\mathbf{I}'' = \kappa(\mathbf{E}'' + \mathbf{E}_{\text{ext}})$ it follows that $\mathbf{I}''' = \mathbf{I}' - \mathbf{I}'' = \kappa \mathbf{E}'''$.

values; hence, during the entire interval $0 \leq t \leq t_1$ being considered the energy

$$U''' = \frac{1}{8\pi} \int_V (\epsilon \mathbf{E}'''^2 + \mu \mathbf{H}'''^2) dV$$

should also remain equal to zero. This can occur only if \mathbf{E}''' and \mathbf{H}''' remain equal to zero at all the points of the volume V . And this means that the two systems of solutions of the initial problem \mathbf{E}' , \mathbf{H}' and \mathbf{E}'' , \mathbf{H}'' , which we assumed to be different, are of necessity identical to each other. Thus, the uniqueness theorem has been proved.

3. It is easy to see that when considering the entire infinite space, the setting of the values of the field vectors on the boundary surface S can be replaced by the superposition of the following conditions at infinity:

$$ER^2 \text{ and } HR^2 \text{ remain finite when } R \rightarrow \infty \quad (7.14)$$

Indeed, it follows from conditions (7.14) that the integral of Poynting's vector over an infinitely remote surface vanishes. This circumstance makes it possible to prove, on the basis of Eq. (7.12), that Eq. (7.13) may be applied to the entire infinite space. We can see that the unambiguous nature of the solutions of the field equation also follows from Eq. (7.13).

The conditions (7.14) coincide with the previous conditions (1.85) and (4.59). For a constant field, they express the fact that if all the charges and currents inducing a field are in a restricted region of space V , then the intensity of the field at infinity should decrease not slower than the reciprocal of the square of the distance R from an arbitrarily selected initial reference point in V .

The conditions (7.14), however, which can be used for a constant field, *cannot be applied to a varying field*. For example, in Sec. 7.9, we shall see that the field of radiation of an oscillator diminishes at infinity inversely proportional to the *first*, and not the second, power of the distance R . In this case, the flux of Poynting's vector through an infinitely remote surface does not vanish, but equals a quite definite finite value*.

It should be noted in passing that when considering the field of radiation we can often limit ourselves to problems of the following kind. Assume that up to the moment $t = 0$ the field outside of a certain finite region of space equalled zero or was stationary and complied with conditions (7.14). Next during the time τ from the

* The conditions (7.14) are sufficient, but not necessary for the vanishing of the flux of Poynting's vector through an infinitely remote surface. This flux will vanish even in the much weaker condition when the product $[\mathbf{E}\mathbf{H}]$ diminishes more rapidly than $1/R^2$ at $R \rightarrow \infty$. In the field of an oscillator, however, this condition is also not observed.

moment $t = 0$ to the moment $t = \tau$, perturbations occurred such as the motion of bodies, the closing and opening of current circuits, and the connection of varying extraneous e.m.f.'s \mathbf{E}_{ext} ; from the moment $t = \tau$, the quantities ε , μ , and κ at all the points of the field again acquired constant values.

As we shall see in Sec. 7.7, disturbances of the electromagnetic field propagate with the velocity of light c . Therefore, if all the charges and currents are concentrated inside a sphere having the finite radius R_0 , then outside the sphere S having the radius $R_0 + ct$ the field will retain an undisturbed value up to the moment t , i.e. will comply with the conditions (7.14). Thus, in this case on the basis of what has been proved above, the field at any moment $t > \tau$ is unambiguously determined by presetting the initial values of the vectors \mathbf{E} and \mathbf{H} at all points of space for the moment of time t_0 complying with the inequality $\tau \leq t_0 < t$ (cf. the end of Sec. 7.6)*.

7.4 Differential Equations for the Potentials of an Electromagnetic Field

1. Having convinced ourselves that Maxwell's equations are unambiguous, we must try to find a way for the actual solution of these equations. We have seen that for a stationary electromagnetic field this task is appreciably facilitated by the introduction of auxiliary quantities—the potentials φ and \mathbf{A} . We shall now show that by appropriately modifying the definition of the scalar and the vector potentials, we can also use these potentials for solving Maxwell's equations in the general case of a varying field. We shall assume for simplicity that *both the permittivity ε and the permeability μ are constant over the entire length of the total field*** and that there are no surface charges and no surface currents in the field. In these conditions, the vectors \mathbf{E} and \mathbf{H} and their first derivatives remain continuous everywhere.

We can retain Eq. (5.27) as the definition of the vector potential \mathbf{A} :

$$\mathbf{B} = \mu\mathbf{H} = \text{curl } \mathbf{A} \quad (7.15)$$

* It should be noted that the results of this section are substantially connected with the assumption that all electric currents consist only of conduction currents whose density \mathbf{j} , according to Eq. (V), is unambiguously determined by presetting κ , \mathbf{E} , and \mathbf{E}_{ext} . If we also take into consideration, for example, the currents created by the motion of free electrons in a vacuum (for instance in a cathode-ray tube or an x-ray tube), then to determine the field in accordance with the initial conditions it is evidently necessary to supplement Maxwell's system of equations of a *field* with equations of *mechanics*, i.e. equations of the motion of electrons.

** The results of the following sections are easily generalized for the presence of a stepwise change in ε and μ on separate interfaces between different media. In the general case of an arbitrary dependence of ε and μ on the coordinates of a point, however, the task of solving Maxwell's equations becomes exceedingly complicated.

from which, according to Eq. (A.42₂), Eq. (III) follows. Using Eq. (7.15) in Eq. (II), we get

$$\text{curl } \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = \text{curl} \left(-\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right)$$

or

$$\text{curl} \left(\mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right) = 0$$

This equation will be satisfied if we assume that

$$\mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \text{grad } \varphi \quad (7.16)$$

where φ is an arbitrary scalar because the curl of the gradient of a scalar identically equals zero [Eq. (A.42₁)].

On the basis of Eqs. (7.15), (7.16), and (A.42₃), Eq. (I), i.e.

$$\text{curl } \mathbf{H} = \frac{4\pi \mathbf{j}}{c} + \frac{\epsilon}{c} \frac{\partial \mathbf{E}}{\partial t}$$

becomes

$$\begin{aligned} \frac{1}{\mu} \text{curl curl } \mathbf{A} &= \frac{1}{\mu} (\text{grad div } \mathbf{A} - \nabla^2 \mathbf{A}) = \\ &= \frac{4\pi \mathbf{j}}{c} - \frac{\epsilon}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \frac{\epsilon}{c} \text{grad } \frac{\partial \varphi}{\partial t} \end{aligned} \quad (7.17)$$

Upon properly dealing with the quantities \mathbf{A} and φ , we can simplify this equation.

Indeed, up to now we determined only the curl of \mathbf{A} and the gradient of φ . Now, however, we can additionally require that*

$$\text{div } \mathbf{A} = -\frac{\epsilon \mu}{c} \frac{\partial \varphi}{\partial t} \quad (7.18)$$

Having formed a gradient from both sides of this equation, we see that two terms of Eq. (7.17) mutually cancel, so that it becomes

$$\nabla^2 \mathbf{A} - \frac{\epsilon \mu}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi \mu}{c} \mathbf{j} \quad (7.19)$$

* We shall determine the degree of arbitrariness in determining the potentials φ and \mathbf{A} with the preset field intensities \mathbf{E} and \mathbf{H} at the end of Sec. 7.6.

Of the fundamental equations of a field, it remains for us to satisfy Eq. (IV). Introducing Eq. (7.16) into it, we get

$$\operatorname{div} \mathbf{D} = \varepsilon \operatorname{div} \mathbf{E} = -\frac{\varepsilon}{c} \frac{\partial}{\partial t} \operatorname{div} \mathbf{A} - \varepsilon \operatorname{div} \operatorname{grad} \varphi = 4\pi\rho$$

Dividing this equation by ε and introducing the value of $\operatorname{div} \mathbf{A}$ into it from Eq. (7.18), we have

$$\frac{\varepsilon\mu}{c^2} \frac{\partial^2 \varphi}{\partial t^2} - \operatorname{div} \operatorname{grad} \varphi = \frac{4\pi}{\varepsilon} \rho$$

or on the basis of Eq. (A.40)

$$\nabla^2 \varphi - \frac{\varepsilon\mu}{c^2} \frac{\partial^2 \varphi}{\partial t^2} = -\frac{4\pi}{\varepsilon} \rho \quad (7.20)$$

2. Equations (7.18), (7.19), and (7.20) make it possible to determine the values of the scalar and vector potentials of an electromagnetic field according to the given distribution of the charges and conduction currents. Knowing φ and \mathbf{A} , we can find \mathbf{E} and \mathbf{H} with the aid of Eqs. (7.15) and (7.16). We shall note that although the scalar potential φ as for stationary fields depends only on the distribution of the charges, and the vector potential \mathbf{A} on the distribution of the conduction currents, the intensity of the electric field, however, depends not only on the gradient of the scalar potential, but also on the derivative of the vector potential with respect to time. The law of electromagnetic induction manifests itself in this circumstance. When a field is stationary and all the derivatives with respect to time vanish, the above equations, as should be expected, acquire the form of the equations for a stationary field that we have previously established [compare Eqs. (7.16), (7.18), (7.19), and (7.20) with Eqs. (1.59), (5.38), (5.39), and (2.27), respectively].

Thus, knowing ρ and \mathbf{j} in their dependence on the coordinates and time, we can determine first φ and \mathbf{A} , and then also \mathbf{E} and \mathbf{H} . As we have already indicated in Sec. 7.1 (p. 457), however, the concept of an electric charge in the classical theory of the field has the nature, in essence, of an auxiliary term denoting the source of the electric induction vector \mathbf{D} . In other words, from the viewpoint of this theory, ρ and \mathbf{j} must in essence be considered functions of the required quantities \mathbf{E} and \mathbf{H} , i.e. required quantities in turn. And indeed, according to the uniqueness theorem, to find the values of the vectors \mathbf{E} and \mathbf{H} at any moment it is sufficient to know the initial values of these vectors for $t = 0$. Having determined \mathbf{E} and \mathbf{H} , we can obviously *calculate* the values of the quantities ρ and \mathbf{j} for any moment of time and any point.

The actual solution of this "overall" problem, however, is extremely complicated. We shall therefore assume in the following that we have in some way or other learned how ρ and \mathbf{j} depend on the coordinates

and the time for the entire space*. In this case, using the system of equations established in the present section, we can indeed determine the intensity of the electromagnetic field at any point of space and at any moment of time, and do this *unambiguously* if only we take into consideration certain additional conditions which we shall mention in Sec. 7.6.

Our main task is to solve the system of equations (7.18), (7.19), and (7.20) determining the values of the potentials because having determined φ and \mathbf{A} , we can find \mathbf{E} and \mathbf{H} by simple differentiation. **3.** Both the scalar potential φ and each of the components A_x , A_y , and A_z of the vector potential in an arbitrary system of Cartesian coordinates, in accordance with Eqs. (7.19) and (7.20), satisfy an equation of the kind

$$\nabla^2 s - \frac{\epsilon\mu}{c^2} \frac{\partial^2 s}{\partial t^2} = -4\pi\chi(x, y, z, t) \tag{7.21}$$

where $\chi(x, y, z, t)$ is, by assumption, a known function of the coordinates and time, and s signifies one of the quantities φ , A_x , A_y , and A_z .

Introducing the notation

$$v = \frac{c}{\sqrt{\epsilon\mu}} \tag{7.22}$$

we can write Eq. (7.21) as follows:

$$\nabla^2 s - \frac{1}{v^2} \frac{\partial^2 s}{\partial t^2} = -4\pi\chi \tag{7.23}$$

We shall see in the following that v equals the speed of propagation of electromagnetic disturbances.

Equations of the type of (7.23) are called *d'Alembert equations*. When $\chi = 0$, a d'Alembert equation acquires the form of the so-called *wave equation*

$$\nabla^2 s = \frac{1}{v^2} \frac{\partial^2 s}{\partial t^2} \tag{7.24}$$

with which we shall repeatedly have to deal in the following. Finally, when s is independent of time (a stationary field), the d'Alembert equation degenerates into the *Poisson equation* (1.65) which we already know:

$$\nabla^2 s = -4\pi\chi(x, y, z)$$

* In essence, apart from the *initial* distribution of the charges ρ for the moment $t = 0$, it is sufficient to know only \mathbf{j} as a function of the coordinates and time because these data can be used to determine the value of ρ at each point of space for every moment of time with the aid of the continuity equation (IVa). It follows from this, however, that \mathbf{j} and ρ are not independent functions of time and therefore cannot be given independently of each other.

7.5 Solution of the Wave Equation and the D'Alembert Equation

1. We shall not stop here to treat the classical ways of solving these equations that are quite strict from the mathematical viewpoint*, but shall use much simpler reasoning that is not distinguished, however, by any special mathematical strictness and has in essence only a guiding nature. We shall impart complete authenticity to this solution, however, upon checking the found solution by substituting it into the initial equations and proving the unambiguity of these solutions.

To find the solution of the d'Alembert equation (7.23), let us first assume that for any value of t the quantity χ equals zero at all points of the field except for only a vanishingly small region around a certain point Q at which χ equals the given function of time $\chi(t)$. For brevity, we shall say here that χ differs from zero only at the point Q itself, which can be called the *source* of the field. Thus, outside of this point, the quantity s should satisfy the wave equation (7.24).

Let us first of all set ourselves the task of finding a *spherically symmetrical* solution of this wave equation, i.e. such a solution which in a polar system of coordinates having its centre at Q depends only on the radius-vector R , but not on the polar and longitudinal angles θ and α . In this case, $\nabla^2 s = \text{div grad } s$ is determined by an equation like Eq. (A.21) so that the wave equation (7.24) becomes

$$\frac{\partial^2 s}{\partial R^2} + \frac{2}{R} \frac{\partial s}{\partial R} = \frac{1}{v^2} \frac{\partial^2 s}{\partial t^2}$$

Multiplying it by R , we get

$$R \frac{\partial^2 s}{\partial R^2} + 2 \frac{\partial s}{\partial R} = \frac{\partial^2 (Rs)}{\partial R^2} = \frac{R}{v^2} \frac{\partial^2 s}{\partial t^2} = \frac{1}{v^2} \frac{\partial^2 (Rs)}{\partial t^2}$$

or

$$\frac{\partial^2 u}{\partial R^2} = \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} \quad (7.25)$$

where we have temporarily introduced the symbol $u = Rs$.

The general solution of Eq. (7.25) is known to have the form

$$Rs = u = f\left(t - \frac{R}{v}\right) + \varphi\left(t + \frac{R}{v}\right) \quad (7.26)$$

* They can be found in any classical course of electricity. See, for example, Lorentz, H. A. *Theory of Electrons*, 2nd ed. New York, Dover Publications (1952); see also Kochin, N. E. *Vektornoe ischislenie i nachala tenzornogo ischisleniya* (Vector Calculus and the Fundamentals of Tensor Calculus), § 21. Moscow, Nauka (1960); see also the footnote to p. 70.

where f and φ are arbitrary (but having derivatives) functions of the arguments indicated in the parentheses*.

2. It is easy to see that the first term of this expression is a spherical wave propagating from the origin of coordinates Q with the velocity v . Indeed, the function $f\left(t - \frac{R}{v}\right)$ has at each given moment t at each given distance R from the point Q the same value that it had at the moment $t - 1$ at the distance $R - v$ from Q (because $t - 1 - \frac{R - v}{v} = t - \frac{R}{v}$). And this means that the values of the quantity u propagate from the field source point Q in the form of a spherical wave having the velocity v .

In a similar way, we can see that the second term of Eq. (7.26) is a spherical wave having the same velocity v arriving from infinity and converging at the field source Q like at a focus.

In this section and in the first half of the following one, we shall limit ourselves to considering the first term of the general solution (7.26), i.e. we shall assume that

$$s = \frac{u}{R} = \frac{1}{R} f\left(t - \frac{R}{v}\right) \tag{7.27}$$

* We can become convinced that Eq. (7.26) satisfies Eq. (7.25) by direct substitution. To show, however, that *any* solution of Eq. (7.25) must have the form of Eq. (7.26), let us introduce the new variables ζ and η instead of t and R :

$$\zeta = t - \frac{R}{v} \quad \text{and} \quad \eta = t + \frac{R}{v}$$

Hence

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \zeta} + \frac{\partial u}{\partial \eta} \quad \text{and} \quad \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial \zeta^2} + 2 \frac{\partial^2 u}{\partial \zeta \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}$$

Similarly

$$\frac{\partial^2 u}{\partial R^2} = \frac{1}{v^2} \left(\frac{\partial^2 u}{\partial \zeta^2} - 2 \frac{\partial^2 u}{\partial \zeta \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} \right)$$

Introducing these expressions into Eq. (7.25), we get

$$-\frac{\partial^2 u}{\partial \eta \partial \zeta} = \frac{\partial^2 u}{\partial \eta \partial \zeta}$$

whence

$$\frac{\partial^2 u}{\partial \eta \partial \zeta} = 0$$

This means that in the general case u can consist of two addends each of which is an arbitrary function of only *one* of the independent variables ζ and η . Equation (7.26) is a mathematical formulation of this conclusion.

We shall discuss when it is expedient to take into consideration the second term of this general solution in the second half of Sec. 7.6.

If in accordance with Sec. 7.4 we understand s to be the potential (scalar or vector) of an electromagnetic field, then we should understand s determined by Eq. (7.27) to be the potential of the field induced by a charge or current at the point Q . According to Eq. (7.27), the values of the potential of the charges and currents at Q propagate from this point in the form of a spherical wave having the velocity v whose amplitude diminishes in inverse proportion to the distance. 3. Equation (7.27) cannot hold for all points of space because when $R = 0$ it becomes equal to infinity, i.e. loses all meaning. In addition, when $\chi \neq 0$ the required function should satisfy the d'Alembert equation instead of the wave one.

To find the solution of this equation, let us recall the solution of a similar problem in electrostatics. Outside of electric charges, the electrostatic potential complies with the Laplace equation $\nabla^2\varphi = 0$ whose spherically symmetrical solution $\varphi = q/R$ is similar to Eq. (7.27). The complete solution of the electrostatic problem, i.e. the solution of the Poisson equation $\nabla^2\varphi = -4\pi\rho$, can be obtained by summation of the spherically symmetrical solutions of the Laplace equation in the form of the integral

$$\varphi = \int \frac{\rho dV}{R}$$

this integral retaining a finite value at all points of space. In view of the analogy between the Laplace and wave equations on the one hand, and the Poisson and d'Alembert equations on the other, we can expect the solution of the latter equation to be expressed by the sum of solutions like Eq. (7.27). Owing to the similar part played by the functions ρ and χ , we can assume that

$$s = \int \frac{\chi \left(t - \frac{R}{v} \right)}{R} dV \quad (7.28)$$

In this case, the potential of the field induced by the "charge" $\chi(t) dV$ of the infinitely small element of volume dV will be expressed by the formula

$$s = \frac{\chi \left(t - \frac{R}{v} \right)}{R} dV$$

coinciding in its form with Eq. (7.27) and similar to the corresponding equation of electrostatics

$$\varphi = \frac{\rho dV}{R}$$

We can prove that Eq. (7.28) is indeed a solution of the d'Alembert equation (7.23) by directly introducing Eq. (7.28) into Eq. (7.23) (see below).

Let $x, y,$ and z be the coordinates of the point P at which we are looking for the value of the function s , let $x', y',$ and z' be the running coordinates of an arbitrarily arranged volume element dV' , and R be the distance from dV' to the point P . We shall introduce the notation

$$t' = t - \frac{R}{v} \tag{7.29}$$

and shall call t' the *effective time* in dV' with respect to the time at P . Equation (7.28) can therefore be written as follows:

$$s(x, y, z, t) = \int \frac{\chi(x', y', z', t')}{R} dV' \tag{7.30}$$

where we should understand dV' to denote the product $dx' dy' dz'$.

4. Before checking the solution of Eq. (7.30) by substitution, we shall conduct a similar calculation for a simpler case; let us introduce into the Poisson equation its solution

$$\varphi(x, y, z) = \int \frac{\rho(x', y', z')}{R} dV' \tag{7.31}$$

which we obtained in Sec. 1.12 in a mathematically irreproachable way.

When calculating, $\nabla^2\varphi$ can be differentiated with respect to the variables $x, y,$ and z inside the integral taken over $x', y',$ and z' provided that the second derivatives of the integrand are finite and continuous throughout the entire region of integration. If the density of the charge ρ equals zero at the point of observation P and in a finite region around this point, then this condition is observed (because R remains finite throughout the entire domain of integration), and we can differentiate inside the integral. Owing to $\rho(x'y'z')$ being independent of the coordinates $x, y,$ and z of the point of observation P , in this case in accordance with the Poisson equation we get

$$\nabla^2\varphi = \int \rho(x', y', z') \cdot \nabla^2 \frac{1}{R} \cdot dV' = 0$$

because according to Eq. (1.74) we have $\nabla^2 \frac{1}{R} = 0$.

If, however, ρ at the point P differs from zero, then when calculating $\nabla^2\varphi$ we can perform only *one* and not two consecutive differentiations inside the integral*. We shall use Eq. (1.64):

$$\nabla^2\varphi = \text{div grad } \varphi$$

and, performing the first differentiation inside the integral, we get

$$\begin{aligned} \text{grad } \varphi &= \int \rho(x', y', z') \text{grad} \left(\frac{1}{R} \right) dV' = \\ &= - \int \frac{\rho(x', y', z') \mathbf{R}}{R^3} dV' \end{aligned}$$

where \mathbf{R} is a vector conducted from the point x', y', z' to the point of observation x, y, z . Further, on the basis of Eq. (A.18), we can express the divergence in the form of the surface integral

$$\text{div } \mathbf{a} = \lim_{V \rightarrow 0} \frac{\oint \mathbf{a} \cdot d\mathbf{S}}{V}$$

where, by definition, the surface S of the volume V should include the point of observation P at which the value of $\text{div } \mathbf{a}$ is determined. In our case, $\mathbf{a} = \text{grad } \varphi$ and, consequently,

$$\nabla^2\varphi = \lim_{V \rightarrow 0} \frac{\oint \text{grad } \varphi \cdot d\mathbf{S}}{V}$$

In calculating the surface integral, we get, upon changing the sequence of integration over dV' and $d\mathbf{S}$,

$$\begin{aligned} \oint \text{grad } \varphi \cdot d\mathbf{S} &= - \oint d\mathbf{S} \int \frac{\rho(x', y', z') \mathbf{R}}{R^3} dV' = \\ &= - \int \rho(x', y', z') dV' \oint \frac{\mathbf{R} d\mathbf{S}}{R^3} \end{aligned}$$

* Indeed, the derivatives of φ , for instance with respect to x , are

$$\frac{\partial \varphi}{\partial x} = \int \frac{(x' - x) \rho}{R^3} dV' \quad \text{and} \quad \frac{\partial^2 \varphi}{\partial x^2} = \int \frac{\rho}{R^3} \left\{ \frac{3(x' - x)^2}{R^2} - 1 \right\} dV'$$

If after differentiation we introduce instead of x', y' , and z' the polar coordinates R, θ , and α with the origin at the point x, y, z and with the polar axis directed along the x -axis, then dV' acquires the form $R^2 dR d\Omega$, where $d\Omega = \sin \theta d\theta d\alpha$ and

$$\frac{\partial \varphi}{\partial x} = \int \rho \cos \theta dR d\Omega, \quad \frac{\partial^2 \varphi}{\partial x^2} = \int \frac{\rho}{R} (3 \cos^2 \theta - 1) dR d\Omega$$

Thus, the integrand in the second of these integrals becomes equal to infinity when $R = 0$.

where \mathbf{R} is a vector conducted from the point x', y', z' to the surface element dS . We calculated the vector flux \mathbf{R}/R^3 through an arbitrary closed surface in Sec. 1.3. It equals 4π or zero depending on whether the source point x', y', z' of the vector \mathbf{R} is within or without the volume V enclosed by the surface S . Hence,

$$\oint \text{grad } \varphi \, dS = -4\pi \int \rho \, dV$$

where integration should obviously be extended over the volume V enclosed by the surface S . Consequently,

$$\nabla^2 \varphi = -4\pi \lim_{V \rightarrow 0} \frac{\int \rho \, dV}{V} = -4\pi \rho_P$$

where ρ_P is the value of ρ at the centre of the volume V , i.e. at the point of observation P at which we are determining the value of the quantity $\text{div grad } \varphi = \nabla^2 \varphi$. We have thus proved that Eq. (7.31) for φ does indeed satisfy the Poisson equation.

5. Let us now pass over to verification of Eq. (7.30). First, we shall separate near the point of observation P a small volume V_0 , for instance a sphere having the radius R_0 with its centre at P , and divide the integral of Eq. (7.30) into two integrals: over the volume of the sphere V_0 , which in the following we shall make approach zero, and over the remaining "external" space:

$$s = s_1 + s_2, \quad s_1 = \int_{V_0} \frac{\chi(x', y', z', t')}{R} \, dV',$$

$$s_2 = \int_{\text{ext}} \frac{\chi(x', y', z', t')}{R} \, dV'$$

In the second of these integrals, the distance R exceeds the finite quantity R_0 . Hence, we can directly differentiate inside the integral. Since the function

$$\frac{1}{R} \chi \left(x', y', z', t - \frac{R}{v} \right)$$

according to Eq. (7.27), satisfies the wave equation (7.24), then

$$\nabla^2 s_2 - \frac{1}{v^2} \frac{\partial^2 s_2}{\partial t^2} = \int_{\text{ext}} \left[\nabla^2 \left(\frac{\chi}{R} \right) - \frac{1}{v^2} \frac{\partial^2}{\partial t^2} \frac{\chi}{R} \right] dV' = 0$$

Further,

$$\frac{1}{v^2} \frac{\partial^2 s_1}{\partial t^2} = \frac{1}{v^2} \int_{V_0} \frac{1}{R} \frac{\partial^2 \chi}{\partial t^2} \, dV'$$

With an unrestricted reduction of the radius R_0 of the sphere V_0 , this integral obviously tends to zero because $\partial^2\chi/\partial t^2$ is a finite quantity, and

$$\int \frac{1}{R} dV = \int \frac{1}{R} \cdot 4\pi R^2 dR = 4\pi \int R dR$$

Thus, the left-hand side of the wave equation reduces to

$$\begin{aligned} \nabla^2 s - \frac{1}{v^2} \frac{\partial^2 s}{\partial t^2} &= \nabla^2 s_1 = \text{div grad } s_1 = \\ &= \lim_{V \rightarrow 0} \frac{\oint \text{grad } s_1 dS}{V} \end{aligned} \quad (7.32)$$

Further,

$$\text{grad } s_1 = \int_{V_0} \text{grad} \frac{\chi}{R} dV' = \int_{V_0} \left(\frac{1}{R} \text{grad } \chi + \chi \text{grad} \frac{1}{R} \right) dV'$$

Upon an unlimited reduction of the sphere V_0 , the integral of the first term in the parentheses tends to zero so that it may immediately be discarded. Thus,

$$\oint \text{grad } s_1 dS = \int_{V_0} dV' \oint \chi \text{grad} \frac{1}{R} dS = - \int_{V_0} dV' \oint \frac{\chi \mathbf{R}}{R^3} dS$$

Upon passing over to the limit, when both the sphere V_0 and the surface S become infinitesimals, we may ignore the dependence of the function $\chi \left(x', y', z', t - \frac{R}{v} \right)$ on R , i.e. on the distance between the point x', y', z' of the volume V_0 and the surface element dS and equate R in the argument of this function χ to zero. The function $\chi(x', y', z', t)$ can be put outside the surface integral.

Thus, we arrive at an expression of the type we have just considered (we assume that the surface S is completely inside of the sphere V_0):

$$\oint \text{grad } s_1 dS = - \int_{V_0} \chi dV' \oint \frac{\mathbf{R}}{R^3} dS = - 4\pi \int \chi dV'$$

whence it follows on the basis of Eq. (7.32) that

$$\nabla^2 s - \frac{1}{v^2} \frac{\partial^2 s}{\partial t^2} = - 4\pi \chi_p$$

Q.E.D.

We shall return to the unambiguity of the solution of the d'Alembert equation in the following section.

**7.6 Delayed and Advanced Potentials.
Gauge Invariance**

1. The results of the last section permit us to directly write integral expressions for the potentials of an electromagnetic field determined by the differential equations (7.19) and (7.20).

Comparing these equations with Eqs. (7.21) and (7.23), on the basis of Eq. (7.28) we get directly

$$\left. \begin{aligned} \varphi &= \frac{1}{\varepsilon} \int \frac{\rho \left(t - \frac{R}{v} \right)}{R} dV \\ A_x &= \frac{\mu}{c} \int \frac{j_x \left(t - \frac{R}{v} \right)}{R} dV \end{aligned} \right\} \quad (7.33)$$

and similar expressions for A_y and A_z so that the vector \mathbf{A} will be expressed by the formula

$$\mathbf{A} = \frac{\mu}{c} \int \frac{\mathbf{j} \left(t - \frac{R}{v} \right)}{R} dV \quad (7.34)$$

and by Eq. (7.22)

$$v = \frac{c}{\sqrt{\varepsilon\mu}}$$

Using the notation of Sec. 7.5, particularly Eq. (7.29), we can write Eqs. (7.33) and (7.34) as follows:

$$\varphi(x, y, z, t) = \frac{1}{\varepsilon} \int \frac{\rho(x', y', z', t')}{R} dV' \quad (7.35)$$

$$\mathbf{A}(x, y, z, t) = \frac{\mu}{c} \int \frac{\mathbf{j}(x', y', z', t')}{R} dV' \quad (7.36)$$

where dV' signifies the product $dx' dy' dz'$.

When ρ and \mathbf{j} are independent of time, i.e. in a stationary field, Eqs. (7.33) and (7.34), as should be expected, coincide with the previously derived equations (1.46) (for $\varepsilon = 1$) and (5.39).

2. Thus, to calculate, for example, the value of the scalar potential φ at the arbitrary point P at the moment t , it is necessary, according to Eq. (7.33), to divide the entire space into volume elements dV and for each element dV determine the charge

$$dq = \rho \left(t - \frac{R}{v} \right) dV$$

that was in it at the moment $t - \frac{R}{v}$, where R is the distance from dV to P . After next dividing this charge dq by ϵR and taking the sum of the expressions obtained over all the elements of the volume, we shall get φ . The value of \mathbf{A} is determined in a similar way.

Thus, the potentials of a varying field are determined absolutely similarly to the potentials of a stationary field. The only, but very significant addition here is that at each moment t the potential of the field induced at the distance R from the element of volume dV by the charges and currents of the element are determined not by the density of these charges and currents at the moment t , but by the preceding density of these charges and currents (at the moment $t - \frac{R}{v}$). We can thus say that the potentials φ and \mathbf{A} of the charges and currents of each element of volume dV spread from dV in all directions with the velocity $v = c/\sqrt{\epsilon\mu}$, their intensity diminishing inversely proportional to the distance R . This is why the quantities φ and \mathbf{A} determined by Eqs. (7.33) and (7.34) are called the *delayed potentials* of an electromagnetic field.

3. In our preceding treatment, we have meanwhile taken no account of Eq. (7.18) relating the possible values of φ and \mathbf{A} in a definite way. We can convince ourselves by direct calculations that our solutions (7.33) and (7.34) also satisfy this equation.

Equation (7.18) includes $\text{div } \mathbf{A}$. When calculating $\text{div } \mathbf{A} = \nabla \cdot \mathbf{A}$ we can reverse the sequence of differentiation with respect to x, y , and z and of integration with respect to x', y' , and z' :

$$\text{div } \mathbf{A} = \nabla \cdot \mathbf{A} = \frac{\mu}{c} \int \nabla \cdot \frac{\mathbf{j}(x', y', z', t')}{R} dV' \quad (7.37)$$

By Eq. (A.43₂), we have

$$\nabla \cdot \frac{\mathbf{j}(x', y', z', t')}{R} = \frac{1}{R} \cdot \nabla \cdot \mathbf{j}(x', y', z', t') + \mathbf{j}(x', y', z', t') \cdot \nabla \frac{1}{R}$$

Since the argument of the vector \mathbf{j} depends on x, y , and z only through the effective time t' , then

$$\nabla \cdot \mathbf{j}(x', y', z', t') = \frac{\partial j_x}{\partial t'} \frac{\partial t'}{\partial x} + \frac{\partial j_y}{\partial t'} \frac{\partial t'}{\partial y} + \frac{\partial j_z}{\partial t'} \frac{\partial t'}{\partial z} = \frac{\partial \mathbf{j}}{\partial t'} \cdot \nabla t'$$

Taking into account that by Eq. (7.29)

$$\nabla t' = -\frac{1}{v} \nabla R$$

we finally get

$$\nabla \cdot \frac{\mathbf{j}}{R} = \mathbf{j} \cdot \nabla \frac{1}{R} - \frac{1}{vR} \frac{\partial \mathbf{j}}{\partial t'} \cdot \nabla R \quad (7.38)$$

On the other hand, it is obvious that

$$\nabla' \frac{\mathbf{j}}{R} = \left(\nabla' \frac{\mathbf{j}}{R} \right)_{t'=\text{const}} + \frac{1}{R} \frac{\partial \mathbf{j}}{\partial t'} \cdot \nabla' t'$$

where ∇' , unlike ∇ , stands for differentiation with respect to the coordinates x' , y' , and z' . After performing simple transformations, we get

$$\nabla' \frac{\mathbf{j}}{R} = \mathbf{j} \cdot \nabla' \frac{1}{R} + \frac{1}{R} (\nabla' \mathbf{j})_{t'=\text{const}} - \frac{1}{vR} \frac{\partial \mathbf{j}}{\partial t'} \cdot \nabla' \mathbf{R} \quad (7.39)$$

Now taking into consideration that the equation $\text{grad}_a f(R) = -\text{grad}_q f(R)$ [see Eqs. (A.8) and (A.9)] when using our present notation becomes

$$\nabla' f(R) = -\nabla f(R)$$

and comparing Eqs. (7.38) and (7.39), we obtain

$$\nabla \frac{\mathbf{j}}{R} = -\nabla' \frac{\mathbf{j}}{R} + \frac{1}{R} (\nabla' \mathbf{j})_{t'=\text{const}}$$

Hence, denoting $\nabla' \mathbf{j}$ by $\text{div}' \mathbf{j}$, etc., we get the following equation from Eq. (7.37):

$$\text{div } \mathbf{A} = -\frac{\mu}{c} \int \text{div}' \frac{\mathbf{j}}{R} \cdot dV' + \frac{\mu}{c} \int \frac{dV'}{R} (\text{div}' \mathbf{j})_{t'=\text{const}}$$

The first of these integrals can be transformed with the aid of Gauss's theorem into an integral over the surface S bounding the volume V^* :

$$\int \text{div}' \frac{\mathbf{j}}{R} dV' = \oint \frac{j_n}{R} dS$$

If we extend integration over the entire infinite space, then this integral vanishes if only all the electric currents are concentrated in a finite region of space. Hence, we finally have

$$\text{div } \mathbf{A}(x, y, z, t) = \frac{\mu}{c} \int \frac{dV'}{R} (\text{div}' \mathbf{j}(x', y', z', t'))_{t'=\text{const}}$$

Let us now turn to the right-hand side of Eq. (7.35). In view of Eq. (7.29) for the arbitrary function $f(t')$, we have

$$\frac{\partial f(t)}{\partial t} = \frac{\partial f(t')}{\partial t'} \frac{\partial t'}{\partial t} = \frac{\partial f(t')}{\partial t'}$$

* This transformation cannot be directly done in the initial expression (7.37) because the volume integration and spatial differentiation (the formation of a divergence) in it are performed with respect to the coordinates of different points.

Therefore, differentiating Eq. (7.35) with respect to time inside the integral, we get

$$-\frac{\epsilon\mu}{c} \frac{\partial\varphi(x, y, z, t)}{\partial t} = -\frac{\mu}{c} \int \frac{dV'}{R} \frac{\partial\rho(x', y', z', t')}{dt'}$$

On the other hand, the continuity equation (IVa) can be written as follows:

$$\operatorname{div} \mathbf{j}(x, y, z, t)_{t=\text{const}} = -\frac{\partial\rho(x, y, z, t)}{\partial t}$$

because upon spatial differentiation (the formation of a divergence) the time t should be considered a constant parameter.

Substituting primed quantities for the unprimed ones in this equation (which is a simple change of the notation), we see that the delayed potentials do indeed satisfy Eq. (7.18)

$$\operatorname{div} \mathbf{A} = -\frac{\epsilon\mu}{c} \frac{\partial\varphi}{\partial t}$$

Q.E.D

4. Let us now turn to the question of the *unambiguity* of the solutions (7.33) and (7.34) of the system of equations (7.18)-(7.20) which we have found.

If we do not take any additional conditions into account, then these solutions are ambiguous. We can recall, for example, that when obtaining Eq. (7.28) we discarded the second term in the general solution (7.26) of Eq. (7.25).

If we, on the contrary, would retain only the second term of this general solution and would discard the first one, we could be able to repeat all the preceding calculations with a single difference consisting in the replacement everywhere of the argument $t - R/v$ with $t + R/v$. The result would be the solution of Eqs. (7.18)-(7.20) in the form of the so-called *advanced potentials* of an electromagnetic field:

$$\left. \begin{aligned} \varphi &= \frac{1}{\epsilon} \int \frac{\rho\left(t + \frac{R}{v}\right)}{R} dV \\ \mathbf{A} &= \frac{\mu}{c} \int \frac{\mathbf{j}\left(t + \frac{R}{v}\right)}{R} dV \end{aligned} \right\} \quad (7.40)$$

relating the value of the potential at the point P at the moment t to the space distribution of the charges and currents in the *following* moments of time $t + R/v$.

Further, since the non-homogeneous differential equations (7.18)-(7.20) determining the potentials φ and \mathbf{A} are *linear*, then the general solution of these equations can be presented in the form of the sum of an arbitrary partial solution of these non-homogeneous equations and the general solution of the corresponding homogeneous equations

$$\operatorname{div} \mathbf{A} = -\frac{c}{v^2} \frac{\partial \varphi}{\partial t}, \quad \nabla^2 \mathbf{A} = \frac{1}{v^2} \frac{\partial^2 \mathbf{A}}{\partial t^2}, \quad \nabla^2 \varphi = \frac{1}{v^2} \frac{\partial^2 \varphi}{\partial t^2} \quad (7.41)$$

Both the delayed potentials (7.33) and the advanced potentials (7.40) are different partial solutions of the non-homogeneous equations (7.18)-(7.20), therefore the general solution of these equations differs from them by the arbitrary solution of Eqs. (7.41).

Consequently, in particular, the advanced potentials themselves can be presented as the sum of the delayed potentials plus a certain solution of the homogeneous equations (7.41).

This ambiguity of the solution of the system of differential equations we are interested in can be eliminated in accordance with the known general rule only by giving definite *initial and boundary conditions*. Only the setting of these conditions separates the single solution corresponding to the given concrete posing of a physical problem from among an infinite combination of solutions of the system of differential equations.

We can show, for example, that the *general* solution of Eq. (7.20) for the internal space of an arbitrary volume V confined by the closed surface S can be written in the form

$$\begin{aligned} \varphi(t) = & \frac{1}{\epsilon} \int_V \frac{[\rho]}{R} dV + \frac{1}{4\pi\epsilon} \oint_S \left\{ \frac{1}{R} \left[\frac{\partial \varphi}{\partial n} \right] - [\varphi] \frac{\partial}{\partial n} \left(\frac{1}{R} \right) + \right. \\ & \left. + \frac{1}{vR} \frac{\partial R}{\partial n} \left[\frac{\partial \varphi}{\partial t} \right] \right\} dS \end{aligned} \quad (7.42)$$

where \mathbf{n} is an outward normal to S , while the brackets signify that the values of the quantities in them must be taken for the effective moment of time $t' = t - R/v$. In other words, if we know the values of the quantities in the right-hand part of Eq. (7.42), then this equation determines the values of φ at an arbitrary point of the volume V *unambiguously*. The same general solution of Eq. (7.20) can also be written, however, in a form differing from Eq. (7.42), first in the sign of the last term of the integrand of the surface integral and, second, in that the values of the quantities in brackets must be taken not for the *preceding* moment of time $t' = t - R/v$, but for the *following* moment $t' = t + R/v$. The first form of the solution corres-

ponds to delayed potentials, and the second to advanced ones*. For a stationary field, both solutions coincide (when $\varepsilon = 1$) with the previously proved formula (1.79).

Thus, the equations of an electromagnetic field, like those of mechanics, allow us to determine both the future according to the past and present, and the past according to the present and the future. More precisely, let us assume that we know the values of ρ in the volume V and the value of φ , $\partial\varphi/\partial n$, and $\partial\varphi/\partial t$ on the surface S of this volume within the time interval from t_1 to t_2 , and let us assume that we are interested in the field at the arbitrary point P of this volume V . Let R_2 be the distance from P to the remotest point of the surface S from it, and let R_1 be the smaller of the following two distances: (1) from P to the nearest point of the surface S , and (2) from P to the nearest point of the volume V from P at which the charge density ρ differed from zero at least during part of the time interval from t_1 to t_2 . Assume that $t_2 - t_1$ is greater than $(R_2 - R_1)/v$. Hence, using the delayed potentials and Eq. (7.42), we can determine the value of φ at the point P for any moment of the "future" interval of time from $t_1 + R_2/v$ to $t_2 + R_1/v$ ** . Using the advanced potentials and the corresponding modification of Eq. (7.42), we can determine φ at the point P in the "past" interval of time from $t_1 - R_1/v$ to $t_2 - R_2/v$.

5. The majority of the problems which we encounter in theoretical and experimental physics on one hand and in technical physics on the other require the use of *delayed*, and not advanced, potentials. Indeed, the problem usually consists in determining the field induced by a system of charges and currents, and this problem can be defined more precisely as follows (cf. the end of Sec. 7.3). Up to a certain moment t_0 , the field either was absent or was stationary, and the potentials φ and \mathbf{A} complied with the conditions (1.85):

$$R\varphi \text{ and } R^2 \text{ grad } \varphi \text{ remain finite when } R \rightarrow \infty$$

and with similar conditions for \mathbf{A} [Eq. (4.34)]. Next at the moment t_0 , varying currents appeared, the charges were displaced, etc. and it

* The proof of Eq. (7.42) for delayed potentials is given, for example, by H. Lorentz (Lorentz, H. A. *Theory of Electrons*, 2nd ed. New York, Dover Publications, 1952). The proof of the analogous formula for the advanced potentials is absolutely similar to that given by Lorentz. It is only necessary to replace

$$F\left(t + \frac{R}{c}\right) \text{ in his calculations with } F\left(t - \frac{R}{c}\right).$$

** Indeed, for this determination, it is necessary, particularly, to know the values of the quantities φ , $\partial\varphi/\partial n$, and $\partial\varphi/\partial t$ for each point of the surface S during

the time interval from $t_1 + \frac{R_2}{v} - \frac{R}{v}$ to $t_2 + \frac{R_1}{v} - \frac{R}{v}$, where R is the distance to this point of the surface from the point P . Since $R_1 \leq R \leq R_2$ and $R_2 - R_1 < v(t_2 - t_1)$, then this entire time interval is confined between t_1 and t_2 .

is necessary to determine their field at the moment of time $t > t_0$. In these conditions, this field is determined by the expressions (7.33) and (7.34) for the *delayed* potentials.

Indeed, when determining the value of the potential φ at the moment t by Eq. (7.42), we can remove the surface S to such a great distance R from the point P of the field being studied that R will comply with the inequality

$$t - \frac{R}{v} < t_0$$

In this case, the quantities $[\partial\varphi/\partial n]$, $[\varphi]$, and $[\partial\varphi/\partial t]$ in Eq. (7.42) should be given the values which they had before the moment t_0 when the field appeared. As a result, the entire surface integral in Eq. (7.42) will vanish.

Thus, in the conditions indicated, the solution of Eq. (7.19) is determined *unambiguously* and is expressed by the first term of the general solution (7.42) coinciding with our formula (7.33). Similarly, in these conditions, the value of the vector potential \mathbf{A} is also unambiguously determined by Eq. (7.34).

In the following in accordance with the nature of the problems we shall consider, we shall have to do only with delayed potentials and not with advanced ones.

6. In our preceding treatment, we gave little attention to the fact that the scalar and the vector potentials of a field are only auxiliary concepts, and that only the *intensities* of an electric and a magnetic field \mathbf{E} and \mathbf{H} have a direct physical meaning. We know that the energy of a field, the ponderomotive forces, current densities, etc. are unambiguously determined by the field intensities \mathbf{E} and \mathbf{H} (with preset values of ϵ and μ). Therefore, two fields described by the same values of \mathbf{E} and \mathbf{H} , but with different values of the potentials φ and \mathbf{A} are physically identical. Then what is the arbitrariness in determining the potentials φ and \mathbf{A} at given intensities \mathbf{E} and \mathbf{H} (or \mathbf{E} and \mathbf{B})_m?

Let φ and \mathbf{A} satisfy Eqs. (7.15) and (7.16):

$$\mathbf{B} = \text{curl } \mathbf{A} \quad \text{and} \quad \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \text{grad } \varphi$$

Since the curl of a gradient is identically equal to zero, then if we add to \mathbf{A} the gradient of an arbitrary scalar χ ,

$$\mathbf{A}' = \mathbf{A} + \text{grad } \chi \tag{7.43}$$

the previous value of the magnetic induction will correspond to the new value of the vector potential \mathbf{A}' :

$$\text{curl } \mathbf{A}' = \text{curl } \mathbf{A} = \mathbf{B}$$

If χ is independent of time, then the value of the electric intensity \mathbf{E} will not change when \mathbf{A}' is substituted for \mathbf{A} . If χ does depend on time,

then the value of \mathbf{E} remains unchanged only provided that we also substitute φ' for φ together with our substitution of \mathbf{A}' for \mathbf{A} , and

$$\varphi' = \varphi - \frac{1}{c} \frac{\partial \chi}{\partial t} \quad (7.44)$$

Indeed, with this condition, we have

$$-\frac{1}{c} \frac{\partial \mathbf{A}'}{\partial t} - \text{grad } \varphi' = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \text{grad } \varphi = \mathbf{E}$$

Thus, the intensity and induction of a field remain constant upon the simultaneous addition to the vector potential of the gradient of an arbitrary scalar and subtraction from the scalar potential of the derivative of the same scalar with respect to time divided by c . The invariance of a field with respect to this class of transformations of the potentials is called *gauge invariance*. Particularly, if χ is independent of the coordinates, then the gauge invariance consists in the possibility pointed out in Sec. 1.8 of adding to the scalar potential an arbitrary additive constant (that can depend on time).

Previously, we used a definite gauging or normalization of the potentials, i.e. eliminated arbitrariness in determining the potentials of a field by additional requirements. For example, we imposed the requirement that $\text{div } \mathbf{A} = 0$ and the conditions for infinity (4.34) on the vector potential of a constant magnetic field, and we imposed the condition (7.18) on the potentials of a varying field. Only in these conditions [\mathbf{A} is expressed by the integral of Eq. (4.28) with a constant magnetic field] do the potentials of a varying field satisfy the d'Alembert equations (7.19) and (7.20), etc. Because of the gauge invariance, however, these conditions are not at all obligatory. Moreover, the solution of individual concrete problems is often facilitated by special gauging of the potentials that is expedient for the given problem but differs from the gauging adopted in this book.

The requirement of the gauge invariance of equations of theoretical physics, i.e. the requirement that the physical content of these equations depend only on the intensity of the electromagnetic field and remain constant upon all the transformations of the field potentials by means of Eqs. (7.43) and (7.44) plays an important part in the electron and quantum theories.

7.7 Velocity of Propagation of Electromagnetic Disturbances. Conditions for a Quasistationary State

1. The physical content of Eqs. (7.33) and (7.34) determining the values of the delayed potentials, and of Eqs. (7.15) and (7.16) establishing the relationship between these potentials and the field intensity

consists in the following: an electromagnetic field is induced by charges and conduction currents and propagates from the place of its induction with a finite velocity $v=c/\sqrt{\epsilon\mu}$. It thus follows from the cited equations, particularly, that the displacement currents, which play such an important part in the mechanism of propagation of a field, cannot exist independently of the motion of charges. Further, in a vacuum ($\epsilon = \mu = 1$), the velocity of propagation of a field must equal c , i.e. must equal numerically the value of the electromagnetic constant determining the force of ponderomotive interaction of currents and indeed having the dimension of velocity [see Eq. (4.9)].

It is exactly this acknowledgment of the *finite velocity of propagation of a field* that contains the most important and *fundamental distinction of the actual content of the so-called theories of short-range action*, and first of all Maxwell's theory, from the *theories of instantaneous long-range action* of the beginning of the nineteenth century. Therefore, the question of the correctness of one of these theories in principle can be solved, for example, by staging the following very simple *experimentum crucis*.

Let the charges q and q' be at rest at a distance of R from each other up to the moment $t = 0$ after which we shall begin to remove q' from q . According to the theory of instantaneous long-range action, the force of attraction $F = qq'/R^2$ exerted on the charge q by the charge q' should begin to diminish *at the same moment* $t=0$. According to Maxwell's equations, this force must remain constant up to the moment $t = R/c$ (we assume that the charges are in a vacuum) because the force F experienced by the charge q , which is at the point P , is determined by the field intensity *at the same point* P ($\mathbf{F} = q\mathbf{E}$). By displacing the charge q' , which is at the point P' , we *directly* change the field only at this point P' . This *change* in the electric field at P' (a displacement current), according to Eq. (I), induces a magnetic field *at adjacent points* of space*, which, in turn, according to Eq. (II), causes the appearance of a curl of the vector \mathbf{E} , i.e. a change in this vector (because in a static field $\text{curl } \mathbf{E} = 0$ when $t < 0$) again only at *adjacent points* of space, etc. As a result, the change, or as is said, the *disturbance of the field* induced at P' by displacement of the charge q' , will propagate to the point P in R/c s, and only at this moment will it cause a change in the force F acting on the charge q .

We are now not at all interested in the circumstance that for practical and technical reasons this very simple experiment is doomed to failure. It is only important that the question of the velocity of propagation of a field can be solved *experimentally* and that it was actually solved in favour of a finite value of this velocity.

* Naturally, in addition to the change in the electric field, the motion of the charge q' also directly induces near the point P' a magnetic field that also propagates to a remote distance.

Moreover, the corollaries following from Maxwell's theory of a field which we have set out, and particularly Eq. (7.22), can also be verified quantitatively because the value of the constant c can be determined by independent electrodynamic measurements (see Secs. 4.2 and 4.18). The most probable value of this constant is (see Sec. 4.18)

$$c = (2.9978 \pm 0.0001) \times 10^{10} \text{ cm/s}$$

which within the limits of experimental errors does indeed coincide with the directly measured velocity of propagation of a field in a vacuum. As shown by an analysis of the most accurate measurements performed up to 1973, the velocity of light in a vacuum* is

$$c = [2.997\,924\,58(1.2)] \times 10^{10} \text{ cm/s}$$

The numerical coincidence of the above values is a convincing proof of both the electromagnetic nature of light and of the correctness of Maxwell's equations, at least when applied to a vacuum.

2. We shall return to all these important questions on a later page, meanwhile we shall use the results obtained to substantiate the conditions for the quasistationary varying field listed in Sec. 6.3. We can now give a more exact definition of the quasistationary state as follows: by a quasistationary field is meant a field whose potentials φ and \mathbf{A} and *magnetic intensity* \mathbf{H} at each given moment t with sufficient accuracy coincide with the corresponding quantities in the field of *stationary charges and steady currents* whose densities ρ and \mathbf{j} equal the *instantaneous* (at the moment t) densities of the varying charges and currents being considered.

From a consideration of the expressions for the delayed potentials (7.33) and (7.34), we see that the following condition must be observed first of all for a quasistationary state: during the time $t = R/v$ needed for propagation of the electromagnetic field over the distance R , the values of the quantities ρ and \mathbf{j} should undergo only vanishingly small changes. By R here is meant the distance from the point being considered to the most remote element of the charge or current.

The value of R can evidently always be chosen so great that at even very slow changes of the field this condition is not observed. In the theory of quasistationary currents, however, we usually limit ourselves to a consideration of the field in the direct proximity to charged or current-carrying conductors. In this case, the upper limit of the distance R can be assumed to equal the distance l between the two elements of charges or currents that are the most remote from each other and whose field we still want to take into account. If, in addition, we have to do with periodic currents having the period T , then the condition for a quasistationary state will obviously consist

* The velocity of light in air is measured directly. The value of the velocity of light in a vacuum is determined from the relationship $v = c/\sqrt{\epsilon}$, where ϵ is the permittivity of air equal to 1.000 58.

in the inequality

$$\tau = \frac{l}{v} \ll T$$

or since v is comparable with c , in the inequality

$$\frac{l}{c} \ll T \tag{7.45}$$

If this condition of a quasistationary state is observed, then the value of the field *potentials* near the charges and currents, i.e. at distances from them complying with the requirement

$$\frac{R}{c} \ll T \tag{7.46}$$

can be determined with sufficient accuracy by Eqs. (1.46) and (5.39) derived for a stationary field (and not taking the displacement currents into consideration).

As regards the *electric field intensity* \mathbf{E} , then even with the same *instantaneous* value of the potentials φ and \mathbf{A} the vector \mathbf{E} of a varying quasistationary field (unlike the vector \mathbf{H}) may appreciably differ from the vector \mathbf{E} of the corresponding constant field (induction phenomena!). This difference is formally connected with the fact that the expression for \mathbf{E} [Eq. (7.16)] includes the *derivative* of the vector potential \mathbf{A} with respect to time.

3. The doubt may appear whether, for example with respect to the vector \mathbf{H} , the identity of the relationship between \mathbf{H} and \mathbf{A} guarantees the same value of \mathbf{H} in a quasistationary field and in a constant field characterized by the same instantaneous value of the potential \mathbf{A} . We know that even when taking spatial derivatives of φ and \mathbf{A} their time dependence may play a significant part. For instance, if in the expression of the derivative

$$\begin{aligned} \frac{\partial \varphi \left(t - \frac{R}{c} \right)}{\partial x} &= \left[\frac{\partial \varphi \left(t - \frac{R}{c} \right)}{\partial x} \right]_{t - \frac{R}{c} = \text{const}} - \\ &- \frac{x}{cR} \frac{\partial \varphi \left(t - \frac{R}{c} \right)}{\partial t} \end{aligned} \tag{7.47}$$

we substitute t for $t - R/c$ in the argument of the function φ , the expression obtained will nevertheless differ from $\partial \varphi(t)/\partial x$ in the second term.

This difference is not appreciable, however, when the conditions (7.45) and (7.46) are observed. For a periodic field as regards the order of magnitude, $\frac{1}{c} \frac{\partial \varphi}{\partial t}$ is comparable with $\frac{\varphi}{cT}$. Hence, in view of expression (7.45), we have $\frac{1}{c} \frac{\partial \varphi}{\partial t} \ll \frac{\varphi}{l}$. On the other hand, the potential

φ (or at least the time-dependent component of this potential) changes along the system by a value comparable with this potential φ itself (particularly, the variable component of φ in general changes its sign along the system). Hence, as regards its order of magnitude, the quantity $\partial\varphi/\partial x$ approximately equals φ/l so that near the system the second term of Eq. (7.47) is indeed small in comparison with the first one, Q.E.D.

Further, at great distances from the system in comparison with its dimensions, we have $[\partial\varphi/\partial x] \sim \varphi/R$, while $\frac{1}{c} \frac{\partial\varphi}{\partial t} \sim \frac{\varphi}{cT}$. Therefore, as long as the condition (7.46) is observed, the second term of Eq. (7.47) remains small in comparison with the first one, whereas at distances of $R \gg cT$ in the so-called wave zone (see Sec. 7.9), on the contrary, the first term is vanishingly small in comparison with the second one.

In Sec. 7.9 when considering the field of an oscillator we shall conduct the proof given here quite strictly.

4. The condition (7.45) alone, however, is insufficient for the application of all the laws of a stationary magnetic field to the *magnetic* field of quasistationary currents because in deriving these laws the *closed nature* of the steady currents is of a considerable significance. Thus, a field is "completely" quasistationary only when both the above "basic" and the "additional" conditions of a quasistationary state are observed simultaneously. These conditions consist in the requirement that the density of the displacement currents be so small in comparison with that of the conduction currents that the latter can be considered as closed (see Sec. 6.13).

7.8 Oscillator.

Delayed Potentials of an Oscillator Field

1. We saw in Sec. 2.1 that the field of an arbitrarily complicated, but as a whole neutral system of fixed electric charges is expressed very simply for great distances from the system with the aid of the electric moment vector \mathbf{p} of this system. We shall now reason similarly for the field of a neutral system of *moving* charges, limiting ourselves for simplicity to system in a vacuum ($\epsilon = \mu = 1$).

Assume that the charges of our system are inside a certain volume V and do not leave the confines of this volume. Let l signify the linear dimensions of the volume V (for instance the distance between two of its points that are the most remote from each other). Let us choose within the volume V an arbitrary point O which we shall conditionally call the *centre* of our system of charges. Let \mathbf{R}_0 , finally, be the radius-vector conducted from O to the point of observation P . We shall limit ourselves to the consideration of only three points of the field,

the distance from which to the system is considerably greater than its dimensions l :

$$R_0 \gg l \tag{7.48}$$

If \mathbf{R} as previously stands for a radius-vector conducted from an arbitrary point $Q(x'y'z')$ of our system to the point P , then

$$\mathbf{R} = \mathbf{R}_0 - \mathbf{R}'$$

where \mathbf{R}' is the distance from Q to the centre of the system O (see Fig. 27 in Sec. 2.1 in which, however, \mathbf{R}_i corresponds to what is now \mathbf{R}' and \mathbf{R}'_i corresponds to \mathbf{R}), and it is obvious that $R' \leq l \ll R_0$. Disregarding the second and higher powers of R' , we get

$$R = \sqrt{R_0^2 - 2\mathbf{R}_0\mathbf{R}' + R'^2} = R_0 - \frac{\mathbf{R}_0\mathbf{R}'}{R_0}$$

2. Let us first consider the scalar potential of our system of charges. The integrand in the right-hand side of Eq. (7.35) contains a function of the distance R :

$$\frac{\rho\left(x', y', z', t - \frac{R}{c}\right)}{R}$$

Expressing R in it through $R_0 - \mathbf{R}'\mathbf{R}_0/R_0$ and expanding into a Taylor series, we get

$$\begin{aligned} & \frac{\rho\left(x', y', z', t - \frac{R}{c}\right)}{R} = \\ & = \frac{\rho\left(x', y', z', t - \frac{R_0}{c}\right)}{R_0} - \\ & - \frac{\mathbf{R}'\mathbf{R}_0}{R_0} \frac{\partial}{\partial R_0} \frac{\rho\left(x', y', z', t - \frac{R_0}{c}\right)}{R_0} + \dots \end{aligned} \tag{7.49}$$

If the dimensions of the system l and also the radius-vector \mathbf{R}' are sufficiently small, then in the first approximation we can limit ourselves to the above two terms of the expansion, discarding the term with higher powers of \mathbf{R}' . Leaving the consideration of the

conditions when this approximation is lawful to the end of the present section and using Eq. (7.49) in Eq. (7.35), we get (with $\varepsilon = 1$)

$$\varphi(x, y, z, t) = \int \frac{\rho\left(x', y', z', t - \frac{R_0}{c}\right)}{R_0} dV' - \int \frac{\mathbf{R}\mathbf{R}_0}{R_0} \frac{\partial}{\partial R_0} \frac{\rho\left(x', y', z', t - \frac{R_0}{c}\right)}{R_0} dV'$$

In the first of these integrals, we can put R_0 , which does not depend on the position of the point $Q(x', y', z')$, outside the integral. Since the integral

$$\int \rho\left(x', y', z', t - \frac{R_0}{c}\right) dV'$$

equals the total charge of the system at the moment $t - R_0/c$, then owing to the presumed neutrality of the system it vanishes, and together with it the entire first term of the expression for φ .

The second integral, owing to R_0 being independent of x', y' , and z' , can be written as follows:

$$- \frac{\mathbf{R}_0}{R_0} \frac{\partial}{\partial R_0} \left\{ \frac{1}{R_0} \int \mathbf{R}' \cdot \rho\left(x', y', z', t - \frac{R_0}{c}\right) \cdot dV' \right\}$$

The integral

$$\int \mathbf{R}' \rho\left(x', y', z', t - \frac{R_0}{c}\right) dV' = \mathbf{p}\left(t - \frac{R_0}{c}\right) \quad (7.50)$$

is evidently the value of the vector of the electric moment of the system \mathbf{p} at the time $t - R_0/c$ [see Eq. (2.1) from which Eq. (7.50) differs only in the transition from a system of point charges to space charges].

Thus, the scalar potential equals

$$\varphi = - \frac{\mathbf{R}}{R} \frac{\partial}{\partial R} \left\{ \frac{\mathbf{p}\left(t - \frac{R}{c}\right)}{R} \right\} \quad (7.51)$$

(we have discarded the subscript 0 of R_0). This equation can be written in the following final form:

$$\varphi = - \operatorname{div}_a \left[\frac{\mathbf{p}\left(t - \frac{R}{c}\right)}{R} \right] \quad (7.52)$$

where the subscript a of the divergence symbol signifies differentiation with respect to the point of observation P . Indeed, on the basis of Eqs. (A.43₂) and (A.10), we have

$$\begin{aligned} \operatorname{div}_a \left\{ \frac{\mathbf{p} \left(t - \frac{R}{c} \right)}{R} \right\} &= \mathbf{p} \operatorname{grad}_a \frac{1}{R} + \frac{1}{R} \operatorname{div} \mathbf{p} = \\ &= -\frac{\mathbf{p}\mathbf{R}}{R^3} + \frac{1}{R} \operatorname{div} \mathbf{p} \end{aligned} \quad (7.53)$$

Further, since the argument of \mathbf{p} depends on the coordinates only through R , then

$$\begin{aligned} \operatorname{div} \mathbf{p} &= \frac{\partial p_x}{\partial x} + \frac{\partial p_y}{\partial y} + \frac{\partial p_z}{\partial z} = \\ &= \frac{x}{R} \frac{\partial p_x}{\partial R} + \frac{y}{R} \frac{\partial p_y}{\partial R} + \frac{z}{R} \frac{\partial p_z}{\partial R} = \frac{\mathbf{R}}{R} \cdot \frac{\partial \mathbf{p}}{\partial R} \end{aligned} \quad (7.54)$$

Introducing Eqs. (7.53) and (7.54) into Eq. (7.52), we see that it coincides with Eq. (7.51).

We can understand by R in Eq. (7.52) the distance from the point of observation to an *arbitrary* point of the volume V occupied by our system of charges because the choice of the position of the “centre” of the system O inside this volume was not restricted above by any condition. This is also obvious directly because of the small value of the dimension l of this volume in comparison with R indicated above. It is significant that the value of the vector of the electric moment \mathbf{p} of an arbitrary neutral system of charges also does not depend on the choice of the point O (we have already proved this in Sec. 2.1).

When \mathbf{p} is independent of time, Eq. (7.52) with the aid of Eq. (A.43₂) can be transformed so that it will coincide with the previous formula (1.50) determining the potential of a static dipole.

3. Let us now pass over to the vector potential \mathbf{A} expressed by Eq. (7.36). Expanding the integrand of this formula, similar to Eq. (7.49), into a Taylor series,

$$\begin{aligned} \frac{\mathbf{j} \left(x', y', z', t - \frac{R}{c} \right)}{R} &= \frac{\mathbf{j} \left(x', y', z', t - \frac{R_0}{c} \right)}{R_0} - \\ &- \frac{\mathbf{R}\mathbf{R}_0}{R_0} \frac{\partial}{\partial R_0} \frac{\mathbf{j} \left(x', y', z', t - \frac{R_0}{c} \right)}{R_0} + \dots \end{aligned}$$

we may limit ourselves to only the first term of this expansion [for the scalar potential φ we had to retain the second term of the

expansion because the integral of the first term of Eq. (7.49) vanished]. Hence, Eq. (7.36) becomes (with $\mu = 1$)

$$\begin{aligned} \mathbf{A} &= \frac{1}{c} \int \frac{\mathbf{j} \left(x', y', z', t - \frac{R}{c} \right)}{R} dV' = \\ &= \frac{1}{cR_0} \int \mathbf{j} \left(x', y', z', t - \frac{R_0}{c} \right) dV' \end{aligned} \quad (7.55)$$

For *current loops*, the integral $\int \mathbf{j} dV'$ taken over the entire volume of the currents identically equals zero [see the proof of Eq. (4.135) following from condition (4.136) in Sec. 4.16]. Hence, in the *approximation being considered both the vector potential and the magnetic field of the system equal zero*. For *unclosed varying currents*, this integral differs from zero and, as we shall now show, equals the derivative of the electric moment \mathbf{p} of the system with respect to time.

Differentiating Eq. (7.50) with respect to time and using the continuity equation (IVa),

$$\begin{aligned} \frac{\partial}{\partial t} \rho \left(x', y', z', t - \frac{R_0}{c} \right) &= \\ &= - \operatorname{div}' \mathbf{j} \left(x', y', z', t - \frac{R_0}{c} \right) \end{aligned}$$

where div' signifies differentiation with respect to the coordinates x', y' , and z' of the point Q (we remind our reader that R_0 does not depend on these coordinates), we get

$$\frac{\partial}{\partial t} \mathbf{p} \left(t - \frac{R_0}{c} \right) = - \int \mathbf{R}' \operatorname{div}' \mathbf{j} \left(t - \frac{R_0}{c} \right) dV'$$

Let us multiply this equation by an arbitrary unit constant vector \mathbf{a} ($a = 1$). We can now transform the integrand with the aid of Eqs. (A.43₂) and (A.11):

$$\begin{aligned} (\mathbf{aR}') \operatorname{div}' \mathbf{j} &= \operatorname{div}' \{ \mathbf{j}(\mathbf{aR}') \} - \mathbf{j} \operatorname{grad}'(\mathbf{aR}') = \\ &= \operatorname{div}' \{ \mathbf{j}(\mathbf{aR}') \} - \mathbf{a} \mathbf{j} \end{aligned}$$

Thus,

$$\mathbf{a} \frac{\partial \mathbf{p} \left(t - \frac{R_0}{c} \right)}{\partial t} = - \int \operatorname{div}' \{ \mathbf{j}(\mathbf{aR}') \} dV' + \mathbf{a} \int \mathbf{j} \left(t - \frac{R_0}{c} \right) dV'$$

Gauss's theorem can be used to transform the first integral into one over the surface S confining the volume V . Since all the electric

charges, according to our condition, are inside the volume V , then no currents flow through its boundary surface S , i.e. $\mathbf{j} = 0$ on it. Hence,

$$\int \operatorname{div} \{(\mathbf{aR}') \mathbf{j}\} dV' = \oint (\mathbf{aR}') j_n dS = 0$$

Since the next-to-last equation holds with any selection of the direction of the constant vector \mathbf{a} , we finally get

$$\frac{\partial}{\partial t} \mathbf{p} \left(t - \frac{R_0}{c} \right) = \int \mathbf{j} \left(x', y', z', t - \frac{R_0}{c} \right) dV' \tag{7.56}$$

Using this expression in Eq. (7.55) and omitting the subscript of R_0 , we get an expression for the vector potential

$$\mathbf{A} = \frac{1}{cR} \frac{\partial}{\partial t} \mathbf{p} \left(t - \frac{R}{c} \right) \tag{7.57}$$

4. Thus, both the scalar and the vector potentials of an arbitrary neutral system at great distances from it are unambiguously determined by the vector \mathbf{p} of the electric moment of this system. A conventional dipole—a combination of two point charges of opposite signs—is the simplest system of this kind. Unlike a static dipole, a dipole whose moment \mathbf{p} changes with time is often called an *oscillator* or a vibrator. Thus, *the field of a neutral system of charges at great distances from it coincides with the field of an oscillator whose moment equals that of the system.* Owing to this circumstance, the studying of the field of an oscillator plays a very important part in the theory of electricity. In certain conditions which we shall now consider, a radio telegraph aerial can be replaced with an equivalent oscillator, a luminous body—with a combination of elementary oscillators, etc.

5. The formulas of this section may be applied only if in expansion (7.49) and in a similar expansion of the integrand of the formula for \mathbf{A} we can disregard the following terms of the expansion in comparison with the preceding ones. Let us now consider the conditions in which this approximation is lawful. It will be sufficient to limit ourselves to the case when the vectors \mathbf{R}_0 and \mathbf{R}' have the same direction. Here, $R = R_0 - R'$, and the expansion (7.49) acquires a simpler form:

$$\begin{aligned} \frac{\rho \left(t - \frac{R}{c} \right)}{R} &= \frac{\rho \left(t - \frac{R_0}{c} \right)}{R_0} - R' \frac{\partial}{\partial R_0} \frac{\rho \left(t - \frac{R_0}{c} \right)}{R_0} + \\ &+ \frac{1}{2} R'^2 \frac{\partial^2}{\partial R_0^2} \frac{\rho \left(t - \frac{R_0}{c} \right)}{R_0} \dots \end{aligned}$$

where

$$\frac{\partial}{\partial R_0} \rho \left(t - \frac{R_0}{c} \right) = - \frac{\rho \left(t - \frac{R_0}{c} \right)}{R_0^2} - \frac{1}{c R_0} \frac{\partial}{\partial t} \rho \left(t - \frac{R_0}{c} \right)$$

Differentiation of multipliers such as $1/R_0^n$ increases the power of R_0 in the denominator so that when $R' \ll R_0$ we can disregard with certainty the following terms obtained in the differentiation of these multipliers. Thus, the entire question of the validity of our simplifications, as can easily be seen, consists in the relative value of terms such as

$$\frac{R'^n}{c^n} \frac{\partial^n}{\partial t^n} \rho \left(t - \frac{R_0}{c} \right)$$

Let us assume for simplicity that ρ is a periodic function of time:

$$\rho \left(x', y', z', t - \frac{R_0}{c} \right) = f(x', y', z') \cos \omega \left(t - \frac{R_0}{c} \right)$$

Hence, as regards the order of magnitude, differentiation of ρ with respect to t will consist in multiplying ρ by ω , while the ratio of the consecutive terms of the expansion being considered with respect to the order of magnitude will equal $R'\omega/c$. Thus, we can ignore the following terms of the expansion in comparison with the preceding ones provided that $R'\omega/c \ll 1$. Since $R' \leq l$ and $\omega = 2\pi/T$, where T is the period of the oscillations, then this condition is equivalent to the condition that

$$\frac{l}{cT} \ll 1 \tag{7.58}$$

that coincides with expression (7.45). Thus, the applicability of the formulas of the present section is restricted, first, by condition (7.48) requiring that the regions of the field being considered be at a sufficiently remote distance from the system of charges and currents inducing this field, and, second, by the requirement that this system comply with the fundamental condition for the quasistationary state (7.45). Naturally, with this condition, the field of the system will be quasistationary only in direct proximity from it, but not far away.

6. Let us consider in conclusion a very simple system equivalent to an oscillator.

From the viewpoint of the electron theory, the simplest form of an oscillator is a combination of one electron and one proton, the distance between which periodically changes with time. In Maxwell's macroscopic theory of the field, the simplest model of an oscillator is the so-called *Hertzian oscillator*, two metal spheres K and K' whose charges q and q' at each given moment are equal in magnitude

and opposite in sign ($q' = -q$) are connected by a conductor having the length l (Fig. 80). If we consider the vector \mathbf{l} to be directed from K' to K , then the moment of the dipole formed by these charges will be

$$\mathbf{p} = q\mathbf{l}$$

because, by definition, the vector \mathbf{p} is directed from the negative charge to the positive one (the direction of \mathbf{p} coincides with \mathbf{l} when $q > 0$ and is opposite to it when $q < 0$). If, after charging the oscillator,

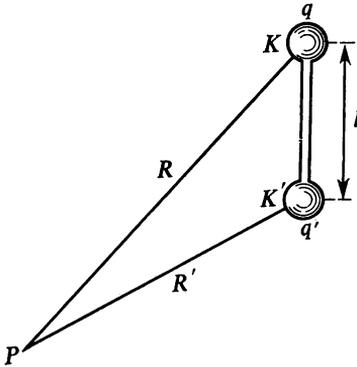


Fig. 80

we leave it to itself, then damping electric oscillations will appear in it similar to the oscillations in a circuit with a capacitor treated in Sec. 6.14. These oscillations will be attended by periodic recharging of the spheres, i.e. by periodic changing of their charges in magnitude and in sign. In this case, the moment of the oscillator \mathbf{p} can be assumed equal to

$$\mathbf{p} = \mathbf{p}_0 f(t)$$

where \mathbf{p}_0 = constant vector directed along the oscillator axis
 $f(t)$ = periodic attenuating function of time.

The current in the oscillator obviously equals the rate of change of the values of the charges q and q' . If we agree to consider the current positive when its direction coincides with the vector \mathbf{l} , then

$$I = \frac{\partial q}{\partial t}$$

whence

$$\mathbf{\Pi} = \frac{\partial q}{\partial t} \mathbf{l} = \frac{\partial \mathbf{p}}{\partial t} \tag{7.59}$$

This equation is a particular case of the general formula (7.56) as applied to a system that we can approximately consider similar

to a length of a straight current, and can easily be obtained from Eq. (7.56) with the aid of Eq. (4.11):

$$\mathbf{j} dV = dI ds$$

7. If the electric moment \mathbf{p} of the system equals zero or is constant in time, then the field of radiation of the system is determined by the senior terms of the expansion of the quantities ρ/R and \mathbf{j}/R in R'/R_0 which we have previously discarded. Let us consider as a very simple example a system of varying but closed currents, i.e. varying currents complying with the condition $\text{div } \mathbf{j} = 0$.

According to the continuity equation (IVa), in this case $\partial\rho/\partial t = 0$, i.e. the distribution of the charges and, consequently, the electric moment of the system \mathbf{p} are constant in time. The scalar potential of the field of the system is also constant and will not interest us. As regards the vector potential of the field of the system, then, limiting ourselves in the expansion of \mathbf{j}/R in R'/R_0 [see Eq. (7.53)] to the first of the terms whose integral differs from zero when $\mathbf{p} = 0$, we obtain

$$\mathbf{A} = -\frac{1}{c} \int \frac{\mathbf{R}'\mathbf{R}_0}{R_0} \frac{\partial}{\partial R_0} \left\{ \frac{\mathbf{j} \left(x', y', z', t - \frac{R_0}{c} \right)}{R_0} \right\} dV' \quad (7.60)$$

Differentiation with respect to R_0 yields

$$\mathbf{A} = \frac{1}{c} \int \frac{\mathbf{R}'\mathbf{R}_0}{R_0^3} \left\{ \mathbf{j} \left(t - \frac{R_0}{c} \right) + \frac{R_0}{c} \frac{\partial}{\partial t} \mathbf{j} \left(t - \frac{R_0}{c} \right) \right\} dV' \quad (7.61)$$

The integral of the first term of the sum coincides in form with the second term of Eq. (4.130). It was shown in Sec. 4.16 that if in Eq. (4.130) we extend integration over the entire volume of the currents [i.e. comply with the condition (4.136)], then the first term of this equation vanishes, while the second one, according to Eq. (4.137), is expressed through the magnetic moment of the system of currents. Accordingly, the integral of the first term of the sum in Eq. (7.61) is*

$$\frac{\left[\boldsymbol{\mu} \left(t - \frac{R_0}{c} \right) \cdot \mathbf{R}_0 \right]}{R_0^3}$$

where $\boldsymbol{\mu}$ stands for the magnetic moment of the system [see Eq. (4.133)]:

$$\boldsymbol{\mu} = \frac{1}{2c} \int [\mathbf{R}' \cdot \mathbf{j}] dV' \quad (7.62)$$

* It was assumed in Sec. 4.16 that the currents are steady and that consequently $\text{div } \mathbf{j} = 0$. We assumed above that the varying currents which we are considering also comply with this condition. If, however, $\text{div } \mathbf{j} \neq 0$, then upon the transformation of the integral in Eq. (7.61), additional terms corresponding to the so-called quadrupole electric moment of the system will appear in the expression for \mathbf{A} apart from Eq. (7.63).

The second term of the integrand in Eq. (7.61) is obtained from the first by differentiation with respect to t and multiplying by R_0/c . Therefore, discarding the subscript 0 of R_0 , we finally get

$$\mathbf{A} = \frac{\left[\boldsymbol{\mu} \left(t - \frac{R}{c} \right) \cdot \mathbf{R} \right]}{R^3} + \frac{1}{cR^2} \left[\frac{\partial}{\partial t} \boldsymbol{\mu} \left(t - \frac{R}{c} \right) \cdot \mathbf{R} \right] \quad (7.63)$$

Since the scalar potential of the system being considered equals zero, then not only the magnetic, but also the electric field of the system is expressed through the vector potential \mathbf{A} , i.e. is determined by the magnetic moment $\boldsymbol{\mu}$ of the system. Such a system is called a *magnetic dipole* or a magnetic oscillator. A very simple system equivalent to such an oscillator is a closed wire contour in which a varying current is induced. Unlike a linear unclosed Hertzian oscillator, such a contour is called a loop in radio engineering.

If a system has both a varying electric and a varying magnetic moments, the field of the latter at great distances may usually be disregarded in comparison with the field of the electric moment. Indeed, expressing the current density \mathbf{j} in Eq. (7.62) through the velocity \mathbf{v} and the density ρ of the electric charges, i.e. $\mathbf{j} = \rho\mathbf{v}$, we get

$$\boldsymbol{\mu} = \frac{1}{2c} \int \rho [\mathbf{R}'\mathbf{v}] dV'$$

whereas the electric moment of the system is

$$\mathbf{p} = \int \rho \mathbf{R}' dV'$$

Consequently, if only because of special features of a system its electric moment is not very small, then with respect to the order of magnitude $\mu \sim \frac{v}{c} p$. Since even the true (and not only the mean) velocity of the electrons v is much smaller than c in the majority of cases, then we also have $\mu \ll p$.

7.9 Field of an Oscillator. Its Radiation

1. For our further study of the field of an oscillator, it is convenient to introduce the so-called *Hertz vector* determined by the equation

$$\mathbf{P}(t, R) = \frac{\mathbf{p} \left(t - \frac{R}{c} \right)}{R} \quad (7.64)$$

As follows from this equation, the value of the Hertz vector at the moment t for a point at the distance R from an oscillator is

determined by the value of the electric moment of the oscillator at $t - \frac{R}{c}$. It is significant that the Hertz vector complies with the wave equation

$$\nabla^2 \mathbf{P} = \frac{1}{c^2} \frac{\partial^2 \mathbf{P}}{\partial t^2} \quad (7.65)$$

In other words, each of the components of the vector \mathbf{P} in an arbitrary system of Cartesian coordinates complies with a wave equation of the type of Eq. (7.24), where for our case we must assume that $v = c$. This follows from the fact that each of the components of this vector according to Eq. (7.64) coincides in form with the spherically symmetrical solution (7.27) of the wave equation which we obtained in Sec. 7.5.

Introducing Eq. (7.64) into Eqs. (7.52) and (7.57) for φ and \mathbf{A} and discarding the subscript a of the symbol div , we get

$$\varphi(t) = -\text{div } \mathbf{P}(t), \quad \mathbf{A}(t) = \frac{1}{c} \frac{\partial \mathbf{P}(t)}{\partial t} \quad (7.66)$$

Passing over from potentials to the field intensity, from Eq. (7.15), assuming in it that the permeability $\mu = 1$, we obtain

$$\mathbf{H} = \text{curl } \frac{1}{c} \frac{\partial \mathbf{P}}{\partial t} = \frac{1}{c} \frac{\partial \text{curl } \mathbf{P}}{\partial t} \quad (7.67)$$

Equation (7.16) on the basis of Eq. (A.42₃) becomes

$$\mathbf{E} = -\frac{1}{c^2} \frac{\partial^2 \mathbf{P}}{\partial t^2} + \text{grad div } \mathbf{P} = -\frac{1}{c^2} \frac{\partial^2 \mathbf{P}}{\partial t^2} + \nabla^2 \mathbf{P} + \text{curl curl } \mathbf{P}$$

Since the sum of the first two terms in the right-hand side, according to Eq. (7.65), equals zero, then we finally have

$$\mathbf{E} = \text{curl curl } \mathbf{P} \quad (7.68)$$

Thus, we have reduced the task of determining \mathbf{E} and \mathbf{H} to the calculation of the curl of the vector \mathbf{P} and its derivatives.

2. Let us assume that only the absolute (or, more exactly, the algebraic) value of the vector of the electric moment \mathbf{p} of an oscillator changes with time. In this case

$$\mathbf{p} = \mathbf{p}_0 f(t) \quad (7.69)$$

where $\mathbf{p}_0 =$ constant vector directed along the oscillator axis
 $f(t) =$ arbitrary scalar function of time.

This assumption is not an appreciable limitation because the moment \mathbf{p} of an arbitrary oscillator can be resolved into three mutually perpendicular components having a constant direction and the field of each of these components may be considered separately.

Introducing, further, the temporary symbol

$$\Phi(R, t) = \frac{f\left(t - \frac{R}{c}\right)}{R}$$

we can write in accordance with Eq. (7.64) that

$$\mathbf{P} = \mathbf{p}_0 \Phi(R, t)$$

In view of the constancy of the vector \mathbf{p}_0 , on the basis of Eqs. (A.43₃) (A.8) and (A.9), we get

$$\text{curl } \mathbf{P} = [\text{grad } \Phi \cdot \mathbf{p}_0] = \left[\frac{\partial \Phi}{\partial R} \frac{\mathbf{R}}{R} \cdot \mathbf{p}_0 \right] = \frac{1}{R} \frac{\partial \Phi}{\partial R} [\mathbf{R} \mathbf{p}_0]$$

Let us introduce a spherical system of coordinates R , α , and θ with its origin in the oscillator Q and with a polar axis parallel to \mathbf{p}_0 (Fig. 81). At every point of the field, we shall resolve all the vectors we encounter into components with mutually perpendicular directions corresponding to the *growth* of the spherical coordinates R , θ , and α .

It is obvious that the vector product $[\mathbf{R} \mathbf{p}_0]$ at every point P of the field will be directed along a tangent to the arc of a parallel circle passing through this point P , and in the direction of *diminishing* of the longitude α (if, as usual, we choose the direction of growth of the angle α so that it will form a right-handed system with the positive

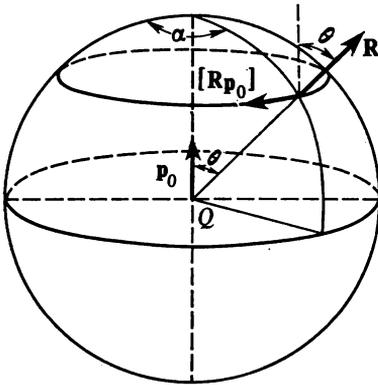


Fig. 81

direction of the polar axis). The absolute value of the product $[\mathbf{R} \mathbf{p}_0]$ will be $R p_0 \sin \theta$, where θ is the polar angle of the point P . Hence, the components of the vector $[\mathbf{R} \mathbf{p}_0]$ in the direction of growth of the coordinates R , θ , and α will be

$$[\mathbf{R} \mathbf{p}_0]_R = [\mathbf{R} \mathbf{p}_0]_\theta = 0, [\mathbf{R} \mathbf{p}_0]_\alpha = -R p_0 \sin \theta$$

Accordingly,

$$\operatorname{curl}_R \mathbf{P} = \operatorname{curl}_\theta \mathbf{P} = 0$$

$$\operatorname{curl}_\alpha \mathbf{P} = -\sin \theta p_0 \frac{\partial \Phi}{\partial R} = -\sin \theta \frac{\partial P}{\partial R}$$

where $\partial P / \partial R$ is a derivative of the *absolute* value of the Hertzian vector. Using this in Eq. (7.67), we get

$$\begin{aligned} H_R &= H_\theta = 0 \\ H_\alpha &= -\sin \theta \frac{1}{c} \frac{\partial^2 P}{\partial t \partial R} \end{aligned} \quad (7.70)$$

To determine \mathbf{E} , we must once more calculate the curl of $\operatorname{curl} \mathbf{P}$. On the basis of Eq. (A.31), we get

$$\left. \begin{aligned} E_R &= \frac{1}{R \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \operatorname{curl}_\alpha \mathbf{P}) = \\ &= -\frac{1}{R \sin \theta} \frac{\partial}{\partial \theta} \left(\sin^2 \theta \frac{\partial P}{\partial R} \right) = \\ &= -\frac{2 \cos \theta}{R} \frac{\partial P}{\partial R} \\ E_\theta &= -\frac{1}{R} \frac{\partial}{\partial R} (R \operatorname{curl}_\alpha \mathbf{P}) = \\ &= \frac{\sin \theta}{R} \frac{\partial}{\partial R} \left(R \frac{\partial P}{\partial R} \right) \\ E_\alpha &= 0 \end{aligned} \right\} \quad (7.71)$$

It follows from Eqs. (7.70) and (7.71) that *the electric and magnetic vectors of the field of an oscillator are mutually perpendicular, the magnetic lines of force coinciding with the parallel circles of our spherical system of coordinates, while the electric lines of force are in meridian planes.*

3. Up to now we made no special assumptions on the form of the functions $f\left(t - \frac{R}{c}\right)$ or $\mathbf{p}\left(t - \frac{R}{c}\right)$. Let us now assume that an oscillator performs *undamped harmonic oscillations*, i.e. that

$$f(t) = \cos \omega t, \quad \mathbf{p}\left(t - \frac{R}{c}\right) = \mathbf{p}_0 f\left(t - \frac{R}{c}\right) = \mathbf{p}_0 \cos \omega \left(t - \frac{R}{c}\right)$$

or in the complex form (see Sec. 6.5)

$$\mathbf{p}\left(t - \frac{R}{c}\right) = \mathbf{p}_0 \exp\left[i\omega \left(t - \frac{R}{c}\right)\right] \quad (7.72)$$

where ω is the cyclic frequency of oscillations.

In this case, by Eq. (7.64)

$$P(t, R) = \frac{p_0 \exp \left[i\omega \left(t - \frac{R}{c} \right) \right]}{R}$$

whence

$$\frac{\partial P}{\partial R} = p_0 \left(-\frac{1}{R^2} - \frac{i\omega}{cR} \right) \exp \left[i\omega \left(t - \frac{R}{c} \right) \right] = -\left(\frac{1}{R} + \frac{i\omega}{c} \right) P$$

and

$$\begin{aligned} \frac{\partial}{\partial R} \left(R \frac{\partial P}{\partial R} \right) &= -\frac{\partial}{\partial R} \left\{ \left(1 + \frac{i\omega R}{c} \right) P \right\} = \\ &= \left(\frac{1}{R} + \frac{i\omega}{c} - \frac{\omega^2 R}{c^2} \right) P \end{aligned}$$

Differentiation of P with respect to t obviously consists in multiplication by $i\omega$.

Consequently, in this case the non-zero components of the vectors \mathbf{E} and \mathbf{H} equal

$$\left. \begin{aligned} H_\alpha &= \sin \theta \frac{1}{c} \frac{\partial}{\partial t} \left(\frac{1}{R} + \frac{i\omega}{c} \right) P = \\ &= \frac{i\omega}{c} \sin \theta \left(\frac{1}{R} + \frac{i\omega}{c} \right) P \\ E_R &= 2 \cos \theta \left(\frac{1}{R^2} + \frac{i\omega}{cR} \right) P \\ E_\theta &= \sin \theta \left(\frac{1}{R^2} + \frac{i\omega}{cR} - \frac{\omega^2}{c^2} \right) P \end{aligned} \right\} \quad (7.73)$$

4. Such are the accurate expressions for the components of the electromagnetic vectors in the field of an oscillator performing harmonic oscillations. They are quite complicated, and therefore we shall limit ourselves to a more detailed consideration of only two extreme cases—the field in direct proximity to an oscillator and the field at a considerable distance from it.

What, however, should serve here as a measure of the distance? The ratio of the absolute values of the terms of the polynomials that are multipliers in the right-hand sides of Eqs. (7.73) is determined by the ratio of $1/R$ to $\omega/c = 2\pi/Tc$, where T is the period of oscillations of the oscillator. We shall see in the following that $Tc = \lambda$,

where λ is the length of the electromagnetic wave emitted by the oscillator, so that $\omega/c = 2\pi/\lambda$. Accordingly, we shall consider that points are close to an oscillator if the distance from the latter to them complies with the requirement

$$\frac{1}{R} \gg \frac{\omega}{c} = \frac{2\pi}{\lambda}$$

or

$$R \ll \frac{\lambda}{2\pi}$$

i.e. the distance from the oscillator to these points is small in comparison with the wavelength of the emitted waves*.

At these distances from the oscillator

$$\omega \left(t - \frac{R}{c} \right) = \omega t - \frac{\omega R}{c} \approx \omega t$$

so that

$$P(t, R) = \frac{p \left(t - \frac{R}{c} \right)}{R} \approx \frac{p(t)}{R}$$

Thus, as should be expected, near an oscillator its field at each given time t is determined by the simultaneous value of the moment of the oscillator $p(t)$ and its derivative $\partial p(t)/\partial t$.

Further, when $1/R \gg \omega/c$, we can with sufficient accuracy limit ourselves only to the first terms of the polynomials in Eqs. (7.73), i.e. we can assume that

$$\left. \begin{aligned} H_{\alpha} &= \frac{i\omega}{c} \sin \theta \frac{1}{R} P = \frac{i\omega}{c} \sin \theta \frac{p(t)}{R^2} = \\ &= \frac{\sin \theta}{R^2} \frac{1}{c} \frac{\partial p(t)**}{\partial t} \\ E_R &= \frac{2 \cos \theta}{R^2} P = \frac{2 \cos \theta}{R^3} p(t) \\ E_{\theta} &= \frac{\sin \theta}{R^2} P = \frac{\sin \theta}{R^3} p(t) \end{aligned} \right\} \quad (7.74)$$

* Naturally, R should nevertheless be so much greater than the length l of the oscillator that it could be considered as a dipole [see Eq. (7.48)]. Both requirements can be complied with simultaneously only if $l \ll \lambda$. This condition is not observed for a simple Hertzian oscillator. There do exist, however, types of oscillators that comply with this condition.

** Because $i\omega p(t) = \partial p(t)/\partial t$.

Comparing these expressions with Eq. (1.62), we see that at each given moment of time t the electric field near an oscillator, as should be expected, coincides with the field of a *static dipole* whose electric moment \mathbf{p} equals the instantaneous value of the moment of the oscillator $\mathbf{p}(t)$. As regards the magnetic field, then since $H_\theta = H_R = 0$, we can write the expression for \mathbf{H} in the following vector form:

$$\mathbf{H} = \frac{1}{cR^3} \left[\frac{\partial \mathbf{p}}{\partial t} \mathbf{R} \right]$$

If an oscillator can be considered similar to an element of a straight current, then on the basis of Eq. (7.59) we get

$$\mathbf{H} = \frac{I}{c} \frac{[\mathbf{IR}]}{R^3} \tag{7.75}$$

Thus, near an oscillator, its magnetic field coincides with the field of an equivalent element of current having the length l determined by the Biot and Savart equation (4.2). It must be noted that near an oscillator its electric field diminishes inversely proportional to the cube, and the magnetic field inversely proportional to the square of the distance from the oscillator.

5. Let us now pass over to considering the field far from an oscillator, i.e. assume that

$$\frac{1}{R} \ll \frac{\omega}{c} = \frac{2\pi}{\lambda}$$

or

$$R \gg \frac{\lambda}{2\pi}$$

The regions of a field, the distance from which to an oscillator complies with this condition, i.e. the distance from which is great in comparison with the wavelength of the oscillator, are called the *wave zone* of the oscillator. The meaning of this term will become clear on a later page.

For the wave zone, all the terms of the polynomials in Eqs. (7.73) containing R in their denominator are vanishingly small in comparison with the terms not containing R , so that we can presume with sufficient accuracy that

$$H_\alpha = \frac{i\omega}{c} \sin \theta \frac{i\omega}{c} P = - \frac{\omega^2}{c^2} \sin \theta \frac{P \left(t - \frac{R}{c} \right)}{R}$$

$$E_R = 0, E_\theta = - \frac{\omega^2}{c^2} \sin \theta P =$$

$$= - \frac{\omega^2}{c^2} \sin \theta \frac{P \left(t - \frac{R}{c} \right)}{R} = H_\alpha$$

Using here the expression for p from Eq. (7.72) and limiting ourselves to the real part of the solution, we finally obtain

$$\left. \begin{aligned} E_R = E_\alpha = H_R = H_\theta = 0 \\ E_\theta = H_\alpha = -\frac{\omega^2 \sin \theta}{c^2 R} p_0 \cos \omega \left(t - \frac{R}{c} \right) \end{aligned} \right\} \quad (7.76)$$

which can also be written in the following form:

$$E_\theta = H_\alpha = \frac{\sin \theta}{R} \frac{1}{c^2} \frac{\partial^2 p \left(t - \frac{R}{c} \right)}{\partial t^2} \quad (7.77)$$

It must be noted that the last expression [like Eqs. (7.74) and (7.75)] remains true with *any* form of the time dependence of the electric moment of an oscillator (including, for example, the case of damped oscillations). Indeed, any function of time can be expanded into a Fourier series or integral, i.e. can be presented in the form of the sum of sinusoidal functions to each of which Eq. (7.77) can be applied. Since this equation does not include ω , it also remains applicable to the entire sum, i.e. to an arbitrary $p(t)$ *

Hence, as follows from Eqs. (7.76) and (7.77), *in the wave zone of an oscillator the intensities of the electric and the magnetic fields numerically equal each other and diminish inversely proportional to the first power of the distance from the oscillator***. The intensity of the field also depends, apart from R , on the polar angle θ of the point of the field being considered: on the continuation of the oscillator axis ($\theta = 0$ and $\theta = \pi$) the field equals zero; it reaches its maximum value in the equatorial plane of the oscillator ($\theta = \pi/2$). At each point of a wave zone, *the vectors \mathbf{E} , \mathbf{H} and \mathbf{R} are mutually perpendicular and form a right-handed system* (Fig. 82); \mathbf{E} is directed along the arc of a meridian, and \mathbf{H} along the arc of a parallel circle.

Finally, and this is the most important, the phase $\omega \left(t - \frac{R}{c} \right)$ of the field vectors \mathbf{E} and \mathbf{H} propagates in the direction of the radius-vector with the velocity c . In other words, the phase of these vectors at a distance of $R + dR$ from the oscillator lags by $\tau = dR/c$ s behind the phase which these vectors have at the distance R from the oscillator $\left(\text{because } t - \frac{R+dR}{c} = t - \tau - \frac{R}{c} \right)$.

Any periodic process not concentrated at one point spread over a definite region of space, whose phase propagates with a definite

* Equation (7.77) is naturally applicable only for such distances R from the oscillator for which the condition $1/R \ll \omega/c$ is observed for all the frequencies ω present in the expansion of the function $p(t)$ and having appreciable values. A similar remark relates to the applicability of Eqs. (7.74) and (7.75).

** Whereas near the oscillator they diminish inversely proportional to the third and second powers of the distance.

velocity, is called a *wave**. Hence, an electromagnetic wave having the velocity c propagates in the wave zone of an oscillator. The distance between two consecutive points at which the field vectors

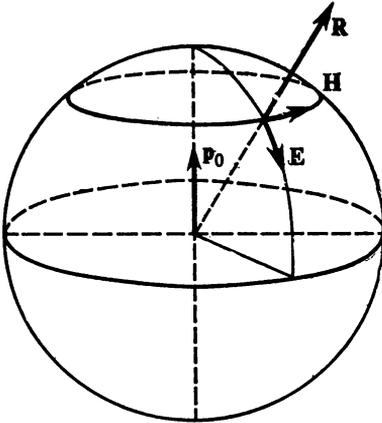


Fig. 82

change synchronously (i.e. in time) measured along the direction of propagation of a wave is called the wavelength λ . This distance equals

$$\lambda = cT = \frac{2\pi c}{\omega}$$

because

$$\begin{aligned} \cos \omega \left(t - \frac{R + \lambda}{c} \right) &= \cos \left\{ \omega \left(t - \frac{R}{c} \right) - 2\pi \right\} = \\ &= \cos \omega \left(t - \frac{R}{c} \right) \end{aligned}$$

The wave emitted by an oscillator is called *spherical* because the phase of the wave at each point of the field depends, apart from the time, only on the distance from this point to the centre of emission (i.e. to the oscillator), while the direction of propagation of the wave coincides with that of radius-vectors conducted from the centre of emission**.

6. The energy flux in the wave zone of an oscillator has the same direction because, as it is easy to see, the direction of Poynting's

* It should be noted in passing that waves of this type are only a particular case of a broad class of processes covered by the concept of a wave (non-monochromatic and standing waves, waves attenuating in time, wave pulses, etc.).

** We remind our reader that the absolute values of the vectors E and H of a spherical wave depend not only on R , but also on the polar angle θ .

vector \mathbf{S} [Eq. (7.11)] coincides at each point of this zone with the direction of the radius-vector \mathbf{R} . The absolute value of Poynting's vector is

$$S = \frac{c}{4\pi} |[\mathbf{EH}]| = \frac{c}{4\pi} EH = \frac{1}{4\pi} \frac{\sin^2 \theta}{R^2 c^3} \left(\frac{\partial^2 p \left(t - \frac{R}{c} \right)}{\partial t^2} \right)^2 \quad (7.78)$$

Consequently, the total energy flux Σ through the surface of a sphere of the radius R with its centre in the oscillator, an element of whose surface is

$$R^2 \sin \theta \, d\alpha \, d\theta$$

is determined by the expression

$$\begin{aligned} \Sigma &= \int_0^{2\pi} d\alpha \int_0^\pi d\theta \, SR^2 \sin \theta = \\ &= \frac{1}{2c^3} \left(\frac{\partial^2 p}{\partial t^2} \right)^2 \int_0^\pi \sin^3 \theta \, d\theta = \frac{2}{3c^3} \left(\frac{\partial^2 p \left(t - \frac{R}{c} \right)}{\partial t^2} \right)^2 \end{aligned} \quad (7.79)$$

Particularly, if an oscillator is performing harmonic oscillations of the cyclic frequency $\omega = 2\pi/T$, so that*

$$p \left(t - \frac{R}{c} \right) = p_0 \cos \omega \left(t - \frac{R}{c} \right)$$

then the total energy flux through the surface of a sphere during one period will be

$$\int_0^T \Sigma \, dt = \frac{2\omega^4 p_0^2}{3c^3} \int_0^T \cos^2 \omega \left(t - \frac{R}{c} \right) dt = \frac{p_0^2 \omega^4 T}{3c^3} = \frac{2\pi p_0^2 \omega^3}{3c^3}$$

Expressing ω through the wavelength λ , we get

$$\int_0^T \Sigma \, dt = \frac{2\pi p_0^2}{3} \left(\frac{2\pi}{\lambda} \right)^3$$

Finally, the mean emission of an oscillator in a unit time is

$$\Sigma = \frac{1}{T} \int_0^T \Sigma \, dt = \frac{c p_0^2}{3} \left(\frac{2\pi}{\lambda} \right)^4 \quad (7.80)$$

* When calculating the square of p , it is necessary to discard the imaginary part of the complex expression (7.72) (cf. Sec. 6.5, p. 398).

7. Thus, an oscillator continuously emits energy into the space surrounding it, and, according to Eq. (7.80), the mean rate of radiating energy $\bar{\Sigma}$ is proportional to the square of the amplitude of its electric moment p_0^2 and inversely proportional to the fourth power of the wavelength λ^4 . The latter circumstance explains, for example, the fact that for radio telegraphy it is necessary to use comparatively short electromagnetic waves from scores of metres to scores of kilometres long; on the contrary, the emission of slowly varying currents used in heavy current engineering (hundreds and thousands of hertz, which wavelengths of thousands and tens of thousands of kilometres correspond to) remains virtually unnoticed.

This nature of the dependence of the radiation of an oscillator on the wavelength also explains, for example, the blue colour of the sky. The light of the Sun penetrating through the atmosphere is dispersed by molecules of the air that can be considered similar to elementary oscillators. The dispersion of the light is due to the fact that under the action of the light waves these oscillators perform "forced" oscillations. Since the period of free oscillations of oscillators corresponding to air molecules appreciably differs from the period of visible light (the absence of resonance), then the amplitude p_0 of the forced oscillations of the oscillators depends only slightly on the period (or length) of the light wave. Therefore the intensity of the dispersed light, i.e. the intensity of the forced emission of these oscillators, is inversely proportional to λ^4 . Thus, short-wave light (blue) is dispersed to a greater extent than, for instance, red light, and this is what causes the blue colour of the sky.

8. During one period, an identical amount of energy flows through any closed surface confining an oscillator (including surfaces outside of the wave zone, i.e. near the oscillator). This can be shown by direct calculation; particularly, for spherical surfaces of a wave zone that are concentric to the oscillator this follows from the fact that Eq. (7.80) does not include the radius of the sphere. Also, the correctness of this statement follows directly from our assumption that there are neither conductors nor electric charges in the space surrounding an oscillator; hence the electromagnetic energy emitted by the oscillator cannot transform into other kinds of energy and must be transferred *without any losses* to remote regions of space. At the same time, the nature of the dependence of \mathbf{E} and \mathbf{H} on the distance R in the wave zone of an oscillator becomes clear. For the total emission through a spherical surface concentric to an oscillator to be independent of its radius, the density of the energy flux S must be inversely proportional to R^2 (because the surface area of a sphere is proportional to R^2). On the other hand for the wave

zone, in view of the vectors \mathbf{E} and \mathbf{H} being perpendicular and numerically equal to each other, we have

$$S = \frac{c}{4\pi} EH = \frac{c}{4\pi} E^2 = \frac{c}{4\pi} H^2$$

Consequently, E and H should diminish in the wave zone inversely proportional to the first power of R . Near the oscillator, i.e. outside of its wave zone, the nature of the field is much more complicated, but the mean amount of energy flowing out through the surface of the sphere during a period is also independent of its radius.

We shall note in conclusion that an oscillator can perform sustained oscillations only if these oscillations are maintained by periodic e.m.f.'s acting from outside. Otherwise, the oscillations of the oscillator will be damped not only owing to the conversion of its electromagnetic energy into Joule heat, but also owing to radiation, i.e. owing to the carrying off of energy by the electromagnetic waves it emits. If the damping of the oscillations is sufficiently small, however, then for a *limited number* of periods the amplitude of the oscillations can be considered virtually constant, and the results obtained for sustained oscillations may be applied to the field of damped ones.

Problem 37. In radio engineering, the loss of energy by a system (for example by an aerial) due to radiation is customarily characterized by the so-called "radiation resistance" R_λ of the system, determined from the equation

$$\bar{\Sigma} = R_\lambda \bar{I}^2$$

[cf. Eq. (4.127)], where I is the current in the system. Use Eqs. (7.59) to show that the radiation resistance of a Hertzian oscillator is

$$R_\lambda = \frac{2}{3c} \left(\frac{2\pi l}{\lambda} \right)^2$$

or in practical units

$$R'_\lambda = 20 \left(\frac{2\pi l}{\lambda} \right)^2 \Omega$$

where l = length of the oscillator
 λ = wavelength of the radiation.

Problem 38. Show on the basis of Eq. (7.63) that in a wave zone the field of a varying magnetic dipole $\boldsymbol{\mu}$ differs from the field of an electric dipole (an oscillator) having an equal moment

$$\left. \begin{aligned} \mathbf{p} = \boldsymbol{\mu} \text{ by the replacement of } \mathbf{E} \text{ with } -\mathbf{H} \text{ and of } \mathbf{H} \text{ with } -\mathbf{E}, \\ \text{i.e. that in the wave zone of a magnetic dipole we have} \\ E_R = E_\theta = H_R = H_\alpha = 0 \\ E_\alpha = H_\theta = -\frac{\sin \theta}{c^2 R} \frac{\partial^2}{\partial t^2} \mu \left(t - \frac{R}{c} \right) \end{aligned} \right\} \quad (7.81)$$

7.10 Electromagnetic Nature of Light. Plane Waves in a Dielectric

1. We saw in Sec. 7.7 that the velocity of propagation of electromagnetic waves in a vacuum coincides with the velocity of light, and in Sec. 7.9 when considering the particular example of the electromagnetic waves emitted by an oscillator that these waves, like light waves, are *transverse* ones, i.e. that the vectors \mathbf{E} and \mathbf{H} of the wave field are perpendicular to the direction of propagation of the wave.

This coincidence of the most important properties of light and electromagnetic waves makes us presume that the former are only a particular case of the latter and differ from invisible electromagnetic waves only in their frequency or wavelength. If this is true, then by a "light vector" (a formal concept used in wave optics) we must obviously understand either the electric or the magnetic vector of an electromagnetic light wave because both \mathbf{E} and \mathbf{H} are perpendicular to the direction of wave propagation (and this is what is required from the light vector). In other words, the direction of polarization, for example, of linearly polarized light should be determined by the direction of the vectors \mathbf{E} and \mathbf{H}^* .

These assumptions are indeed confirmed by deeper studying of the properties of electromagnetic waves and a comparison of them with the properties of light. Consequently, at present we can say with certainty that the establishment of the electromagnetic nature

* Experimental optics operates with the concept of the *plane of polarization* of light. Linearly polarized light can be obtained by reflecting unpolarized light at the proper angle from a reflecting surface. The plane of incidence of the light on this reflecting surface, by definition, is called the plane of polarization of the reflected polarized light.

In the last century, the question of whether the light vector of a linearly polarized wave is parallel or perpendicular to the plane of polarization of this wave was discussed quite animatedly. The theories of G. Green, F. Neumann and others were based on the first assumption, and A. Fresnel's theory on the second. From the viewpoint of the electromagnetic theory of light, this question simply consists in whether to call the intensity of the magnetic or of the electric field of the wave the light vector. From O. Winer's experiments [see G. S. Landsberg, *Optika* (Optics). Moscow, Gostekhizdat, 1957, p. 90] it follows that the magnetic vector \mathbf{H} of a light wave is parallel, and the electric vector \mathbf{E} is perpendicular to its plane of polarization.

of light is among the most substantiated and important achievements of physics in the 19th century.

In the present section, we shall treat one of the very simple questions of the theory of electromagnetic waves—the propagation of *plane monochromatic waves* in homogeneous *dielectrics*.

2. A wave is called *plane* if at any moment of time at all the points of any plane perpendicular to the direction of the wave, the field vectors have the same value. In other words, if we choose the z -axis along the direction of the wave, then the vectors \mathbf{E} and \mathbf{H} of the field of a plane wave should depend only on the coordinate z , and not on the coordinates x and y . The consideration of such plane waves has a quite definite physical meaning because, for example, at a sufficiently remote distance from an oscillator a limited portion of the spherical wave which it emits can quite accurately be considered as plane.

A wave is called *monochromatic* (from the Greek for single-coloured; this term has been taken from optics) if the field of the wave is a harmonic (sinusoidal) function of time. Hence, the complex expressions for the vectors of the field of a plane monochromatic wave should have the form

$$\mathbf{E} = \mathbf{A}(z) e^{i\omega t} \quad \text{and} \quad \mathbf{H} = \mathbf{B}(z) e^{i\omega t} \quad (7.82)$$

where the (complex, in general) vectors $\mathbf{A}(z)$ and $\mathbf{B}(z)$ depend only on the coordinate z . It is quite natural that only the real part of these expressions has a proper physical meaning (Sec. 6.5, p. 398).

3. Let us assume that the dielectric we are considering is homogeneous (the permittivity ε and permeability μ are constant) and is deprived of free electric charges ($\rho = 0$). Assuming that in Maxwell's equation (I) we have $\mathbf{j} = 0$, we get

$$\frac{\varepsilon}{c} \frac{\partial \mathbf{E}}{\partial t} = \text{curl} \mathbf{H} \quad (\text{I})$$

Differentiating this equation with respect to time and then introducing into it the value of $\partial \mathbf{H} / \partial t$ from Eq. (II)

$$\frac{\mu}{c} \frac{\partial \mathbf{H}}{\partial t} = - \text{curl} \mathbf{E} \quad (\text{II})$$

we get on the basis of Eq. (A.42₃)

$$\begin{aligned} \frac{\varepsilon}{c} \frac{\partial^2 \mathbf{E}}{\partial t^2} &= \text{curl} \frac{\partial \mathbf{H}}{\partial t} = - \frac{c}{\mu} \text{curl} \text{curl} \mathbf{E} = \\ &= - \frac{c}{\mu} (\text{grad} \text{div} \mathbf{E} - \nabla^2 \mathbf{E}) \end{aligned}$$

Since when $\rho = 0$ and $\varepsilon = \text{const}$, Eq. (IV) becomes

$$\text{div} \mathbf{D} = \varepsilon \text{div} \mathbf{E} = 0$$

then, consequently,

$$\frac{\epsilon\mu}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = \nabla^2 \mathbf{E} \quad (7.83)$$

Equations (I) and (II) are symmetrical relative to \mathbf{E} and \mathbf{H} (up to the sign). Hence, in an absolutely similar way, we obtain

$$\frac{\epsilon\mu}{c^2} \frac{\partial^2 \mathbf{H}}{\partial t^2} = \nabla^2 \mathbf{H} \quad (7.84)$$

4. The correctness of Eqs. (7.83) and (7.84) is limited only by the requirement of homogeneity of the medium and the absence in it of conduction currents and free charges. For plane monochromatic waves, these equations on the basis of Eq. (7.82) can be written in the following form (after cancelling $e^{i\omega t}$):

$$\begin{aligned} -\frac{\epsilon\mu\omega^2}{c^2} \mathbf{A}(z) &= \nabla^2 \mathbf{A}(z) = \frac{\partial^2 \mathbf{A}(z)}{\partial z^2} \\ -\frac{\epsilon\mu\omega^2}{c^2} \mathbf{B}(z) &= \nabla^2 \mathbf{B}(z) = \frac{\partial^2 \mathbf{B}(z)}{\partial z^2} \end{aligned}$$

or

$$\frac{\partial^2 \mathbf{A}}{\partial z^2} + k^2 \mathbf{A} = 0, \quad \frac{\partial^2 \mathbf{B}}{\partial z^2} + k^2 \mathbf{B} = 0$$

where we have introduced the symbol

$$k^2 = \frac{\epsilon\mu\omega^2}{c^2}, \quad k = \omega \frac{\sqrt{\epsilon\mu}}{c} \quad (7.85)$$

The solutions of these equations are known to have the following form:

$$\mathbf{A} = \mathbf{A}_0 e^{-ikz} + \mathbf{A}'_0 e^{ikz}, \quad \mathbf{B} = \mathbf{B}_0 e^{-ikz} + \mathbf{B}'_0 e^{ikz}$$

where \mathbf{A}_0 , \mathbf{A}'_0 , \mathbf{B}_0 , and \mathbf{B}'_0 are arbitrary integration constants. Using these expressions in Eq. (7.82), we get

$$\mathbf{E} = \mathbf{A}_0 e^{i(\omega t - kz)} + \mathbf{A}'_0 e^{i(\omega t + kz)}$$

$$\mathbf{H} = \mathbf{B}_0 e^{i(\omega t - kz)} + \mathbf{B}'_0 e^{i(\omega t + kz)}$$

The first terms of these expressions represent the wave propagating in the positive direction of the z -axis, and the second—the wave propagating in the opposite direction. Without appreciably restricting the general nature of our reasoning, we can limit ourselves to a consideration of only one of these waves, for example the first one, and assume that

$$\mathbf{E} = \mathbf{A}_0 e^{i(\omega t - kz)}, \quad \mathbf{H} = \mathbf{B}_0 e^{i(\omega t - kz)} \quad (7.86)$$

\mathbf{A}_0 and \mathbf{B}_0 are the amplitudes of the vectors \mathbf{E} and \mathbf{H} . The fact that these amplitudes are independent of the coordinates signifies that *the propagation of plane waves in a dielectric is not connected with a change in their intensity*. The amplitudes \mathbf{A}_0 and \mathbf{B}_0 , in general, are complex quantities.

5. The velocity of a wave, according to Eq. (7.86), equals ω/k because at the moment t_0 the values of the field vectors in the plane $z = z_0$ coincide with the values which these vectors had at the moment $t_0 - 1$ in the plane $z = z_0 - \omega/k$. This follows from the equality of the corresponding phases:

$$\omega t_0 - kz_0 = \omega(t_0 - 1) - k\left(z_0 - \frac{\omega}{k}\right)$$

According to Eq. (7.85), this velocity is

$$v = \frac{\omega}{k} = \frac{c}{\sqrt{\epsilon\mu}} \quad (7.87)$$

which coincides with the general results obtained in Secs. 7.4-7.6 [Eq. (7.22)].

It must be noted that the quantity k is related very simply to the wavelength λ . Using the value $\omega = 2\pi/T$ in Eq. (7.87), we get

$$k = \frac{\omega}{v} = \frac{2\pi}{Tv} = \frac{2\pi}{\lambda} \quad (7.88)$$

Thus k equals the number of waves accommodated on a length of 2π centimetres, and is therefore called the *wavelength constant*.

6. To simplify our further calculations, we shall note that in accordance with Eq. (7.86), the differentiation of the vectors of a plane wave with respect to z consists in multiplying them by $-ik$. Since, on the other hand, these vectors do not depend on x and y , then their symbolic multiplication by the differential nabla operator consists in multiplication by the ordinary vector $-ikk$. Hence, as applied to these vectors,

$$\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} = -ikk$$

(do not confuse the unit vectors \mathbf{i} and \mathbf{k} along the axes of coordinates with the square root of -1 , i and the wavelength constant k).

If the z -axis of the coordinate system does not coincide with the direction of propagation of the wave, it is evidently sufficient to replace \mathbf{k} with the unit vector \mathbf{n} coinciding with this direction:

$$\nabla = -ik\mathbf{n} \quad (7.89)$$

On the basis of this relationship, Maxwell's equations (III) and (IV) become

$$\operatorname{div} \mathbf{B} = \nabla \mu \mathbf{H} = -ik\mu \mathbf{nH} = 0$$

$$\operatorname{div} \mathbf{D} = \nabla \epsilon \mathbf{E} = -ik\epsilon \mathbf{nE} = 0$$

whence it follows that the vectors \mathbf{E} and \mathbf{H} are perpendicular to \mathbf{n} , i.e. are perpendicular to the direction of the wave. Thus, *plane electromagnetic waves, like spherical waves, are transverse ones.*

Differentiation of the vectors \mathbf{E} and \mathbf{H} with respect to time, according to Eq. (7.86), consists in multiplying them by $i\omega$. Consequently, Eq. (II) with the aid of Eq. (7.89) can be written as follows:

$$\frac{i\omega\mu}{c} \mathbf{H} = - \text{curl } \mathbf{E} = - [\nabla \mathbf{E}] = ik[\mathbf{nE}]$$

Inserting into this equation the value of k from Eq. (7.85) and dividing it by $i\omega\sqrt{\mu}/c$, we get

$$\sqrt{\mu}\mathbf{H} = \sqrt{\varepsilon} [\mathbf{nE}] \tag{7.90}$$

[A consideration of Eq. (I) gives nothing new and leads to the same results.] It follows from this equation, first, that *the vectors \mathbf{E} and \mathbf{H} are mutually perpendicular*, and, second, that the mutually perpendicular vectors \mathbf{n} , \mathbf{E} , and \mathbf{H} form a right-handed system (Fig. 83;

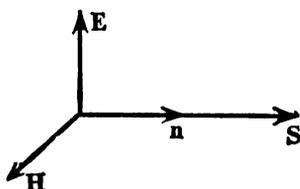


Fig. 83

cf. Fig. 82). Further, since \mathbf{n} and \mathbf{E} are perpendicular, we get the following relationship between the absolute values of the vectors \mathbf{E} and \mathbf{H} :

$$\sqrt{\mu}H = \sqrt{\varepsilon} |[\mathbf{nE}]| = \sqrt{\varepsilon} E \tag{7.91}$$

Thus, the ratio of the absolute values of the vectors \mathbf{E} and \mathbf{H} does not depend on the time, i.e. these vectors have the *same phase* and change synchronously.

7. Turning to the determination of the non-linear functions of the field vectors (the energy, Poynting's vector, etc.), we must first pass over to the *real parts* of the complex expressions (7.86) (see Sec. 6.5, p. 398). In the following formulas, we shall accordingly consider the vectors \mathbf{E} and \mathbf{H} to be real.

By Eq. (7.91), *the density of the magnetic energy* in the field of a wave is equal to the density of the electric energy:

$$u_m = \frac{\mu H^2}{8\pi} = \frac{\varepsilon E^2}{8\pi} = u_e$$

Consequently,

$$\dot{u} = u_m + u_e = \frac{\varepsilon E^2}{4\pi} = \frac{\mu H^2}{4\pi} \tag{7.92}$$

It follows from inspection of Fig. 83 that the direction of Poynting's vector

$$\mathbf{S} = \frac{c}{4\pi} [\mathbf{E}\mathbf{H}]$$

i.e. the direction of the energy flux in a wave coincides with the direction of its propagation. Since the vectors \mathbf{E} and \mathbf{H} are perpendicular, we have

$$S = \frac{c}{4\pi} EH$$

Expressing H through E with the aid of Eq. (7.91) and using Eqs. (7.92) and (7.87), we get

$$S = \frac{c\sqrt{\varepsilon}}{4\pi\sqrt{\mu}} E^2 = \frac{c}{\sqrt{\varepsilon\mu}} u = uv \quad (7.93)$$

whence

$$S dt = uv dt$$

Thus, the amount of energy $S dt$ flowing during the time element dt through a unit area perpendicular to the vector \mathbf{S} (i.e. perpendicular to the direction of the wave) equals the amount of energy $uv dt$ contained in a cylinder having the altitude $v dt$ resting on this area. From the viewpoint of physics, this means that *the rate of energy flow equals v , i.e. coincides with the phase velocity of the wave**.

7.11 Reflection and Refraction of Plane Waves in Dielectrics

1. The unfoundedness of the mechanical theories of light, developed in the 19th century and based on the notion of an elastic light ether, was revealed especially clearly in the unsuccessful attempts of these theories to explain the very simple phenomena of

* The phase velocity is the velocity of propagation of a phase of a wave. The equality of the phase velocity and the rate of energy in isotropic media is violated only when dispersion is present, i.e. when the phase velocity of a wave depends on the wavelength. The explanation is that to determine the rate of energy flow it is necessary to consider not a monochromatic wave, but a light pulse bounded in time and in space. In the absence of dispersion, this circumstance does not affect the final result, whereas in dispersing media the rate of the energy flow is found to equal not the phase, but the so-called group velocity of the waves.

the reflection and refraction of light*. On the contrary, to explain these phenomena from the viewpoint of the electromagnetic theory of light, no special assumptions have to be resorted to.

In this section, we shall treat the refraction and reflection of plane monochromatic waves on the interface of two homogeneous dielectrics 1 and 2 whose permittivities are ϵ_1 and ϵ_2 . Further, we shall assume for simplicity that the permeabilities $\mu_1 = \mu_2 = 1$. We shall see below that for *light* waves this assumption does not at all restrict the general nature of our reasoning. In these conditions, the velocity of the waves in the first and the second dielectrics, according to Eq. (7.87), will be, respectively,

$$v_1 = \frac{c}{\sqrt{\epsilon_1}} \quad \text{and} \quad v_2 = \frac{c}{\sqrt{\epsilon_2}} \quad (7.94)$$

In the preceding section, we assumed for simplification of our equations that the direction of the z -axis is selected so that it coincides with the direction of the wave. When beginning to consider a combination of several waves having *different* directions (an incident, reflected, and refracted waves), we must obviously first generalize the formulas of the preceding section for an arbitrary direction of the coordinate axes. Assume that the direction of the wave coincides with that of the unit vector \mathbf{n} forming the angles α , β and γ with the coordinate axes x , y , and z (Fig. 84). In the system of Cartesian

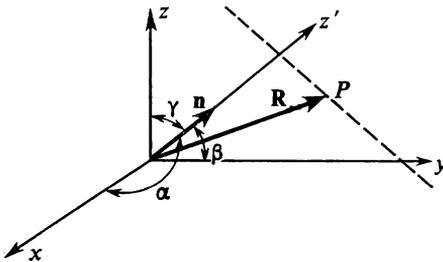


Fig. 84

coordinates x' , y' , and z' whose axis z' coincides with \mathbf{n} , the phase of this wave at the point P should, according to Eq. (7.86), be determined by the expression $\omega t - kz'$, where z' is the coordinate of the point P . This coordinate equals the projection of the radius-vector \mathbf{R} of the point P onto the direction \mathbf{n} of the axis z' . Hence,

$$z' = \mathbf{nR} = x \cos \alpha + y \cos \beta + z \cos \gamma \quad (7.95)$$

* Particularly, it follows from the theory of elasticity that upon the refraction or reflection of *transverse* waves, *longitudinal* compression waves should also appear. To explain the fact of the absence of longitudinal light waves in ether, it was necessary to resort either to the hypothesis of absolutely incompressible ether or to the hypothesis of unstable ether having a negative cubic elasticity. Any of these hypotheses led to further contradictions.

where x , y , and z are the coordinates of the same point P in the initial system of coordinates. Introducing this into Eq. (7.86) we get the required expressions for the vectors of the field of a plane wave propagating in the arbitrary direction \mathbf{n} :

$$\begin{aligned}\mathbf{E} &= \mathbf{E}_0 \exp [i(\omega t - k\mathbf{nR})] \quad \text{and} \\ \mathbf{H} &= \mathbf{H}_0 \exp [i(\omega t - k\mathbf{nR})]\end{aligned}\quad (7.96)$$

where \mathbf{E}_0 and \mathbf{H}_0 are the constant complex amplitudes of the relevant vectors.

2. After these preparatory remarks, we shall begin to solve our contemplated problem. Assume that the plane wave (7.96) propagating in the medium I in the direction \mathbf{n} , impinges on the flat boundary between the media I and 2 . After penetration into the medium 2 , this wave should evidently propagate with a different velocity ($v_2 \neq v_1$) and, as we shall see, in general, in a different direction not coinciding with \mathbf{n} . To determine the amplitude, direction, and phase of this so-called *refracted* wave, it is sufficient to require that the boundary conditions which we listed in Sec. 7.1 be observed on the boundary. We shall find that these conditions can be complied with only if we assume that there is a third, so-called *reflected* wave propagating in the same medium I as the incident wave, but in a direction *from* the boundary.

Let us denote the complex vectors of the field of the incident wave by \mathbf{E} and \mathbf{H} , of the reflected wave by \mathbf{E}_r and \mathbf{H}_r , and, finally, of the refracted wave by \mathbf{E}_g and \mathbf{H}_g and assume, in accordance with Eq. (7.96), that

$$\begin{aligned}\mathbf{E} &= \mathbf{E}_0 \exp [i(\omega t - k\mathbf{nR})], \quad \mathbf{E}_r = \mathbf{E}_{0,r} \exp [i(\omega_r t - \\ &- k_r \mathbf{n}_r \mathbf{R})], \quad \text{and} \quad \mathbf{E}_g = \mathbf{E}_{0,g} \exp [i(\omega_g t - k_g \mathbf{n}_g \mathbf{R})]\end{aligned}$$

We shall not write out similar expressions for \mathbf{H} , \mathbf{H}_r , and \mathbf{H}_g . The first two waves propagate in the medium I so that the resultant intensity of the field in this medium will be

$$\mathbf{E}_1 = \mathbf{E} + \mathbf{E}_r$$

whereas the field in the medium 2 will be

$$\mathbf{E}_2 = \mathbf{E}_g$$

Let us consider a boundary condition, for example condition (II') of the continuity of the tangential components of the vector \mathbf{E} . In the case being considered, it becomes

$$E_t + E_{t,r} = E_{t,g}$$

i.e.

$$\begin{aligned}E_{0,t} \exp [i(\omega t - k\mathbf{nR})] + E_{0,t,r} \exp [i(\omega_r t - k_r \mathbf{n}_r \mathbf{R})] = \\ = E_{0,t,g} \exp [i(\omega_g t - k_g \mathbf{n}_g \mathbf{R})]\end{aligned}\quad (7.97)$$

For such a condition to be observed with any value of the time t , it is first of all essential that

$$\omega = \omega_r = \omega_g \tag{7.98}$$

Indeed, condition (7.97) has the form

$$a \exp(i\omega t) + b \exp(i\omega_r t) = c \exp(i\omega_g t)$$

where a , b , and c do not depend on the time.

Differentiation of this equation with respect to t yields

$$\omega a \exp(i\omega t) + \omega_r b \exp(i\omega_r t) = \omega_g c \exp(i\omega_g t)$$

Excluding from the last two equations $c \exp(i\omega_g t)$, we get

$$a(\omega - \omega_g) \exp(i\omega t) = b(\omega_g - \omega_r) \exp(i\omega_r t)$$

which can hold only when $\omega = \omega_r$. Excluding from the cited equations, on the other hand, $\exp(i\omega_r t)$, we see that $\omega = \omega_g$. Hence, we indeed have $\omega = \omega_r = \omega_g$, i.e. *the frequency of a wave does not change upon its reflection and refraction.*

In an absolutely similar way we can see that the following equations must be observed:

$$k\mathbf{n}\mathbf{R}' = k_r\mathbf{n}_r\mathbf{R}' = k_g\mathbf{n}_g\mathbf{R}' \tag{7.99}$$

where \mathbf{R}' is an arbitrary vector in the boundary plane. Indeed, the radius-vector \mathbf{R} of an arbitrary point of the boundary can be presented in the form

$$\mathbf{R} = \mathbf{R}_0 + \mathbf{R}'$$

where \mathbf{R}' is in the plane of the boundary between the media, and \mathbf{R}_0 is the radius-vector of an arbitrarily fixed point of this surface. Consequently, the boundary condition can be written in the form

$$\begin{aligned} a' \exp(-ik\mathbf{n}\mathbf{R}') + b' \exp(-ik_r\mathbf{n}_r\mathbf{R}') = \\ = c' \exp(-ik_g\mathbf{n}_g\mathbf{R}') \end{aligned}$$

where the quantities a' , b' , and c' do not depend on the coordinates of the vector \mathbf{R}' . The further proof of the equation (7.99) is similar to that of Eq. (7.98).

3. It is convenient to go over to coordinate expressions for our further calculations. Assume that the z -axis is perpendicular to the boundary between the media 1 and 2. Hence, the vector \mathbf{R}' in this plane will be perpendicular to the z -axis, and Eq. (7.99) on the basis of Eq. (7.95) can be written as follows:

$$\begin{aligned} k(x \cos \alpha + y \cos \beta) = k_r(x \cos \alpha_r + y \cos \beta_r) = \\ = k_g(x \cos \alpha_g + y \cos \beta_g) \end{aligned}$$

where x and y = components of the vector \mathbf{R}'
 $\alpha, \beta, \alpha_r, \beta_r, \alpha_g, \beta_g$ = angles formed by the vectors $\mathbf{n}, \mathbf{n}_r,$ and \mathbf{n}_g
 with the axes x and y , respectively.

To simplify matters, we shall choose the x and y axes so that the direction of propagation of the incident wave \mathbf{n} will be in the plane xz (Fig. 85). In this case, $\cos \beta = 0$. Since the above equation

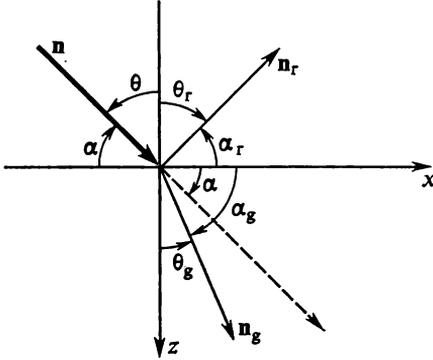


Fig. 85

must be observed at all points of the boundary plane, i.e. with any values of x and y , then it directly follows from it that

$$\cos \beta_r = \cos \beta_g = 0$$

and

$$k \cos \alpha = k_r \cos \alpha_r = k_g \cos \alpha_g \quad (7.100)$$

The first of these equations signifies that *the directions of the reflected and refracted waves \mathbf{n}_r and \mathbf{n}_g are in the plane xz , i.e. in the plane of incidence of the wave.* (By the plane of incidence is meant the plane passing through the direction of the incident wave and through a normal to the boundary.)

Let us now take account of the circumstance that the incident and reflected waves propagate in the medium 1, and the refracted wave in the medium 2, and that, consequently, in accordance with Eqs. (7.87) and (7.98), we have

$$k = \frac{\omega}{v_1}, \quad k_r = \frac{\omega}{v_1} = k, \quad \text{and} \quad k_g = \frac{\omega}{v_2}$$

Equation (7.100) can therefore be written as follows:

$$\frac{1}{v_1} \cos \alpha = \frac{1}{v_1} \cos \alpha_r = \frac{1}{v_2} \cos \alpha_g \quad (7.101)$$

It follows from these equations, first, that

$$\alpha_r = \alpha$$

(Since, by definition, the value of a direction angle cannot exceed π , then the solution $\alpha_r = 2\pi - \alpha$ falls away.)

Introducing, as usual, the angles of incidence and reflection θ and θ_r (as follows from Fig. 85, $\theta = \frac{\pi}{2} - \alpha$ and $\theta_r = \frac{\pi}{2} - \alpha_r$), we can say that *the angle of incidence θ equals the angle of reflection θ_r .*

Further, since $\cos \alpha$ and $\cos \alpha_g$ have the same signs, then the directions of the incident and refracted waves must be in the same quadrant of the plane xz (Fig. 85).

Introducing the angle of refraction $\theta_g = \frac{\pi}{2} - \alpha_g$ and noting that $\cos \alpha = \sin \theta$ and $\cos \alpha_g = \sin \theta_g$, from Eq. (7.101) we get

$$\frac{1}{v_1} \sin \theta = \frac{1}{v_2} \sin \theta_g$$

or*

$$\frac{\sin \theta}{\sin \theta_g} = \frac{v_1}{v_2} \tag{7.102}$$

Thus, *the ratio of the sine of the angle of incidence to the sine of the angle of refraction is a constant quantity* depending only on the properties of the adjacent media 1 and 2. This ratio, which we shall designate by n_{12} , as is well known, is called the *relative refractive index* of the medium 2 to the medium 1. On the basis of Eq. (7.94), we can write that

$$n_{12} = \frac{v_1}{v_2} = \sqrt{\frac{\epsilon_2}{\epsilon_1}} \tag{7.103}$$

4. Hence, all the *geometrical* laws of refraction and reflection of electromagnetic waves coinciding with the relevant laws for light waves follow from the single fact of there being linear conditions of the type of Eq. (7.97) on the boundary between two media. A more detailed consideration of these boundary conditions allows us to

* We must remind our reader that as is known from the elementary course of optics, when $v_1/v_2 = n_{12} < 1$, the angle of refraction θ_g acquires a real value only if the angle of incidence θ does not exceed the so-called critical angle Φ determined by the equation $\sin \Phi = v_1/v_2 = n_{12}$. Otherwise when $\theta > \Phi$ we have $\sin \theta_g = \sin \theta \frac{v_2}{v_1} = \frac{\sin \theta}{\sin \Phi} > 1$, and the quantity θ_g becomes a complex one. As a special treatment of the question, which we shall not stop to consider here, shows, the so-called *total internal reflection of the waves* should be observed here. In this case, the refracted wave is absent, and the electromagnetic field in the second medium differs from zero only in a very thin layer (of the order of magnitude of the wavelength) directly adjoining the boundary (see, for example, Born, M. and Wolf, E. *Principles of Optics*, 3rd ed. Oxford, Pergamon Press, 1965).

These corollaries of theory agree quite well with the data of experiments both in the field of long electromagnetic waves and in that of light waves.

determine the relationship between the squares of the amplitudes of the incident wave and the reflected and refracted waves. These relationships are found to be identical with the so-called *Fresnel formulas* determining the comparative *intensity* of the reflected and refracted light depending on the refractive index, the angle of incidence, and the polarization of the impinging light. This experimentally confirmed derivation of Fresnel's formulas from the general laws of electrodynamics is one of the most important proofs of the electromagnetic nature of light. Not having the possibility of giving it here completely, we shall limit ourselves to a treatment of the simplest case, namely, of the *normal* incidence of a plane wave onto the boundary of dielectrics $z = 0$.

Any plane monochromatic wave can be resolved into a combination of two linearly polarized waves that can be considered separately. Assume that the electric vector of the incident wave is directed along the x -axis:

$$E_x = E_0 \exp \{i(\omega t - kz)\}, \quad E_y = E_z = 0$$

Hence, according to Eq. (7.103), the magnetic vector of the incident wave will be directed along the y -axis and will equal

$$H_x = H_z = 0, \quad H_y = \sqrt{\varepsilon_1} E_0 \exp \{i(\omega t - kz)\}$$

According to the laws of reflection and refraction (7.101) and (7.102), the refracted and reflected waves will be directed along and opposite to the x -axis, respectively. As can easily be seen by considering the boundary conditions, the electric vectors of these waves, as in the incident wave, are directed along the x -axis and their magnetic vectors along the y -axis so that the non-zero components of these vectors are

$$E_{x,r} = E_{0,r} \exp \{i(\omega t + kz)\}, \quad H_{y,r} = -\sqrt{\varepsilon_1} E_{0,r} \exp \{i(\omega t + kz)\}$$

$$E_{x,g} = E_{0,g} \exp \{i(\omega t - k_g z)\}, \quad H_{y,g} = \sqrt{\varepsilon_2} E_{0,g} \exp \{i(\omega t - k_g z)\}$$

The expression for $H_{y,r}$ is preceded by a minus sign in accordance with the fact that the reflected wave propagates opposite to the z -axis [see Eq. (7.90)].

Let us introduce these expressions for \mathbf{E} and \mathbf{H} into the boundary conditions. Condition (7.97) of the continuity of the tangential components of the vector \mathbf{E} on the boundary plane $z = 0$ after cancelling $\exp(i\omega t)$ yields

$$E_0 + E_{0,r} = E_{0,g}$$

According to this, the continuity condition for the tangential components of the vector \mathbf{H} gives

$$\sqrt{\varepsilon_1} E_0 - \sqrt{\varepsilon_1} E_{0,r} = \sqrt{\varepsilon_2} E_{0,g}$$

The components of the field vectors normal to the boundary equal zero, and therefore the boundary conditions (III') and (IV') relating to them are always complied with identically.

Solving the obtained equations and using the ratio (7.103) we get

$$E_{0,r} = \frac{1 - n_{12}}{1 + n_{12}} E_0 \quad \text{and} \quad E_{0,g} = \frac{2}{1 + n_{12}} E_0 \quad (7.104)$$

Passing over to the real parts of the complex expressions and considering E_0 to be real, we can write

$$E_x = E_0 \cos(\omega t - kz), \quad E_{x,r} = \frac{1 - n_{12}}{1 + n_{12}} E_0 \cos(\omega t + kz)$$

$$E_{x,g} = \frac{2}{1 + n_{12}} E_0 \cos(\omega t - k_g z)$$

Denoting by \bar{S} , \bar{S}_r and \bar{S}_g the mean (during a period) energy flux densities in the incident, reflected, and refracted waves, on the basis of Eq. (7.93) we get

$$\left. \begin{aligned} \bar{S} &= \frac{c}{4\pi} \sqrt{\varepsilon_1 E_0^2 \cos^2(\omega t - kz)} = \frac{c}{8\pi} \sqrt{\varepsilon_1} E_0^2 \\ \bar{S}_r &= \frac{c}{8\pi} \sqrt{\varepsilon_1} \left(\frac{1 - n_{12}}{1 + n_{12}} \right)^2 E_0^2 \\ \bar{S}_g &= \frac{c}{8\pi} \sqrt{\varepsilon_2} \left(\frac{2}{1 + n_{12}} \right)^2 E_0^2 \end{aligned} \right\} \quad (7.105)$$

It is simple to verify with the aid of Eq. (7.103) that these expressions comply with the law of conservation of energy, i.e. that the mean (during a period) energy flux in the incident wave equals the sum of the mean energy fluxes in the refracted and reflected waves*:

$$\bar{S} = \bar{S}_r + \bar{S}_g \quad (7.106)$$

The *reflection factor* r of the boundary between two media is the ratio of the intensity of the reflected wave to that of the incident one, or, in other words, the ratio of the mean energy fluxes during a period in the reflected and the incident waves:

$$r = \frac{\bar{S}_r}{\bar{S}} = \left(\frac{1 - n_{12}}{1 + n_{12}} \right)^2 \quad (7.107)$$

When $n_{12} = 1$, i.e. when $v_1 = v_2$, no reflection naturally occurs.

* Naturally, the law of conservation of energy is observed not only for the mean values of the energy flux during a period, but also for its instantaneous values.

Thus, we have determined all the parameters characterizing the reflected and the refracted waves for the normal incidence of the primary wave onto the interface. The expressions we have obtained coincide with Fresnel's corresponding formulas.

5. Apart from the derivation of the laws of refraction and reflection of light that were known before its electromagnetic nature was established, the theory which we have set out makes it possible to establish a direct relationship between the refractive index of light n and the permittivity of the medium ϵ [Eq. (7.103)]. Particularly, the absolute refractive index for a dielectric (relative to a vacuum) for which $\epsilon = 1$ is

$$n = \sqrt{\epsilon} \quad (7.108)$$

For some dielectrics (for instance air, CO, and benzene), Eq. (7.108) is indeed confirmed by experiments. For many substances however, this formula does not at all correspond to the results of measurements, as can be seen from the following data:

Substance	water	methyl alcohol	ethyl alcohol
n	1.33	1.34	1.36
$\sqrt{\epsilon}$	9.0	5.7	5.0

Moreover, the very fact of the existence of the *dispersion* of light, i.e. the dependence of the refractive index on the wavelength, proves the unfitness of Eq's. (7.103) and (7.108) according to which n ought to have a constant value for all electromagnetic waves.

6. Thus, Maxwell's phenomenological theory of a *macroscopic field* leads, in general, to incorrect values of the refractive index. This contradiction, however, is eliminated very simply by the electron theory of a *microscopic field*. Indeed, when deriving the equations for a macroscopic field in Chapter 2 we assumed dielectric molecules to be similar to electric dipoles. If these dipoles are *quasi-elastic*, they must have a natural period of oscillations. If this natural period is close to that of light waves, then the amplitude of the oscillations of the dipole charges induced by the varying field of a light wave should noticeably depend *not only on the amplitude* of the electric field of the wave \mathbf{E} , *but also on the period* (or length) of the light wave (*resonance*). Hence, both the amplitude of the variable polarization vector \mathbf{P} of a dielectric and the amplitude of the displacement vector $\mathbf{D} = \epsilon\mathbf{E}$ should noticeably depend on the period or length of a light wave. Thus, when taking into consideration the microscopic structure of dielectrics we should arrive at a definite dependence of the value of the permittivity ϵ on the wavelength, and, therefore, according to Eq. (7.108), at an explanation of light dispersion. By introducing the value of the permittivity measured in a constant or a slowly varying field into Eq. (7.108), we can obviously determine the value of the refractive index only

for comparatively long electromagnetic waves whose period is great in comparison with the period of natural oscillations of the dielectric dipoles. For waves of this kind, Eq. (7.108) is indeed confirmed by experiments.

If the molecules of a dielectric can be considered similar to *solid* dipoles, we encounter phenomena of a somewhat different nature. An appreciable portion of the polarization of such dielectrics consists in orientation of the axes of the dipoles (i.e. molecules) in the direction of the field. In fast-varying (high-frequency) fields, the axes of the dielectric molecules, which have a definite moment of inertia, do not manage to follow the rapid changes in the direction of the field. As a result, the dielectric is polarized much weaker than in a constant electric field having the same intensity. Hence, the value of the permittivity (and, therefore, the refractive index) in dielectrics of this class (water, alcohols, and the like) rapidly falls off with a decreasing period of oscillations of the field. This decrease begins at frequencies that are much lower than the frequency of natural oscillations of the quasielastic molecular dipoles (for example in some alcohols already at radio frequencies).

The circumstance that when studying *light* waves we can consider the permeability μ to equal unity mentioned at the beginning of the present section can be explained in an absolutely similar way. Indeed, the susceptibility of *diamagnetics* is always so insignificant that when determining the velocity of light from the equation $v = c/\sqrt{\epsilon\mu}$ we can disregard the difference of μ from unity. The mechanism of *paramagnetic* magnetization is similar to the mechanism of polarization of a dielectric with solid dipoles. Therefore, the paramagnetic susceptibility χ becomes virtually equal to zero, while the permeability μ equals unity at frequencies that are much lower than the frequency of visible light. Finally, the magnetization of *ferromagnetics* is due to a change in the direction of magnetization in separate Weiss domains magnetized to saturation, and also to a change in the dimensions of these domains (see Sec. 5.13). These processes also have no time to follow the changes in the field of a light wave so that in an optical respect ferromagnetics do not virtually differ from diamagnetics*.

7.12 Propagation of Waves in a Conducting Medium. Reflection of Light from a Metal Surface

1. In dielectrics, electromagnetic waves propagate without damping, while in good conductors—metals—they damp so rapidly that even thin layers of metals are not transparent to the waves. The

* Except for such optical phenomena (as for example, the Faraday effect of rotation of the polarization plane of light in a magnetic field) that depend on the magnetization created in a given body by a constant magnetic field

explanation is, naturally, that the energy of a wave propagating in the medium transforms into Joule heat liberated by the conduction currents induced by the field of the wave.

We shall show first of all that the propagation of waves in a *homogeneous* conductor is not connected with the appearance in it of free electric charges. Using expression (V) for the current density in the continuity equation (IVa) and assuming that there are no extraneous e.m.f.'s in a conductor, we get

$$\frac{\partial \rho}{\partial t} = -\operatorname{div} \mathbf{j} = -\kappa \operatorname{div} \mathbf{E} = -\frac{\kappa}{\varepsilon} \operatorname{div} \mathbf{D} = -\frac{4\pi\kappa}{\varepsilon} \rho$$

The solution of this differential equation is

$$\rho = \rho_0 \exp\left(-\frac{4\pi\kappa}{\varepsilon} t\right) \quad (7.109)$$

where ρ_0 is an arbitrary constant.

Consequently, even if we do introduce free space charges into a conductor in some way or other, the density of these charges will diminish with time to zero according to an exponential law. The greater the conductivity κ , the more rapidly will this dissipation of the charges occur. An electromagnetic field in general cannot create free space charges in a conductor because if $\rho = 0$ at the moment t_0 , then according to Eq. (7.109) it remains equal to zero during the entire following time.

2. Let us consider a monochromatic wave having the frequency ω in a metal, i.e. let us assume that

$$\mathbf{E} = \mathbf{E}_0(x, y, z)e^{i\omega t} \quad \text{and} \quad \mathbf{H} = \mathbf{H}_0(x, y, z)e^{i\omega t}$$

Using these expressions in Maxwell's equations (I)-(IV), taking advantage of Eqs. (V), and assuming, according to Eq. (7.109), that $\rho = 0$, we get after cancelling $e^{i\omega t}$

$$\left(\frac{i\omega\varepsilon}{c} + \frac{4\pi\kappa}{c}\right) \mathbf{E}_0 = \operatorname{curl} \mathbf{H}_0$$

$$\frac{i\omega\mu}{c} \mathbf{H}_0 = -\operatorname{curl} \mathbf{E}_0, \quad \operatorname{div} \mathbf{H}_0 = \operatorname{div} \mathbf{E}_0 = 0$$

These equations differ from their counterparts for dielectrics only in that in the first of them the factor $i\omega\varepsilon/c$ is replaced with the factor, $i\omega\varepsilon/c + 4\pi\kappa/c$. In other words, these equations will coincide with those for a wave in a dielectric if ε is replaced with ε' in the latter

$$\varepsilon' = \varepsilon - \frac{4\pi\kappa}{\omega} i \quad (7.110)$$

Thus, with respect to the propagation of monochromatic waves a conductor is equivalent to a dielectric with a *complex permittivity*

ϵ' . Therefore, when considering waves in a metal we can use directly the results obtained in Secs. 7.10 and 7.11 for waves in dielectrics, substituting ϵ' for ϵ in the formulas of these sections.

For example, the (complex) wavelength constant k' in accordance with Eq. (7.85) will be determined by the formula

$$k'^2 = \frac{\epsilon'\mu\omega^2}{c^2} = \frac{\epsilon\mu\omega^2}{c^2} - \frac{4\pi\kappa\mu\omega}{c^2} i \quad (7.111)$$

It is good to resolve k' into a real and an imaginary parts:

$$k' = k - is, \quad k^2 - s^2 = \frac{\epsilon\mu\omega^2}{c^2}, \quad 2ks = \frac{4\pi\kappa\mu\omega}{c^2} \quad (7.112)$$

We shall take the positive roots of these equations for k and s . In accordance with Eqs. (7.112) and (7.86), the field of a plane monochromatic wave in a conductor propagating along the z -axis is expressed by the equations

$$\left. \begin{aligned} \mathbf{E} &= \mathbf{E}_0 \exp [i(\omega t - k'z)] = \mathbf{E}_0 \exp (-sz) \exp [i(\omega t - kz)] \\ \mathbf{H} &= \mathbf{H}_0 \exp (-sz) \exp [i(\omega t - kz)] \end{aligned} \right\} \quad (7.113)$$

Thus, the complex nature of the wavelength constant k' corresponds to the presence of absorption: the amplitude of the wave exponentially diminishes as it propagates. When $\kappa = 0$, the imaginary part s of the wavelength constant k' becomes equal to zero, and damping of the waves stops.

In accordance with Eq. (7.90), the vectors \mathbf{E} and \mathbf{H} of a wave in a conductor are mutually perpendicular and form a right-handed system with the direction of propagation of the wave. In a conductor, however, the vectors \mathbf{E} and \mathbf{H} are *different phases*, and not identical ones as in a dielectric. Indeed, substituting ϵ' for ϵ in Eq. (7.90), we get

$$\mathbf{H} = \sqrt{\frac{\epsilon'}{\mu}} [\mathbf{nE}]$$

Since the factor $\sqrt{\epsilon'/\mu}$ is complex, then the phase of the vector \mathbf{H} differs from that of \mathbf{E} . This will be treated in greater detail in Sec. 7.13. 3. In Sec. 6.15 devoted to the skin effect, we also studied a periodic field in a conductor, the only difference from our present consideration being that in Sec. 6.15 we ignored the displacement currents in a conductor in comparison with the conduction currents. Since according to Sec. 6.13, the displacement currents in metals are weak in comparison with the conduction currents up to frequencies corresponding to the infrared part of the spectrum ($\nu = \omega/2\pi \approx \approx 10^{14} \text{ s}^{-1}$), then the results of the present section will only insignificantly differ from those of Sec. 6.15 for lower frequencies.

Indeed, solving Eqs. (7.112) with respect to k and s , we obtain

$$\left. \begin{aligned} k^2 &= \frac{\varepsilon\mu\omega^2}{2c^2} \left\{ \sqrt{1 + \left(\frac{4\pi\kappa}{\varepsilon\omega}\right)^2} + 1 \right\} \\ s^2 &= \frac{\varepsilon\mu\omega^2}{2c^2} \left\{ \sqrt{1 + \left(\frac{4\pi\kappa}{\varepsilon\omega}\right)^2} - 1 \right\} \end{aligned} \right\} \quad (7.114)$$

As we indicated in Sec. 6.13, for metals $\varepsilon v \ll \kappa$, i.e. $\varepsilon\omega \ll 2\pi\kappa$ up to $v \approx 10^{14} \text{ s}^{-1}$. Therefore even for light waves, we can disregard unity in Eq. (7.114) in comparison with $4\pi\kappa/\varepsilon\omega$. Thus, with a sufficient degree of accuracy,

$$k^2 = s^2 = \frac{2\pi\mu\kappa\omega}{c^2} \quad (7.115)$$

which coincides with Eq. (6.116) for p^2 . Thus, Eq. (7.113) practically coincides with the previously found expression (6.117) for the electric vector of a wave in a metal.

As we noted in Sec. 6.15, the depth of penetration of a wave into a metal is determined by the quantity

$$\delta = \frac{1}{p} = \frac{1}{s}$$

because the amplitude of a wave diminishes at this depth e times in comparison with the amplitude at the surface. Since according to Eq. (7.88) the wavelength $\lambda = 2\pi/k$, and since $k = s$, then

$$\delta = \frac{1}{k} = \frac{\lambda}{2\pi} \quad (7.116)$$

Thus, only about one-sixth of a wavelength fits into the distance δ , i.e. there is no *spatial* periodicity of the field of a wave in a metal. We shall give as an illustration the following data on the depth of penetration of fields having various frequencies ω into copper ($\kappa = 5.14 \times 10^{17} \text{ s}^{-1}$), where λ_0 signifies the length of the relevant wave in a vacuum: $\lambda_0 = 2\pi c/\omega$.

λ_0	10^{-4} cm	1 cm	$100 \text{ m} = 10^4 \text{ cm}$
δ	$3.8 \times 10^{-7} \text{ cm}$	$3.8 \times 10^{-5} \text{ cm}$	$3.8 \times 10^{-3} \text{ cm}$

4. The *reflection of light* from a metal surface is much more complicated than reflection on the boundary of dielectrics. For example a linearly polarized wave when reflected from a metal becomes elliptically polarized (if the angle of incidence is not 90°). We shall limit ourselves to a consideration of the very simple case of the normal incidence of a plane monochromatic wave from a vacuum onto the surface of a metal.

When solving this problem, we can use the results of Sec. 7.11. Assuming as in Sec. 7.11 that the permeability of the medium μ equals unity and, in addition, that our metal borders on the vacuum, in Eqs. (7.11) we shall have to substitute 1 for ϵ_1 and ϵ' for ϵ_2 . Particularly, the relative refractive index n' of a metal to a vacuum, according to Eq. (7.19), is

$$n' = \sqrt{\epsilon'} \tag{7.117}$$

i.e. we shall have a complex value. The amplitudes of the electric vector of the reflected and refracted waves with normal incidence of the wave on the metal is determined by Eq. (7.104):

$$E_{0,r} = \frac{1-n'}{1+n'} E_0 \quad \text{and} \quad E_{0,g} = \frac{2}{1+n'} E_0 \tag{7.118}$$

Assuming in Eqs. (7.11) that $k = k_0$ and $k_g = k' = k - is$, we get

$$\left. \begin{aligned} E_x &= E_0 \exp[i(\omega t - k_0 z)] \\ E_{x,r} &= \frac{1-n'}{1+n'} E_0 \exp[i(\omega t + k_0 z)] \\ E_{x,g} &= \frac{2}{1+n'} E_0 \exp[-sz + i(\omega t - kz)] \end{aligned} \right\} \tag{7.119}$$

The real part of these complex expressions equals

$$\left. \begin{aligned} E_x &= E_0 \cos(\omega t - k_0 z) \\ E_{x,r} &= \left| \frac{1-n'}{1+n'} \right| E_0 \cos(\omega t + k_0 z - \varphi) \\ E_{x,g} &= \left| \frac{2}{1+n'} \right| E_0 e^{-sz} \cos(\omega t - kz - \psi) \end{aligned} \right\} \tag{7.120}$$

where the angles φ and ψ should be determined from the relationship

$$\frac{1-n'}{1+n'} = \left| \frac{1-n'}{1+n'} \right| e^{-i\varphi} \quad \text{and} \quad \frac{1}{1+n'} = \frac{1}{|1+n'|} e^{-i\psi} \tag{7.121}$$

Here, for instance, $|1+n'|$ signifies the modulus of the complex quantity $1+n'$. Thus, since n' is the complex quantity, the phases of the reflected and the refracted waves will not coincide on the boundary with the phase of the incident wave as occurs in dielectrics, but will be shifted relative to it through the angles φ and ψ , respectively.

In accordance with Eqs. (7.105) and (7.107), the mean densities of the energy flux in the incident and the reflected wave during a period and the reflection factor r will be

$$\bar{S} = \frac{c}{8\pi} E_0^2, \quad \bar{S}_r = \frac{c}{8\pi} \left| \frac{1-n'}{1+n'} \right|^2 E_0^2, \quad r = \left| \frac{1-n'}{1+n'} \right|^2 \quad (7.122)$$

Since κ for metals is of the order of magnitude of 10^{16} to 10^{17} absolute units (s^{-1}), then $4\pi\kappa/\omega \gg 1$ up to the frequencies of visible light. Hence, according to Eqs. (7.110) and (7.117), the modulus of n' is also much greater than unity. Therefore, the reflection factor of metal surfaces r is close to unity. For example, even for the yellow line of sodium, r is 0.95 for Ag, 0.85 for Au, 0.83 for Al, and 0.74 for Cu.

7.13 Light Pressure. Momentum of an Electromagnetic Field

1. Bodies placed in the field of a light wave are acted upon in this field (as in any electromagnetic field) by mechanical (ponderomotive) forces called *light pressure*. The light pressure is associated with the absorption and reflection of light by a very simple relationship whose investigation leads to very important physical corollaries.

Light propagating in a homogeneous transparent medium exerts no ponderomotive action on this medium* so that the pressure of light is connected either with its absorption or with a change in the direction of its propagation (reflection, refraction, dispersion).

Let us limit ourselves to considering the very simple case of the normal incidence of light on the surface of a metal. The latter is acted upon by pressure in the direction of the incident wave. This pressure is due to the fact that the field of the wave induces periodic conduction currents \mathbf{j} in the metal that are acted upon by a Lorentz force on the part of the magnetic field of the same light wave.

* This statement is not completely correct. First, even a transparent homogeneous medium is acted upon in a light field by the striction forces (see Secs. 2.13 and 6.8).

$$f = \frac{1}{8\pi} \text{grad} \left\{ E^2 \frac{\partial \varepsilon}{\partial \tau} \tau + H^2 \frac{\partial \mu}{\partial \tau} \tau \right\}$$

These forces, however, pulsate with the double period of the light and therefore do not lend themselves to observation in practice. Second, if the passage of light through a medium is attended by turning of its plane of polarization, then the elements of the volume of the medium experience a torsion action (see the footnote on p. 543).

We shall calculate the magnitude of this pressure for the case dealt with in the preceding section: a wave having the frequency ω whose electric vector is directed along the x -axis, propagating in a vacuum along the z -axis, is reflected from the surface of the metal $z = 0$. The electric field \mathbf{E}_g inside the metal is determined by Eq. (7.120). Under the action of this field, currents directed along the x -axis will appear in the metal and their density will be

$$j_x = \kappa E_{x, g} = \frac{2\kappa}{|1+n'|} E_0 e^{-sz} \cos(\omega t - kz - \psi) \quad (7.123)$$

The intensity of the magnetic field inside the metal, according to Eq. (7.90), will be directed along the y -axis. Assuming in Eq. (7.90) that $\mu = 1$ and $\sqrt{\varepsilon} = n'$ [see Eq. (7.117)], we get the following complex expression for $H_{y, g}$ on the basis of Eq. (7.119):

$$H_{y, g} = \frac{2n'}{1+n'} E_0 \exp[-sz + i(\omega t - kz)] \quad (7.124)$$

Let us assume that

$$n' = |n'| e^{-i\chi} \quad (7.125)$$

Hence with the aid of Eq. (7.121), we get

$$\frac{n'}{1+n'} = \frac{|n'|}{|1+n'|} e^{-i(\psi+\chi)}$$

and, consequently, the real part of Eq. (7.124) is

$$H_{y, g} = \frac{2|n'|}{|1+n'|} E_0 e^{-sz} \cos(\omega t - kz - \psi - \chi)$$

The density \mathbf{f} of the ponderomotive forces acting on the currents (7.123) in the magnetic field of a wave is determined by Eq. (5.41):

$$\mathbf{f} = \frac{1}{c} [\mathbf{jH}]$$

(we consider the permeability μ of a metal to equal unity). The vector \mathbf{f} is directed along the z -axis and its absolute value is

$$\begin{aligned} f_z &= \frac{1}{c} \kappa E_{x, g} H_{y, g} = \\ &= \frac{4\kappa |n'|}{c |1+n'|^2} E_0^2 e^{-2sz} \cos(\omega t - kz - \psi) \cos(\omega t - kz - \psi - \chi) \end{aligned}$$

The mean value of this force density during a period will be

$$\bar{f}_z = \frac{2\kappa |n'|}{c |1 + n'|^2} E_0^2 e^{-2sz} \cos \chi$$

According to Eq. (7.125), the quantity $|n'| \cos \chi$ equals the real part of the complex refractive index $n' = \sqrt{\epsilon'}$. On the other hand, according to Eqs. (7.111) and (7.112), when $\mu = 1$ we have

$$\epsilon' = \frac{c^2 k'^2}{\omega^2} \quad \text{and} \quad n' = \sqrt{\epsilon'} = \frac{c}{\omega} k' = \frac{ck}{\omega} - i \frac{cs}{\omega} \quad (7.126)$$

Therefore $|n'| \cos \chi = ck/\omega$ and, consequently,

$$\bar{f}_z = \frac{2k\kappa}{\omega |1 + n'|^2} E_0^2 e^{-2sz}$$

Owing to the presence of the factor e^{-2sz} , the density of the forces \bar{f}_z diminishes very rapidly with an increasing penetration into a metal so that these forces can be considered concentrated on the surface of the metal and can be replaced with the pressure p applied to this surface:

$$p = \int_0^{\infty} \bar{f}_z dz = \frac{k\kappa}{s\omega |1 + n'|^2} E_0^2$$

This pressure p equals the sum of the forces \bar{f}_z related to a unit surface area of the metal. According to Eq. (7.112), when $\mu = 1$ we have $\kappa/s = kc^2/2\pi\omega$, and, therefore, finally

$$p = \frac{k^2 c^2}{2\pi\omega^2 |1 + n'|^2} E_0^2 \quad (7.127)$$

2. Apart from the pressure p connected with the action of the Lorentz force on the currents induced in a conductor, forces also act on the surface of a conductor that depend on the (real) value of its permittivity ϵ and permeability μ . To avoid complications in our calculations, we shall assume that not only μ , but also ϵ for a metal equals unity. Now Eq. (7.127) will determine the *total* light pressure on a metal*.

* When ϵ and μ are arbitrary, apart from the pressure given by Eq. (7.127) we must also take into account the pressure due to ponderomotive forces

$$\mathbf{f} = - \frac{1}{8\pi} (E^2 \nabla \epsilon + H^2 \nabla \mu)$$

The final equation (7.128) in this case too, however, remains correct for the *total* pressure on the surface of a conductor.

We shall show that this pressure p differs only in the factor $1/c$ from the sum of the mean densities of the energy flux in the incident and reflected wave during a period:

$$p = \frac{1}{c} (\bar{S} + \bar{S}_r) = \frac{1+r}{c} \bar{S} \tag{7.128}$$

where \bar{S} , \bar{S}_r , and r are determined by Eqs. (7.122). According to these equations

$$1 + r = \frac{|1 + n'|^2 + |1 - n'|^2}{|1 + n'|^2}$$

Further, according to Eq. (7.126)

$$\begin{aligned} |1 + n'|^2 + |1 - n'|^2 &= \\ &= \left(1 + \frac{ck}{\omega}\right)^2 + 2 \frac{c^2 s^2}{\omega^2} + \left(1 - \frac{ck}{\omega}\right)^2 = 2 \left\{1 + \frac{c^2(k^2 + s^2)}{\omega^2}\right\} \end{aligned}$$

Expressing s^2 through k^2 with the aid of Eq. (7.112) and assuming that $\epsilon = \mu = 1$, we get

$$|1 + n'|^2 + |1 - n'|^2 = \frac{4c^2 k^2}{\omega^2}$$

and, therefore

$$(1 + r)\bar{S} = \frac{4c^2 k^2}{\omega^2 |1 + n'|^2} \bar{S} = \frac{c^2 k^2}{2\pi\omega^2 |1 + n'|^2} cE_0^2$$

which indeed differs from Eq. (7.127) for p only by the factor c . 3. Equation (7.128) allows us to determine the *momentum of an electromagnetic field*.

The evolution which the concept of momentum or impulse has undergone is very similar to that of the concept of energy. Similar to the latter, the concept of momentum was first applied only to mechanical motions and was defined as the product of the mass of a body and its velocity. The law of conservation of mechanical momentum corresponds to the law of conservation of mechanical energy. These laws, however, have only a limited field of application.

With the passage of time, the concepts of energy and momentum were so generalized that they began to cover not only mechanical, but also all other possible kinds of energy and momentum. This made it possible to formulate universal laws of conservation of energy and momentum which take account of the possibility of their conversion from one form into another. Particularly, the establishment of the circumstance that light exerts a pressure on material bodies made it necessary to also ascribe a definite momentum to the field of an electromagnetic wave.

Indeed, we shall proceed from the universal applicability of the law of conservation of momentum*. Let us assume that light impinges on the flat surface of a metal mirror in a vacuum normally to this surface and exerts the pressure p on it. If the area of the mirror is S , then the total force acting on the mirror will be $\mathbf{F} = pS\mathbf{n}$, where the unit vector \mathbf{n} coincides in direction with the incident wave. According to the equations of mechanics, the force \mathbf{F} will cause acceleration of the mirror, and the mechanical momentum of the mirror \mathbf{G}_m will change according to the law

$$\frac{d\mathbf{G}_m}{dt} = \mathbf{F} = pS\mathbf{n}$$

But if the law of conservation of momentum is valid, then \mathbf{G}_m can change only at the expense of a corresponding change in some other form of momentum. Since the reflection and absorption of light are the only processes attending acceleration of the mirror in these conditions, then only light can be the carrier of this other form of momentum. We must therefore assign a definite *momentum* or *impulse* \mathbf{G} to electromagnetic waves, and the change in the momentum of light \mathbf{G} in the process being considered must comply with the condition

$$\frac{d}{dt} (\mathbf{G}_m + \mathbf{G}) = 0 \quad (7.129)$$

* It must be noted that the correctness of the law of conservation of momentum is an essential condition for the application of the law of conservation of energy in an arbitrary inertial frame of reference. Let us consider as a very simple example the elastic collision of two bodies having the masses m_1 and m_2 whose velocities before and after the collision are $\mathbf{u}_1, \mathbf{u}_2$ and $\mathbf{U}_1, \mathbf{U}_2$, respectively. According to the law of conservation of energy,

$$m_1 u_1^2 + m_2 u_2^2 = m_1 U_1^2 + m_2 U_2^2$$

Measuring the velocities of these bodies relative to a different frame of reference S' uniformly moving with the velocity \mathbf{v} relative to the initial frame S , we get $\mathbf{u}'_1 = \mathbf{u}_1 - \mathbf{v}$, $\mathbf{U}'_1 = \mathbf{U}_1 - \mathbf{v}$, etc. Hence, $u_1'^2 = u_1^2 + 2\mathbf{u}_1 \cdot \mathbf{v} + v^2$, etc. Thus, the preceding equation becomes

$$m_1 u_1'^2 + m_2 u_2'^2 + 2\mathbf{v}(m_1 \mathbf{u}'_1 + m_2 \mathbf{u}'_2) = m_1 U_1'^2 + m_2 U_2'^2 + 2\mathbf{v}(m_1 \mathbf{U}'_1 + m_2 \mathbf{U}'_2)$$

Therefore, the law of conservation of energy will hold in the inertial frame of reference S' only if the equation

$$\mathbf{v}(m_1 \mathbf{u}'_1 + m_2 \mathbf{u}'_2) = \mathbf{v}(m_1 \mathbf{U}'_1 + m_2 \mathbf{U}'_2)$$

is observed. Owing to the arbitrary nature of the velocity \mathbf{v} of the frame S' relative to the initial frame S , this condition is equivalent to the equation

$$m_1 \mathbf{u}'_1 + m_2 \mathbf{u}'_2 = m_1 \mathbf{U}'_1 + m_2 \mathbf{U}'_2$$

expressing the law of conservation of momentum.

If we proceeded from relativistic mechanics instead of from classical mechanics, we would arrive at similar relationships.

i.e. it must equal

$$\frac{d\mathbf{G}}{dt} = -pS\mathbf{n}$$

The right-hand side of this equation is, strictly speaking, not the instantaneous value of the force acting on the mirror, but the value of this force averaged over the period of the light wave. Therefore, the left-hand side should also contain the derivative of the mean momentum of the field during a period

$$\frac{d\bar{\mathbf{G}}}{dt} = -pS\mathbf{n}$$

(It is easy to see that the derivative of the mean value of \mathbf{G} with respect to the period equals the mean value of the derivative of \mathbf{G} with respect to the period.)

Let us now use Eq. (7.128) and take into account that the energy flux \mathbf{S} in the incident wave is directed along the vector \mathbf{n} and in the reflected wave is opposite to \mathbf{n} . With the aid of Eq. (7.128), the last equation can be written in the following form:

$$\frac{d\bar{\mathbf{G}}}{dt} = -\frac{1}{c}(\bar{S}\mathbf{n} + \bar{S}_r\mathbf{n})S = -\frac{1}{c}(\bar{S} - \bar{S}_r)S$$

Dividing this equation by the area of the mirror S and introducing the symbol $\bar{\mathbf{g}} = \bar{\mathbf{G}}/S$, we get

$$\frac{d\bar{\mathbf{g}}}{dt} = -\frac{1}{c}\bar{\mathbf{S}} + \frac{1}{c}\bar{\mathbf{S}}_r \tag{7.130}$$

It is natural to interpret this equation (proved for mean quantities during the period of a wave) in the sense that the momentum of an electromagnetic field is distributed in space with the volume density \mathbf{g} whose instantaneous value at each point of the field is [see Eq. (7.11)]

$$\mathbf{g} = \frac{1}{c^2}\mathbf{S} = \frac{1}{4\pi c}[\mathbf{E}\mathbf{H}] \tag{7.131}$$

Indeed, upon this assumption, the incident wave brings to each unit surface area of the mirror in a unit time the entire momentum that is confined in the volume of the wave field having a cross-sectional area of 1 cm^2 and a length of $c \text{ cm}$ (since we have assumed that the wave is propagating in a vacuum, then its velocity is c). This momentum equals

$$c\mathbf{g}_0 = \frac{1}{c}\bar{\mathbf{S}}$$

where the bar over a symbol should signify space averaging over the length of the wave, which, however, is equivalent to time averaging during the period of the wave. Accordingly, the reflected wave carries away in a unit time from a unit surface area of the mirror the momentum

$$c\bar{\mathbf{g}}_r = \frac{1}{c} \bar{\mathbf{S}}_r$$

Thus, in a unit time the momentum $c\bar{\mathbf{g}}_0$ is *absorbed* and the momentum $c\bar{\mathbf{g}}_r$ *appears* anew on a unit area of the mirror, so that the total change in the momentum of the light is

$$-c\bar{\mathbf{g}}_0 + c\bar{\mathbf{g}}_r = -\frac{1}{c} \bar{\mathbf{S}} + \frac{1}{c} \bar{\mathbf{S}}_r$$

which indeed coincides with Eq. (7.130).

Hence, similar to how the pressure of a gas on the walls of a vessel is explained by the fact that the molecules of the gas rebounding from the walls transmit to them a definite momentum, so is the pressure of light on the mirror due to transmission of electromagnetic momentum to the latter by the light reflected from it.

4. We have shown that the assumption expressed by Eq. (7.131) allows us to give a simple physical interpretation of light pressure.

This phenomenon, at any rate, makes us ascribe a definite momentum to the field of a light wave. The assumption, in turn, that the electromagnetic momentum is distributed over the volume of a field with the density \mathbf{g} determined by Eq. (7.131), and that this equation may be applied to an arbitrary electromagnetic field is one of the fundamental postulates of the theory of a field and is completely justified by experiments*.

It must be noted, however, that the pressure of light and the electromagnetic momentum are so small that it is very difficult to measure them directly in experiments. For instance, a mirror at a distance of one metre from a source of light having an intensity of a million candelas is acted upon with a pressure of only 10^{-4} dyn/cm² by the visible light of this source.

* This assumption is an independent postulate of theory only because we take no account of the principle of relativity. Proceeding from this principle, we can show that four quantities, namely one scalar—the density of electromagnetic energy u , two vectors—Poynting's vector \mathbf{S} and the vector of the density of electromagnetic momentum \mathbf{g} , and, finally, the stress tensor of an electromagnetic field \mathbf{T} are different components of a *single tensor in the four-space*, the so-called energy-momentum tensor. The requirements of relativistic invariance restrict the possible form of this tensor to such an extent that, for instance, for a vacuum the quantities u , \mathbf{S} , and \mathbf{g} are unambiguously determined by setting the stress tensor \mathbf{T} in 3-space as a function of \mathbf{E} and \mathbf{H} .

Light pressure was experimentally detected and measured for the first time by P. Lebedev in Moscow in 1901. Both his results and the more accurate ones of following investigators agree with theory within the limits of the errors of experiment (up to 2%).

Light pressure plays a significant part only in two fields that are opposite in their scale—in astronomical phenomena and in atomic ones. For instance, the gravitational attraction of the upper layers of stars to their centre is balanced to a considerable extent by the pressure of the light flux flowing outward from the centre of a star. Among atomic phenomena, we shall mention the “light recoil” which an excited atom is subjected to when it emits light, and also the Compton effect related to light pressure. The latter consists in that gamma-rays transfer part of their momentum to the electrons on which they are scattered, and thus impart great velocities to these electrons.

**7.14 Electromagnetic Angular Momentum.
A Particular Case of a Static Field**

1. Both the concept of the electromagnetic momentum \mathbf{g} and that of the flux of electromagnetic energy (Poynting’s vector \mathbf{S}) were formulated on the basis of studying varying (particularly wave) fields. Let us now use a particular example to see the conclusions which the application of these concepts to static fields leads to.

Let us consider a cylindrical capacitor placed in a uniform magnetic field \mathbf{H} parallel to its axis (Fig. 86). In the space between the capacitor

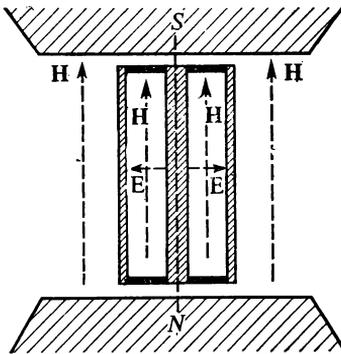


Fig. 86

plates, apart from the magnetic field, there is also a radial electric field having the intensity $\mathbf{E} = 2q\mathbf{r}/lr^2$, where q is the charge on the inner plate of the capacitor numerically equal, but opposite in sign to the charge on its outer plate, l is the length of the capacitor, and \mathbf{r} is the vector distance from a point of the field to the capacitor axis [see Eq. (1.16)].

We assume that the length of the capacitor is so much greater than its diameter that we can disregard the edge effects, i.e. we may use Eq. (1.16) that strictly holds only for an infinitely long capacitor. Further, we shall assume for simplicity that the capacitor is filled with a gas having a permittivity whose difference from unity may be ignored.

Poynting's vector in the space between the plates of the capacitor differs from zero and equals

$$\mathbf{S} = \frac{c}{4\pi} [\mathbf{E}\mathbf{H}] = \frac{cq}{2\pi r^2 l} [\mathbf{r}\mathbf{H}]$$

The lines of Poynting's vector, i.e. the lines of the energy flux, are concentric circles whose planes are perpendicular to the axis of the capacitor.

We thus arrive at the notion of the continuous circulation of energy around closed paths in a *static* electromagnetic field. This notion does not lead to any corollaries that could be directly verified experimentally, and is therefore deprived of a direct physical meaning.

Let us take into consideration, however, that according to Eq. (7.131) the density of the electromagnetic momentum \mathbf{g} is proportional to Poynting's vector \mathbf{S}^* . The statement that a definite momentum having the volume density

$$\mathbf{g} = \frac{1}{c^2} \mathbf{S} = \frac{q}{2\pi cr^2 l} [\mathbf{r}\mathbf{H}] \quad (7.132)$$

is localized in the static field being considered is very essential and leads to corollaries that can (at least in principle) be verified experimentally.

True, the total momentum of the entire *static* field as a whole equals zero by necessity. Knowing the space distribution of the electromagnetic momentum, however, we can determine the *moment* of this *momentum* or the *angular momentum* \mathbf{L} relative to the centre of inertia of the capacitor by the formula

$$\mathbf{L} = \int [\mathbf{R}\mathbf{g}] dV \quad (7.133)$$

which is absolutely similar to the formula

$$\mathbf{L} = \sum_i [\mathbf{R}_i \mathbf{g}_i] = \sum_i m_i [\mathbf{R}_i \mathbf{v}_i]$$

determining the mechanical angular momentum of a system of material points having the mass m_i and the velocity \mathbf{v}_i . If we discharge

* It must be noted that the relationship $\mathbf{g} = \frac{1}{c^2} \mathbf{S}$ follows from the main postulates of the theory of relativity and must be observed for any fields in a vacuum, and also for all forms of energy and momentum in general with a strictly *microscopic* interpretation of them.

our capacitor, then both the electric field \mathbf{E} and the electromagnetic angular momentum \mathbf{L} will vanish. On the basis of the law of conservation of the angular momentum, we must conclude that the system capacitor + magnet inducing the field \mathbf{H} acquires in the course of discharge a *mechanical* angular momentum equal to \mathbf{L} . If, for example, the magnet is fixed in place, while the capacitor can rotate freely about its axis, then in the course of the discharge it should acquire an angular velocity equal to $\omega = L/I$, where I is the moment of inertia of the capacitor relative to its axis.

This conclusion, which can be verified experimentally and which follows from the assumption on the localization of the momentum having the density \mathbf{g} in an electromagnetic field, can be confirmed by direct calculations, and we shall now pass over to them.

2. The application of the law of conservation of angular momentum assumes that the system being considered is not subjected during discharge to the action of external forces (or that the moment of these forces equals zero). This condition will be observed if we assume, for instance, that the capacitor is discharged by bringing close to it a radioactive substance causing ionization of the gas between the capacitor plates. If the density of the discharge current in the gas is \mathbf{j} , then each element of volume of the gas dV will be acted upon by the force $\frac{1}{c} [\mathbf{j}\mathbf{H}] dV$ during discharge. The moment of all the forces acting on the gas will be

$$\mathbf{N} = \frac{1}{c} \int [\mathbf{R}[\mathbf{j}\mathbf{H}]] dV \quad (7.134)$$

where integration should be performed over the entire space between the capacitor plates. Owing to the friction between the gas and the plates, this moment in the long run will be transmitted to the entire capacitor as a whole.

Let us introduce the cylindrical system of coordinates $z, r,$ and α whose z -axis coincides with the axis of the capacitor. If there were no magnetic field, the vector of the current density would be radial ($\mathbf{j} = |j_r|$). The lines of current in the magnetic field \mathbf{H} owing to the Hall effect acquire a helical shape, and the component j_α will differ from zero. The component j_z owing to axial symmetry of the field remains equal to zero: $j_z = 0$.

Further, owing to the same symmetry, the moment \mathbf{N} of the forces applied to the gas will be directed along the z -axis ($N_x = N_y = 0$).

Evaluating the triple vector product in Eq. (7.134), we get

$$\mathbf{N} = \frac{1}{c} \int \{ \mathbf{j}(\mathbf{R}\mathbf{H}) - \mathbf{H}(\mathbf{R}\mathbf{j}) \} dV$$

Taking into account that the field \mathbf{H} is uniform, $j_z = 0$, and that, consequently, $\mathbf{R}\mathbf{j} = \mathbf{r}\mathbf{j} = rj_r$, we have

$$N_z = - \frac{H_z}{c} \int rj_r dV$$

The current flowing at the moment of time t through a cylindrical surface having the radius r and coaxial with the capacitor is

$$I(t) = 2\pi r l \cdot j_r$$

the current being considered positive if it is directed outward from the axis of the capacitor. The current $I(t)$ does not depend on the radius r of the cylindrical surface (we disregard the possibility of accumulating space charges in the space of the capacitor)*. Therefore,

$$N_z = - \frac{H_z I(t) V}{2\pi c l}$$

where V is the volume of the capacitor space. Finally, the current $I(t)$ flowing through the capacitor from its inner to its outer plate equals the rate of diminishing of the charge q on the inner plate of the capacitor:

$$I(t) = - \frac{dq}{dt}$$

Since the vectors \mathbf{N} and \mathbf{H} are directed along the z -axis, we finally have

$$\mathbf{N} = \frac{\mathbf{H}V}{2\pi c l} \frac{dq}{dt}$$

The mechanical angular momentum of the capacitor \mathbf{L}_m under the action of the moment of the forces \mathbf{N} will change according to the law

$$\frac{d}{dt} \mathbf{L}_m = \mathbf{N} = \frac{\mathbf{H}V}{2\pi c l} \frac{dq}{dt} \quad (7.135)$$

On the other hand, the total angular momentum of the field inside the capacitor, according to Eqs. (7.133) and (7.132), is

$$\mathbf{L} = \int [\mathbf{R}\mathbf{g}] dV = \frac{q}{2\pi c l} \int \frac{[\mathbf{R}[\mathbf{r}\mathbf{H}]]}{r^2} dV$$

The triple product inside the integral equals

$$[\mathbf{R}[\mathbf{r}\mathbf{H}]] = \mathbf{r}(\mathbf{R}\mathbf{H}) - \mathbf{H}(\mathbf{R}\mathbf{r})$$

where

$$\mathbf{R}\mathbf{r} = r^2$$

* If we assume that space charges form, then it is necessary to correspondingly change the expression for the intensity of the electric field in the capacitor. As a result, we again arrive at Eq. (7.136).

Consequently,

$$L_z = - \frac{qH_z V}{2\pi cl}$$

Since it follows from considerations of symmetry that the components of the vector \mathbf{L} along the x and y axes equal zero, then upon comparing the last equation with Eq. (7.135), we get

$$\frac{d}{dt} \mathbf{L}_m - \frac{d}{dt} \mathbf{L}$$

or, finally,

$$\frac{d(\mathbf{L} + \mathbf{L}_m)}{dt} = 0 \quad (7.136)$$

We have thus confirmed by direct calculation that the sum of the electromagnetic and mechanical angular momenta remains constant in time so that when a capacitor is discharged in a magnetic field it should acquire the mechanical angular momentum \mathbf{L} equal to \mathbf{L}_m . 3. The above example shows that the concept of the electromagnetic angular momentum is fruitful even for static fields. This concept plays a still greater part when studying varying fields, especially radiation fields.

True, in the sphere of macroscopic phenomena, the experimental measurement of the electromagnetic momentum and angular momentum is a very difficult task owing to the negligible value of the effects related to them*. In the field of atomic phenomena, the exchange of angular momentum between light and a substance is very significant. For example, the emission of light by an excited atom, in general, is connected with a change in the angular momentum of the electrons in the atom, this change as regards its order of magnitude being comparable with the absolute value of the angular momentum of the atom.

7.15 Stress Tensor and Ponderomotive Forces of an Electromagnetic Field

1. In Secs. 2.15 and 6.9, we found an expression for the stress tensor \mathbf{T} in stationary electric and magnetic fields. We shall now show that if we assume the applicability of these expressions to an arbitrary varying electromagnetic field, then the conclusions following from

* Compare with what was said in Sec. 7.13 on light pressure. Experimenters succeeded in measuring the angular momentum transmitted by light to a quartz plate when it passed through this plate; the transmission of the angular momentum is connected with the rotation of the plane of polarization of light in the plate.

this assumption will be in complete accordance with the results of the present chapter.

We saw in Secs. 2.15 and 6.9 that the stress tensor of an electromagnetic field \mathbf{T} can be resolved into the sum of the Maxwellian tensor \mathbf{T}' and the striction stress tensor \mathbf{T}'' . In the following, for simplicity, we shall omit a consideration of the striction stresses \mathbf{T}'' leading only to a redistribution of the ponderomotive forces over the volume of the bodies in a field, but not changing the resultant of these forces, and shall identify the total stress tensor \mathbf{T} with the Maxwellian tensor \mathbf{T}' . For these conditions, the stress tensor \mathbf{T} will be expressed by the sum of Eqs. (2.122) and (6.81), which can be written as follows:

$$T_{ik} = \frac{1}{4\pi} (E_i D_k + H_i B_k) - \frac{1}{8\pi} \delta_{ik} (\mathbf{E}\mathbf{D} + \mathbf{H}\mathbf{B}) \quad (7.137)$$

The subscripts i and k in this expression can take on any of the values of x , y , and z , while the quantity δ_{ik} is determined by the equations

$$\delta_{ii} = \delta_{xx} = \delta_{yy} = \delta_{zz} = 1, \quad \delta_{ik} = 0 \quad \text{when } i \neq k \quad (7.138)$$

It is more convenient, however, to introduce the notation $x = x_1$, $y = x_2$, and $z = x_3$, and give the subscripts i and k the values 1, 2, and 3. Using this notation, the formula (2.117) expressing the density of the ponderomotive forces \mathbf{f} becomes

$$f_i = \sum_{k=1}^3 \frac{\partial T_{ik}}{\partial x_k}$$

Using the values of the components T_{ik} given by Eq. (7.137), we get

$$\begin{aligned} \sum_k \frac{\partial T_{ik}}{\partial x_k} &= \frac{1}{4\pi} \sum_k \left(E_i \frac{\partial D_k}{\partial x_k} + \frac{\partial E_i}{\partial x_k} D_k + \right. \\ &\left. + H_i \frac{\partial B_k}{\partial x_k} + \frac{\partial H_i}{\partial x_k} B_k \right) - \frac{1}{8\pi} \frac{\partial}{\partial x_i} (\mathbf{E}\mathbf{D} + \mathbf{H}\mathbf{B}) \end{aligned}$$

We get the last term on the basis of Eq. (7.138):

$$\begin{aligned} - \frac{1}{8\pi} \sum_k \delta_{ik} \frac{\partial}{\partial x_k} (\mathbf{E}\mathbf{D} + \mathbf{H}\mathbf{B}) &= \\ = - \frac{1}{8\pi} \delta_{ii} \frac{\partial}{\partial x_i} (\mathbf{E}\mathbf{D} + \mathbf{H}\mathbf{B}) &= - \frac{1}{8\pi} \frac{\partial}{\partial x_i} (\mathbf{E}\mathbf{D} + \mathbf{H}\mathbf{B}) \end{aligned}$$

Let us take into consideration that according to Maxwell's equations (III) and (IV)

$$\sum_k \frac{\partial D_k}{\partial x_k} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \text{div } \mathbf{D} = 4\pi\rho$$

$$\sum_k \frac{\partial B_k}{\partial x_k} = \text{div } \mathbf{B} = 0$$

Further,

$$\begin{aligned} \frac{\partial}{\partial x_i} (\mathbf{ED}) &= \sum_k \frac{\partial}{\partial x_i} (E_k D_k) = \\ &= \sum_k \left(\frac{\partial E_k}{\partial x_i} D_k + E_k \frac{\partial D_k}{\partial x_i} \right) \end{aligned}$$

and, therefore,

$$\begin{aligned} \frac{1}{4\pi} \sum_k \frac{\partial E_i}{\partial x_k} D_k - \frac{1}{8\pi} \frac{\partial}{\partial x_i} (\mathbf{ED}) &= \\ &= \frac{1}{4\pi} \sum_k \left(\frac{\partial E_i}{\partial x_k} - \frac{\partial E_k}{\partial x_i} \right) D_k + \\ &+ \frac{1}{8\pi} \sum_k \left(\frac{\partial E_k}{\partial x_i} D_k - E_k \frac{\partial D_k}{\partial x_i} \right) \end{aligned}$$

In a similar way, we can also transform the terms

$$\frac{1}{4\pi} \sum_k \frac{\partial H_i}{\partial x_k} B_k - \frac{1}{8\pi} \frac{\partial}{\partial x_i} (\mathbf{HB})$$

For the case of a fixed isotropic medium that we are considering, the relationship between \mathbf{E} and \mathbf{D} and between \mathbf{H} and \mathbf{B} is determined by Maxwell's equations (V), and therefore

$$\begin{aligned} \sum_k \left(\frac{\partial E_k}{\partial x_i} D_k - E_k \frac{\partial D_k}{\partial x_i} \right) &= - \sum_k E_k^2 \frac{\partial \epsilon}{\partial x_i} = - E^2 \frac{\partial \epsilon}{\partial x_i} \\ \sum_k \left(\frac{\partial H_k}{\partial x_i} B_k - H_k \frac{\partial B_k}{\partial x_i} \right) &= - H^2 \frac{\partial \mu}{\partial x_i} \end{aligned}$$

Consequently, $\sum_k \partial T_{ik} / \partial x_k$ can be written as follows:

$$\begin{aligned} \sum_k \frac{\partial T_{ik}}{\partial x_k} &= \rho E_i + \frac{1}{4\pi} \sum_k \left(\frac{\partial E_i}{\partial x_k} - \frac{\partial E_k}{\partial x_i} \right) D_k + \\ &+ \frac{1}{4\pi} \sum_k \left(\frac{\partial H_i}{\partial x_k} - \frac{\partial H_k}{\partial x_i} \right) B_k - \\ &- \frac{1}{8\pi} \left(E^2 \frac{\partial \epsilon}{\partial x_i} + H^2 \frac{\partial \mu}{\partial x_i} \right) \end{aligned} \tag{7.139}$$

Let us consider the component of Eq. (7.139) along the x -axis. The second term in the right-hand side when $i = 1$ equals

$$\frac{1}{4\pi} \sum_{k=1}^3 \left(\frac{\partial E_1}{\partial x_k} - \frac{\partial E_k}{\partial x_1} \right) D_k = \frac{1}{4\pi} \left(\frac{\partial E_1}{\partial x_2} - \frac{\partial E_2}{\partial x_1} \right) D_2 + \\ + \frac{1}{4\pi} \left(\frac{\partial E_1}{\partial x_3} - \frac{\partial E_3}{\partial x_1} \right) D_3$$

In conventional symbols

$$\frac{\partial E_1}{\partial x_2} - \frac{\partial E_2}{\partial x_1} = \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} = -\text{curl}_z \mathbf{E}$$

and

$$\frac{\partial E_1}{\partial x_3} - \frac{\partial E_3}{\partial x_1} = \text{curl}_y \mathbf{E}$$

Consequently,

$$\sum_{k=1}^3 \left(\frac{\partial E_1}{\partial x_k} - \frac{\partial E_k}{\partial x_1} \right) D_k = -D_y \text{curl}_z \mathbf{E} + D_z \text{curl}_y \mathbf{E} = \\ = [\text{curl } \mathbf{E} \cdot \mathbf{D}]_x = [\text{curl } \mathbf{E} \cdot \mathbf{D}]_1$$

The third term in the right-hand side of Eq. (7.139) can be transformed in a similar way, so that the sum of the second and third terms is

$$\frac{1}{4\pi} \sum_k \left(\frac{\partial E_i}{\partial x_k} - \frac{\partial E_k}{\partial x_i} \right) D_k + \\ + \frac{1}{4\pi} \sum_k \left(\frac{\partial H_i}{\partial x_k} - \frac{\partial H_k}{\partial x_i} \right) B_k = \\ = \frac{1}{4\pi} [\text{curl } \mathbf{E} \cdot \mathbf{D}]_i + \frac{1}{4\pi} [\text{curl } \mathbf{H} \cdot \mathbf{B}]_i$$

On the basis of Maxwell's equations (I) and (II), we have

$$[\text{curl } \mathbf{E} \cdot \mathbf{D}] + [\text{curl } \mathbf{H} \cdot \mathbf{B}] = -\frac{1}{c} \left[\frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{D} \right] + \\ + \frac{1}{c} \left[\frac{\partial \mathbf{D}}{\partial t} \cdot \mathbf{B} \right] + \frac{4\pi}{c} [\mathbf{j} \cdot \mathbf{B}]$$

Further

$$-\frac{1}{c} \left[\frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{D} \right] + \frac{1}{c} \left[\frac{\partial \mathbf{D}}{\partial t} \cdot \mathbf{B} \right] = \\ = \frac{1}{c} \left[\mathbf{D} \cdot \frac{\partial \mathbf{B}}{\partial t} \right] + \frac{1}{c} \left[\frac{\partial \mathbf{D}}{\partial t} \cdot \mathbf{B} \right] = \frac{1}{c} \frac{\partial}{\partial t} [\mathbf{D} \cdot \mathbf{B}]$$

Thus, we finally obtain

$$\sum_k \frac{\partial T_{ik}}{\partial x_k} = \rho E_i + \frac{1}{c} [\mathbf{jB}]_i - \frac{1}{8\pi} \left(E^2 \frac{\partial \varepsilon}{\partial x_i} + H^2 \frac{\partial \mu}{\partial x_i} \right) + \frac{\partial}{\partial t} \left(\frac{1}{4\pi c} [\mathbf{DB}] \right)_i \quad (7.140)$$

For the case of the fixed isotropic medium we are considering, the last term in Eq. (7.140) can be written in the following form:

$$\frac{\partial}{\partial t} \left(\frac{1}{4\pi c} [\mathbf{DB}] \right)_i = \frac{\partial g_i}{\partial t} + \frac{(\varepsilon\mu - 1)}{4\pi c} \frac{\partial}{\partial t} [\mathbf{EH}]$$

where the vector

$$\mathbf{g} = \frac{1}{4\pi c} [\mathbf{EH}]$$

signifies, according to Eq. (7.131), the density of the momentum of an electromagnetic field.

If we also introduce the notation

$$f_i^{(1)} = \rho E_i + \frac{1}{c} [\mathbf{jB}]_i \quad (7.141)$$

$$f_i^{(2)} = -\frac{1}{8\pi} (E^2 \text{grad } \varepsilon + H^2 \text{grad } \mu) + \frac{(\varepsilon\mu - 1)}{4\pi c} \frac{\partial}{\partial t} [\mathbf{EH}] \quad (7.142)$$

then Eq. (7.140) becomes

$$\sum_k \frac{\partial T_{ik}}{\partial x_k} = f_i^{(1)} + f_i^{(2)} + \frac{\partial g_i}{\partial t} \quad (7.143)$$

The first term in the right-hand side, $f_i^{(1)}$, equals the component along the i -axis of the density of the ponderomotive forces acting on the free charges ρ and on the electric currents \mathbf{j} . The second term in the right-hand side in Eq. (7.140), $f_i^{(2)}$ is the density of the ponderomotive forces acting on the medium (i.e. on the dielectrics and magnetics). Indeed, in a vacuum, we have $\varepsilon = \mu = 1$, and therefore, according to Eq. (7.142), in a vacuum $f^{(2)} = 0^*$.

* The last term in Eq. (7.142) is called the density of the Abraham force. Its appearance is partly connected with the force action of a magnetic field on a displacement current. This force was recently detected experimentally [*Nature*, **253**: 339 (1975)]. The presence of the Abraham force gives us grounds for choosing the energy-momentum tensor in a dielectric in the form proposed by Abraham. The problem of the energy-momentum tensor is in the discussion stage for about 70 years already, but now it has been solved to a considerable extent [see *Uspekhi fiz. nauk*, **110**: 253, 309 (1973); **118**, vyp. 1 (1976)].

In addition, in a constant field, the last term in Eq. (7.142) equals zero, and the vector determined by this equation actually coincides with the sum of the second terms of Eqs. (2.106) and (6.75) for the density of the forces acting on dielectrics and magnetics.

2. If we denote by \mathbf{f} the total density of the ponderomotive forces:

$$\mathbf{f} = \mathbf{f}^{(1)} + \mathbf{f}^{(2)} \quad (7.144)$$

where $\mathbf{f}^{(1)}$ and $\mathbf{f}^{(2)}$ are determined by Eqs. (7.141) and (7.142), then Eq. (7.140) becomes

$$f_i + \frac{\partial g_i}{\partial t} = \sum_k \frac{\partial T_{ik}}{\partial x_k} \quad (7.145)$$

In stationary fields, $\partial g_i / \partial t = 0$, and this equation coincides with Eq. (2.117); in varying fields, it differs from Eq. (2.117) by the term $\partial g_i / \partial t$ which takes into consideration the change in the electromagnetic momentum density \mathbf{g} with time. This additional term ensures the constancy of the sum of the momentum of material bodies and an electromagnetic field.

Indeed, let us integrate (7.145) over the entire space assuming that the components of the electromagnetic stress tensor sufficiently rapidly diminish toward infinity. The integral of the right-hand side of Eq. (7.145) vanishes in these conditions because, for example, when $i = 1$ we have

$$\sum_k \frac{\partial T_{1k}}{\partial x_k} = \frac{\partial T_{1x}}{\partial x} + \frac{\partial T_{1y}}{\partial y} + \frac{\partial T_{1z}}{\partial z}$$

and, consequently, according to the equation directly preceding Eq. (2.117) on p. 172, we have

$$\int_V \sum_k \frac{\partial T_{ik}}{\partial x_k} dV = \oint_S T_{in} dS$$

Here the surface integral must be taken over the surface S of the volume V , and when this surface is extended to infinity it will presumably vanish. Thus, we get

$$\int f_i dV + \frac{\partial}{\partial t} \int g_i dV = 0$$

or, passing over from components to vectors and replacing the partial derivative with respect to time with the total one (because the integral over the entire space depending only on t is inside the sign $\partial/\partial t$), we have

$$\frac{d}{dt} \int \mathbf{g} dV + \int \mathbf{f} dV = 0 \quad (7.146)$$

On the other hand, the change in the mechanical momentum \mathbf{G}_m of all the bodies in the space is determined by the resultant of all the ponderomotive forces:

$$\frac{d\mathbf{G}_m}{dt} = \int \mathbf{f} dV$$

Therefore, Eq. (7.146) is equivalent to the equation

$$\frac{d}{dt} \left(\mathbf{G} + \int \mathbf{g} dV \right) = 0 \tag{7.147}$$

which expresses the *law of conservation of the total momentum* (mechanical plus electromagnetic).

The important equation (7.145) establishes the relationship between the density of the ponderomotive forces \mathbf{f} , the momentum of the field \mathbf{g} , and the stress tensors T_{ik} in the most general form.

7.16 An Example of Non-Quasistationary Currents: Waves along a Cable

1. In Chapter 6, we considered in detail the properties of quasistationary varying (alternating) currents. Now we shall consider, using a particular example for this purpose, the properties of *fast-varying* currents to which the theory set out in Chapter 6 cannot be applied at all. We shall conduct our treatment by the method of consecutive approximations based on account being taken of the following circumstance.

As we have seen in Sec. 6.15, fast-varying currents virtually completely concentrate on the surface of conductors (the skin effect) in a layer having the thickness

$$\delta = \frac{1}{p} = \frac{c}{\sqrt{2\pi\kappa\mu\omega}} \tag{7.148}$$

where ω = cyclic frequency of the current
 κ = conductivity of the conductor.

The same quantity δ also determines the depth of penetration of the varying field into the metal. Since with increasing ω the thickness δ tends to zero, then when considering fast-varying currents we can assume in a first approximation that $\delta = 0$, i.e. consider the currents to be *surface* ones, which exceedingly simplifies the solution of the problem. Particularly, the density of the surface current \mathbf{i} is unambiguously determined in this approximation by the tangential components of the magnetic field \mathbf{H} at the outer surface of the conductor

because the field inside the conductor equals zero when $\delta = 0$, and the boundary condition (I') in Sec. 7.1 (p. 459) therefore becomes

$$\frac{4\pi}{c} \mathbf{i} = [\mathbf{nH}] \quad (7.149)$$

where \mathbf{n} is an outward normal to the surface of the conductor.

In this first approximation, however, no account at all is taken of the losses of energy of the current for Joule heat: disregarding the penetration of the field into the conductor, we thus disregard the liberation in it of the Joule heat

$$Q = \int \frac{j^2}{\kappa} dV$$

We can say that in this first approximation the conductor is considered to be *ideal*, i.e. having an infinitely great conductivity κ , or, in other words, that in this approximation the resistance of the conductor is disregarded, and together with it the losses for Joule heat. Indeed, when $\kappa = \infty$, both the thickness δ and the Joule heat Q become equal to zero, which will be seen from the second approximation.

Since in concrete problems encountered in experimental and technical physics the losses for Joule heat are not at all small and cannot be ignored in any way, after finding the first approximation we must pass over to a second one. In this second approximation, the values obtained in the first approximation are taken for the field *outside* of conductors. These values serve as the boundary conditions for determining the field *inside* the conductors. After determining the latter, we can also find the Joule heat liberated by the currents, and together with it the damping of waves along conductors.

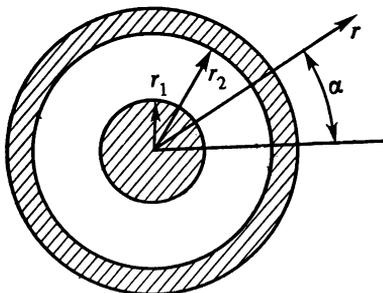


Fig. 87

2. Let us consider as a concrete example the fast-varying currents in a cable consisting of two coaxial cylindrical conductors: an external hollow one with the radius of its internal surface equal to r_2 , and an internal solid one with the radius r_1 (Fig. 87). Assume the space

between these conductors is filled with a homogeneous non-conducting medium having a permittivity ϵ and a permeability μ . In our first approximation, we shall disregard the resistance of the cable, i.e. consider that the cable conductors are ideal. Hence, inside the conductors, the varying field will equal zero, and all the currents will be completely concentrated on the surface of the conductors. In these conditions, the field in the internal space of the cable will evidently be absolutely independent of the one in the external space.

Let us introduce a cylindrical system of coordinates $r, \alpha,$ and z in which the z -axis is directed along the axis of the cable, and r and α are polar coordinates in a plane perpendicular to this axis. Owing to the presumed axial symmetry of the cable, the field in it will also be axially symmetrical, i.e. the field intensities \mathbf{E} and \mathbf{H} and the density of the surface currents \mathbf{i} will not depend on the polar angle α . For the same reason, the direction of the currents in the cable should coincide with that of its axis ($|i_z| = i,$ and $i_\alpha = 0$).

Consequently, the boundary condition (7.149) will become:

when $r = r_1$

$$\frac{4\pi}{c} i_1 = [\mathbf{n}_1\mathbf{H}]_z, \quad [\mathbf{n}_1\mathbf{H}]_\alpha = 0$$

when $r = r_2$

$$\frac{4\pi}{c} i_2 = [\mathbf{n}_2\mathbf{H}]_z, \quad [\mathbf{n}_2\mathbf{H}]_\alpha = 0$$

where i_1 and $i_2 =$ densities of the surface currents on the first and second cylinders, respectively

\mathbf{n}_1 and $\mathbf{n}_2 =$ outward normals to their surfaces.

It is easy to see that these equations can be written in the following form:

when $r = r_1$

$$\left. \begin{aligned} \frac{4\pi}{c} i_1 = H_\alpha, \quad H_z = 0 \end{aligned} \right\}$$

when $r = r_2$

$$\left. \begin{aligned} \frac{4\pi}{c} i_2 = -H_\alpha, \quad H_z = 0 \end{aligned} \right\}$$

(7.150)

Further, it follows from the boundary conditions (II') and (III') and from the assumption that the varying field does not penetrate into the conductors that at the surface of these conductors

$$E_\alpha = E_z = H_r = 0 \tag{7.151}$$

Finally, from the boundary condition (IV') it follows that when $r = r_1$

$$\left. \begin{aligned} D_r &= \varepsilon E_r = 4\pi\sigma_1 \\ \text{and when } r &= r_2 \\ D_r &= \varepsilon E_r = -4\pi\sigma_2 \end{aligned} \right\} \quad (7.152)$$

Conditions (7.151) and part of the conditions (7.150) will be complied with if we assume that *in the entire space* between the cable casings only the radial component E_r of the vector \mathbf{E} and the azimuthal component H_α of the vector \mathbf{H} differ from zero. Having found the corresponding solution of Maxwell's equations, owing to the unambiguity of these equations we can be sure that this solution is the only required solution of the problem we are considering.

3. Thus, let us assume that

$$\left. \begin{aligned} E_\alpha &= E_z = H_z = H_r = 0 \\ E_r &= E_0(r, z) e^{i\omega t} \\ H_\alpha &= H_0(r, z) e^{i\omega t} \end{aligned} \right\} \quad (7.153)$$

where ω is the cyclic frequency of the current, while the functions E_0 and H_0 depend only on r and z , but not on α and t .

Using Eqs. (A.22) and (A.32) which express the divergence and the curl of an arbitrary vector \mathbf{a} in a cylindrical system of coordinates, we see that expressions (7.153) always satisfy Maxwell's third equation

$$\operatorname{div} \mathbf{B} = \mu \operatorname{div} \mathbf{H} = 0$$

(we have assumed that μ and ε are constant quantities), and that Maxwell's equation (IV) becomes

$$\frac{\partial(rE_0)}{\partial r} = 0$$

whence

$$E_0 = \frac{A(z)}{r} \quad (7.154)$$

where $A(z)$ is a function of only the single coordinate z . Further, using expressions (7.153) in the fundamental equations (I) and (II), i.e.

$$\frac{\varepsilon}{c} \frac{\partial \mathbf{E}}{\partial t} = \operatorname{curl} \mathbf{H} \quad \text{and} \quad \frac{\mu}{c} \frac{\partial \mathbf{H}}{\partial t} = -\operatorname{curl} \mathbf{E}$$

we get, after cancelling $e^{i\omega t}$,

$$\left. \begin{aligned} \frac{i\omega\varepsilon}{c} E_0 &= -\frac{\partial H_0}{\partial z} \\ 0 &= \frac{1}{r} \frac{\partial}{\partial r} (rH_0) \\ \frac{i\omega\mu}{c} H_0 &= -\frac{\partial E_0}{\partial z} \end{aligned} \right\} \quad (7.155)$$

The remaining three equations for the components corresponding to the vector equations (I) and (II) are identically satisfied by expressions (7.153).

From Eqs. (7.155₃) and (7.155₁), we obtain

$$\frac{\partial^2 E_0}{\partial z^2} = -\frac{i\omega\mu}{c} \frac{\partial H_0}{\partial z} = -\frac{\omega^2\varepsilon\mu}{c^2} E_0$$

Introducing, according to Eq. (7.85), the symbol

$$k^2 = \frac{\omega^2\varepsilon\mu}{c^2}, \quad k = \frac{\omega\sqrt{\varepsilon\mu}}{c}$$

and taking into consideration Eq. (7.154), we have

$$\frac{\partial^2 A(z)}{\partial z^2} + k^2 A(z) = 0$$

whence

$$A_z = A_0 e^{-ikz} + A'_0 e^{ikz}$$

where A_0 and A'_0 are arbitrary integration constants.

Using this in Eqs. (7.153) and (7.154), we finally get

$$E_r = \frac{A_0}{r} e^{i(\omega t - kz)} + \frac{A'_0}{r} e^{i(\omega t + kz)}$$

The first term of this expression is the wave propagation along the positive direction of the z -axis, and the second one is the wave propagating in the reverse direction. These waves are independent of each other so that we can limit ourselves to a consideration of only one of them, for instance the first one, i.e. assume that $A'_0 = 0$ and

$$E_r = \frac{A_0}{r} e^{i(\omega t - kz)} \quad (7.156)$$

Introducing this equation into Eq. (7.155₃), after simple transformations we get

$$H_z = \sqrt{\frac{\varepsilon}{\mu}} E_r = \sqrt{\frac{\varepsilon}{\mu}} \frac{A_0}{r} e^{i(\omega t - kz)} \quad (7.157)$$

It is easy to see that this expression also satisfies the first two equations (7.155).

It remains for us to establish the relationship between the intensity of the wave field in a cable and the density of the surface charges and currents on its casings. Introducing the expressions for E_r and H_z into Eqs. (7.150) and (7.152), we get

$$\left. \begin{aligned} \frac{4\pi i_1}{c} &= \sqrt{\frac{\varepsilon}{\mu}} \frac{A_0}{r_1} e^{i(\omega t - kz)}, \\ 4\pi\sigma_1 &= \frac{\varepsilon A_0}{r_1} e^{i(\omega t - kz)}, \\ \frac{4\pi i_2}{c} &= -\sqrt{\frac{\varepsilon}{\mu}} \frac{A_0}{r_2} e^{i(\omega t - kz)}, \\ 4\pi\sigma_2 &= -\frac{\varepsilon A_0}{r_2} e^{i(\omega t - kz)} \end{aligned} \right\} \quad (7.158)$$

Passing over to the real parts of the complex expressions and taking into account that the total intensity of the currents I_1 and I_2 flowing in the surfaces of the cylindrical conductors of the cable equals the product of the surface density of the current and the circumference of the conductor, we finally have

$$\begin{aligned} I_1 &= 2\pi r_1 i_1 = \sqrt{\frac{\varepsilon}{\mu}} \frac{cA_0}{2} \cos(\omega t - kz) = \\ &= -2\pi r_2 i_2 = -I_2 \end{aligned}$$

Thus, at each given moment t in each cross section of the cable, the currents flowing through the internal and external conductors of the cable are equal in magnitude and opposite in sign (i.e. in direction).

In a similar way, we shall find that the charges q'_1 and q'_2 per unit length of the cylindrical cable conductors equal

$$\begin{aligned} q'_1 &= 2\pi r_1 \sigma_1 = \frac{\varepsilon A_0}{2} \cos(\omega t - kz) = \\ &= -2\pi r_2 \sigma_2 = -q'_2 \end{aligned}$$

4. Thus, electromagnetic waves propagate in the space between the cable casings. Their properties coincide in many respects with those of spherical and plane waves which we have treated in Secs. 7.9 and 7.10. Indeed, it follows from the above expressions that the vectors \mathbf{E} and \mathbf{H} are mutually perpendicular, normal to the direction of the wave, and form a right-handed system with this direction. Further, the absolute values of these vectors comply with Eq. (7.91), the energy of the field with Eq. (7.92), and so on. Finally, the velocity of propagation of waves in a cable satisfies the previously established equation (7.87)

$$v = \frac{\omega}{k} = \frac{c}{\sqrt{\epsilon\mu}}$$

i.e. is determined by the properties of the non-conducting medium filling the space between the cable casings.

Thus, the conductors forming a cable play, in essence, only the part of *waveguides* determining the direction of the *wave propagating in a dielectric* and preventing the dissipation of electromagnetic energy to the surrounding space: when a wave propagates along a cable its amplitude does not diminish, and, consequently, all the energy of the wave is transmitted along the cable without losses (do not forget that we have disregarded the resistance of the cable). To a known extent, the part of the waveguide can be played not only by a cable, but also by a conventional single-core conductor. This is used in engineering, for instance, in telegraphy with conductors using high-frequency currents.

5. Let us now pass over to the *second approximation* taking into account the penetration of the field into the cable conductors. Of practical interest, however, are not the accurate values of the field in the cable conductors, but only the *losses of energy* and the *damping of a wave* when it propagates along a cable, and this is what we shall limit our treatment to.

If the thickness of the current-conducting layer $\delta = 1/p$ [Eq. (7.148)] is much smaller than the cable radii r_1 and r_2 , then the curvature of the surface of the cable conductors will not affect the distribution of the currents in the conducting layer. We can therefore use the results of Sec. 6.15 relating to the case of a flat conductor surface. According to Eq. (6.119)

$$j = j_0 e^{-p\zeta} \cos(\omega t - p\zeta)$$

where j_0 = amplitude of the space density of the current on the surface of the conductor ($\zeta = 0$)

ζ = coordinate directed normally into the conductor*.

* The phase of the current density, apart from ζ , also depends on the coordinate z along the cable axis. Since, however, the value of the phase does not affect the value of the time averaged quantities q , Q , etc., we shall not take this relationship into consideration.

If we replace this volume distribution of the currents with a surface distribution, then the density of the equivalent surface current i should obviously be assumed equal to

$$\begin{aligned} i &= \int_0^{\infty} j d\zeta = j_0 \int_0^{\infty} e^{-p\zeta} \cos(\omega t - p\zeta) d\zeta = \\ &= \frac{j_0}{p\sqrt{2}} \cos\left(\omega t - \frac{\pi}{4}\right) \end{aligned}$$

where we have extended integration up to $\zeta = \infty$ because at great values of ζ the current density owing to the factor $e^{-p\zeta}$ is vanishingly small. Thus, the amplitude of the equivalent surface current is

$$i_0 = \frac{j_0}{p\sqrt{2}}$$

Further, calculating the quantity of Joule heat liberated in a unit time in the surface layer of a conductor having an area of 1 cm^2 , and averaging the found value over the period of the field, we get

$$\begin{aligned} \bar{q} &= \int_0^{\infty} \frac{j^2 d\zeta}{\kappa} = \frac{j_0^2}{\kappa} \int_0^{\infty} e^{-2p\zeta} \overline{\cos^2(\omega t - p\zeta)} d\zeta = \\ &= \frac{j_0^2}{2\kappa} \int_0^{\infty} e^{-2p\zeta} d\zeta = \frac{j_0^2}{4p\kappa} \end{aligned}$$

or, expressing j_0 through i_0 ,

$$\bar{q} = \frac{i_0^2 p}{2\kappa} \tag{7.159}$$

This formula expresses the heat losses in fast-varying fields through the amplitude of the surface current density to determine which it is sufficient to solve the problem on the propagation of waves along conductors in the first approximation, considering the conductors to be ideal. Particularly, Eq. (6.128) for the ohmic resistance of a cylindrical conductor to fast-varying currents directly follows from Eq. (7.159).

6. For a cable, i_0 is determined by equations (7.158) and, for example, on the inner surface of the cable is

$$i_{01} = \frac{c}{4\pi} \sqrt{\frac{\varepsilon}{\mu}} \frac{A_1}{r_1}$$

Consequently, the heat liberated in a unit time per unit length of the internal conductor of the cable whose surface area is $2\pi r_1$ cm² is

$$Q_1 = 2\pi r_1 \frac{i_{01}^2 p_1}{2\kappa_1} = \frac{\epsilon c^2 A_0^2 p_1}{16\pi \mu \kappa_1 r_1}$$

where the subscript 1 of κ and p signifies that these quantities relate to the internal conductor.

A similar expression is obtained for the heat Q_2 liberated in the external conductor of the cable so that the total value of the heat losses per unit cable length is

$$Q = Q_1 + Q_2 = \frac{\epsilon c^2}{16\pi \mu} A_0^2 \left(\frac{p_1}{\kappa_1 r_1} + \frac{p_2}{\kappa_2 r_2} \right) \tag{7.160}$$

Let us now calculate the mean (during a period) energy flux carried by the wave in a unit time through the cross section of the cable:

$$\begin{aligned} \Sigma &= \int_{r_1}^{r_2} \overline{S_z} 2\pi r \, dr = \\ &= \frac{2\pi c}{4\pi} \int_{r_1}^{r_2} \overline{E_r H_\alpha} r \, dr \end{aligned}$$

where S_z stands for the z -component of Poynting's vector $\mathbf{S} = \frac{c}{4\pi} [\mathbf{E}\mathbf{H}]$ and where it is taken into account that in the first approximation according to Eq. (7.153), $H_r = E_\alpha = 0$.

Passing over in determining the product $E_r H_\alpha$ to the real parts of the complex expressions (7.156) and (7.157), we get

$$\begin{aligned} \overline{E_r H_\alpha} &= \sqrt{\frac{\epsilon}{\mu}} \frac{A_0^2}{r^2} \overline{\cos^2(\omega t - kz)} = \\ &= \sqrt{\frac{\epsilon}{\mu}} \frac{A_0^2}{2r^2} \end{aligned}$$

Therefore,

$$\begin{aligned} \Sigma &= \frac{c}{4} \sqrt{\frac{\epsilon}{\mu}} A_0^2 \int_{r_1}^{r_2} \frac{dr}{r} = \\ &= \frac{c}{4} \sqrt{\frac{\epsilon}{\mu}} A_0^2 \ln \left(\frac{r_2}{r_1} \right) \end{aligned} \tag{7.161}$$

In the first approximation, as we have seen, the amplitude of the wave is constant so that the wave does not attenuate upon its propagation. The heat losses Q , however, should result in damping of the wave, i.e. should result in its amplitude A_0 depending on z . This relationship can be determined from the fact that the heat $Q dz$ liberated in a unit time over a portion of the cable with the length dz must equal the difference between the energy fluxes through the two cross sections of the cable limiting this portion:

$$\Sigma - \left(\bar{\Sigma} + \frac{d\Sigma}{dz} dz \right) = Q dz$$

or

$$\frac{d\Sigma}{dz} = -Q \quad (7.162)$$

Introducing into this equation the values of Q and Σ from Eqs. (7.160) and (7.161), after simple transformations and after inserting the values of the constants p_1 and p_2 from Eq. (7.148), we get

$$\frac{dA_0}{dz} = -sA_0$$

where

$$s = \frac{1}{4} \sqrt{\frac{\omega\epsilon}{2\pi\mu}} \frac{1}{\ln\left(\frac{r_2}{r_1}\right)} \left(\frac{1}{r_1} \sqrt{\frac{\mu_1}{\kappa_1}} + \frac{1}{r_2} \sqrt{\frac{\mu_2}{\kappa_2}} \right) \quad (7.163)$$

Thus, when account is taken of the heat losses, the amplitude of the wave propagating along the cable attenuates according to the law

$$A_0 = e^{-sz} a_0 \quad (7.164)$$

where a_0 is a constant.

It should be noted that the approximate expression (7.163) which we have obtained for the damping factor s coincides in the limiting case of high frequencies ω with the result of accurate solution of the problem, which is very complicated mathematically.

7. According to Eq. (7.163), the wave damping factor s is proportional to the square root of the wave frequency. This is the reason why, for example, when speech is being transmitted over great distances via telephone wires the sounds are distorted because separate harmonics (separate terms of the expansion of the sound into a Fourier series by sinusoidal functions of time) are transmitted with a different intensity. For this, see also Sec. 7.17, p. 563.

It follows from Eq. (7.163) that if the radius r_2 of the inner surface of the external cable conductor appreciably exceeds the radius r_1

of the internal conductor, then the damping factor s depends to a considerably greater extent on the conductivity κ_1 of the internal conductor than on the conductivity κ_2 of the external conductor of the cable. This holds, however, only provided that

$$\frac{1}{r_1} \sqrt{\frac{\mu_1}{\kappa_1}} > \frac{1}{r_2} \sqrt{\frac{\mu_2}{\kappa_2}}$$

Exactly the reverse relationship occurs, for example, when a single-core insulated conductor (a submarine cable) is submerged in the sea, when the part of the external conductor is played by the sea water. Since the conductivity of sea water is incomparably smaller than the conductivity of metals, the damping of high-frequency waves in a single-core submarine cable is completely determined by the conductivity of the water and is virtually independent of the conductivity of the metal conductor.

7.17 Approximate Theory of Fast-Varying Currents. “Telegraph Equation”

1. The investigation of various cases of the propagation of waves in conductors is connected in general with known mathematical difficulties. Therefore in electrical engineering of fast-varying currents when considering such problems, a simplified and, in essence, a quite unstrict method of reasoning is resorted to, which, however, for definite conditions relating to the configuration of conductors and the frequency of waves gives correct results. This method consists in the following.

To apply to fast-varying currents the theory of quasistationary currents that in essence cannot at all be applied to them, we consider not the entire circuit of the current as a whole, but separate small portions of it having the length dz and assume that the theory of quasistationary currents can be applied to each of these separate portions.

If R' is the resistance of a unit length of a conductor, then for the length dz Ohm's law in its form applicable to steady currents,

$$R_{12}I = \varphi_1 - \varphi_2$$

can be written as follows:

$$R' dz I = - \frac{\partial \varphi}{\partial z} dz$$

because $\frac{\partial \varphi}{\partial z} dz$ equals the potential difference across the ends of the length dz . In the approximate theory which we are considering,

it is assumed that for alternating currents a similar relationship holds, namely,

$$R' dz I = - \frac{d\varphi}{dz} dz - \frac{1}{c^2} L' dz \frac{\partial I}{\partial t} \quad (7.165)$$

It differs from the preceding one only in the last term taking into account the induced e.m.f. [cf. Eq. (6.17)]. In this term, L' signifies the self-inductance of a unit length of the conductor (cf. Sec. 6.6, p. 410), and $L' dz$ —the self-inductance of the length dz . The assumption is made here that for a varying field having no potential the first term of the right-hand side of the above equation has a definite physical meaning, namely, that this term takes into account the instantaneous Coulomb field of the electric charges of the conductor. Further, it is assumed that the instantaneous value of the potential of these charges is determined by the relationship

$$\varphi = \frac{q'}{C'}$$

where q' and C' are the charge and the capacitance per unit length of conductor, respectively [cf. Eq. (1.56)]. On the basis of this relationship, Eq. (7.165) after cancelling dz can be written as follows:

$$R'I = - \frac{\partial}{\partial z} \left(\frac{q'}{C'} \right) - \frac{1}{c^2} L' \frac{\partial I}{\partial t} \quad (7.166)$$

Let us use a continuity equation to exclude the quantity q' from this equation. Assume that the portion of the conductor dz is confined between sections S and S' whose coordinates are z and $z + dz$, respectively. The current in the conductor will in general change along its length, i.e. will be a function of the coordinate z . If I units of charge flow in a unit time through the section S , then $I + \frac{\partial I}{\partial z} dz$ units will flow through the section S' . Hence, the value of the charge $q' dz$ in the portion of the conductor being considered will diminish in a unit time by $\frac{\partial I}{\partial z} dz$ units, whence it follows that

$$\frac{\partial q'}{\partial t} = - \frac{dI}{dz}$$

Differentiating Eq. (7.166) with respect to t and then deleting $\partial q'/\partial t$ from it, we finally get

$$\frac{1}{c^2} L' \frac{\partial^2 I}{\partial t^2} + R' \frac{\partial I}{\partial t} = \frac{\partial}{\partial z} \left(\frac{1}{C'} \frac{dI}{dz} \right) \quad (7.167)$$

This is the required approximate equation of a non-quasistationary alternating current. If C' does not depend on z , then Eq. (7.167) becomes

$$\frac{1}{c^2} L' \frac{\partial^2 I}{\partial t^2} + R' \frac{\partial I}{\partial t} = \frac{1}{C'} \frac{\partial^2 I}{\partial z^2} \quad (7.168)$$

Equation (7.168) is called the *telegraph equation* and is used in engineering, for example, when calculating the propagation of telegraph signals through conductors, the distribution of currents in aerials, etc.

2. It can be seen that the above derivation of Eq. (7.167) is based on a number of assumptions that, generally speaking, do not correspond to reality. The very concepts of the capacitance and self-inductance of a unit length have no unambiguous meaning because, for example, the potential of a given point of a conductor, even with stationary distribution of the charges, should depend not only on the linear density of the charges q' at this point, but also on the distribution of the charges along the entire length of the conductors. We can show, however, that if the lateral dimensions l of a system of conductors comply with the inequalities

$$\frac{1}{\kappa} \sqrt{\frac{\omega}{\kappa\mu}} \ll \frac{l}{c} \ll \frac{1}{\omega} \sqrt{\frac{\kappa}{\omega\mu}}$$

where κ and μ are the conductivity and permeability of the conductors, respectively, and if the conductors are sufficiently straight*, then Eq. (7.167) is a good approximation to reality.

We shall not stop for a general treatment of this question, and shall limit ourselves to showing the applicability of Eq. (7.168) to the particular case of a cylindrical cable, which we considered in the preceding section.

3. Let us assume that the quantities L' , C' , and R' are constant in the entire conductor. Let us consider an alternating current having the frequency ω :

$$I = A(z) e^{i\omega t}$$

where $A(z)$ is independent of t . Introducing this expression into Eq. (7.168), cancelling $e^{i\omega t}$ and multiplying by C' , we get

$$\frac{d^2 A(z)}{dz^2} = -u^2 A(z)$$

* The conductors must approximately have the shape of straight cylinders of an arbitrary cross section; the radius of curvature of the axes of the cylinders must be great both in comparison with l and with the length of an electromagnetic wave in a vacuum.

where

$$u^2 = \frac{\omega^2 L' C'}{c^2} - i\omega R' C' \quad (7.169)$$

The general solution of this equation is

$$A(z) = I_0 e^{-iuz} + I'_0 e^{iuz}$$

where I_0 and I'_0 are arbitrary integration constants determining the amplitudes of two waves propagating in a conductor in mutually opposite directions. It is evidently sufficient to consider one of these waves, i.e. to assume, for instance, that $I'_0 = 0$. In this case the current in the cable is

$$I = I_0 \exp [i(\omega t - uz)] \quad (7.170)$$

Assuming that

$$u = k - is \quad (7.171)$$

where k and s are real, we get from Eq. (7.169)

$$k^2 - s^2 = \frac{\omega^2 L' C'}{c^2} \quad \text{and} \quad 2ks = \omega R' C' \quad (7.172)$$

Thus, k and s have the same signs. For definiteness, let us choose a positive sign for them. Using Eq. (7.171) in (7.170), we get

$$I = I_0 \exp [-sz + i(\omega t - kz)]$$

whence it follows that the wave damping factor equals s , and its phase velocity is

$$v = \frac{\omega}{k} = \frac{2s}{R' C'} \quad (7.173)$$

Finally, from Eq. (7.172), we get

$$s^2 = \frac{C' L' \omega^2}{2c^2} \left\{ \sqrt{1 + \left(\frac{c^2 R'}{\omega L'} \right)^2} - 1 \right\} \quad (7.174)$$

In determining the dependence of the quantities v and s on ω , it is necessary to bear in mind that the ohmic resistance of a conductor to fast-varying currents, according to Eqs. (6.128) and (6.116), is directly proportional to the square root of the frequency ω , hence,

$$R' = \rho \sqrt{\omega} \quad (7.175)$$

where ρ is independent of ω^* .

* Unlike R' , the capacitance C' does not depend on ω , while L' tends to a constant limit when $\omega \rightarrow \infty$ [see Sec (6.15)].

We invite our reader to prove that both the phase velocity of propagation of waves v and their damping factor s monotonously grow with an increasing wave frequency ω and that at sufficiently high frequencies complying with the condition

$$\frac{c^2 R'}{\omega L'} = \frac{c^2 \rho}{\sqrt{\omega L'}} \ll 1 \tag{7.176}$$

these quantities approximately equal

$$v = \frac{c}{\sqrt{C'L'}} \quad \text{and} \quad s = \frac{cR'}{2} \sqrt{\frac{C'}{L'}} = \frac{c\rho}{2} \sqrt{\frac{\omega C'}{L'}} \tag{7.177}$$

thus, when $\omega \rightarrow \infty$ the phase velocity tends to a constant limit, whereas the damping factor grows proportional to $\sqrt{\omega}$.

To reduce as much as possible the distortion of speech in long-distance telephone conversations over wires, it is necessary to ensure the observance of condition (7.176) in the region of sonic frequencies because when this condition is observed, the phase velocity of the waves becomes independent of the frequency (the damping factor, however, inevitably grows with increasing frequency). In practice, the observance of condition (7.176) is generally ensured by increasing the self-inductance of the line. For this end, special self-inductance coils (so-called loading coils) are included in the line at definite intervals.

4. It remains for us to show on the example of a cylindrical cable that the calculation of fast-varying currents in it with the aid of the telegraph equation (7.168) coincides with the results which we obtained in Sec. 7.16 on the basis of a stricter theory.

It is not difficult to see that in any section of a cable at any moment of time a fast-varying current flowing through the internal conductor must be equal in magnitude and opposite in direction to the current flowing through the external casing of the cable*. We can therefore apply to the case being considered the results of solving Example 2 in Sec. 6.6**, whence it follows that the self-inductance of a unit

* Indeed, owing to the skin effect, the field does not penetrate into the conductors. Therefore, the circulation $\oint H_s ds$ of the magnetic vector around a circle passing inside the external conductor and enclosing both currents equals zero. Since from Eq. (6.95) we get

$$\oint_L H_s ds = \frac{4\pi}{c} \int_S (\mathbf{j} + \mathbf{j}_{dis}) dS$$

and since the displacement current \mathbf{j}_{dis} can be disregarded in comparison with the conduction current \mathbf{j} (see Sec. 6.13), then the algebraic sum of the currents flowing through any section S of the cable must equal zero at any moment.

** In this example, we considered that the cable consists of two coaxial cylindrical conducting surfaces, which owing to the skin effect is permissible at high frequencies.

length of an ideal cable should be expressed by Eq. (6.59):

$$L' = 2\mu \ln \left(\frac{r_2}{r_1} \right)$$

On the other hand, the cable casings form a cylindrical capacitor whose capacitance per unit length according to Eq. (1.56) is

$$C' = \frac{1}{2 \ln \left(\frac{r_2}{r_1} \right)}$$

if there is no dielectric in the space between the cylinders, and, consequently, is

$$C' = \frac{\varepsilon}{2 \ln \left(\frac{r_2}{r_1} \right)}$$

if there is a dielectric having the permittivity ε between the cylinders. Thus, $C'L' = \varepsilon\mu$.

Further, the resistance of a unit length of the cable consists of the resistance of both of its conductors and, according to Eq. (6.128), equals

$$\begin{aligned} R' &= \frac{p_1}{2\pi r_1 \kappa_1} + \frac{p_2}{2\pi r_2 \kappa_2} = \\ &= \frac{1}{c} \sqrt{\frac{\omega}{2\pi}} \left(\frac{1}{r_1} \sqrt{\frac{\mu_1}{\kappa_1}} + \frac{1}{r_2} \sqrt{\frac{\mu_2}{\kappa_2}} \right) \end{aligned}$$

where κ_1, μ_1 and κ_2, μ_2 are the conductivity and the permeability of the internal and the external conductors of the cable, respectively. Introducing these values of L' , C' , and R' into Eqs. (7.177) that hold for sufficiently high frequencies [condition (7.176)], we get

$$\begin{aligned} v &= \frac{c}{\sqrt{\varepsilon\mu}} \quad \text{and} \quad s = \frac{1}{4} \sqrt{\frac{\omega\varepsilon}{2\pi\mu}} \times \\ &\times \frac{1}{\ln \left(\frac{r_2}{r_1} \right)} \cdot \left(\frac{1}{r_1} \sqrt{\frac{\mu_1}{\kappa_1}} + \frac{1}{r_2} \sqrt{\frac{\mu_2}{\kappa_2}} \right) \end{aligned} \quad (7.178)$$

which indeed completely coincides with Eq. (7.163).

7.18 Free Energy of Ferromagnetics. Hysteresis

1. The question of the energy of magnetized ferromagnetics does not directly relate to the contents of this chapter. We could not

consider this question in the preceding chapters, however, because for its elucidation it is necessary to use the concept of the electromagnetic energy flux with which we have become acquainted only in Sec. 7.2.

Let us consider a stationary ferromagnetic having the volume V and the surface area S surrounded by a non-ferromagnetic medium. Since the ferromagnetic is stationary, the ponderomotive forces do no work on it, and energy from the surrounding space can be transmitted to the volume V occupied by the ferromagnetic through the surface S only in two forms—heat and electromagnetic energy. Hence, the change in the total energy U_{tot} localized inside the volume V during the time dt is

$$dU_{\text{tot}} = Q - dt \oint_S S_n dS \quad (7.179)$$

Here Q signifies the heat absorbed by the ferromagnetic during the time dt from the surroundings, and S_n is the projection of Poynting's vector \mathbf{S} onto the *outward* normal \mathbf{n} to the surface of the ferromagnetic S . Therefore, the second term in the right-hand side equals the amount of electromagnetic energy transmitted to the volume V through the surface S from outside during the time dt (see Sec. 7.2).

The results of Sec. 7.2 can be applied to the non-ferromagnetic medium surrounding the ferromagnetic. Hence, Poynting's vector \mathbf{S} in this medium and, therefore, on the external surface of the ferromagnetic, is

$$\mathbf{S} = \frac{c}{4\pi} [\mathbf{E}\mathbf{H}]$$

To relate the values of Poynting's vector on the surface of a ferromagnetic to the electromagnetic field inside of it, we shall take advantage of the fact that Maxwell's fundamental equations can also be applied to ferromagnetics. It is not possible to apply only the equation $\mathbf{B} = \mu\mathbf{H}$ and expression (VI) for the electromagnetic energy. Multiplying Maxwell's equation (I) by $c\mathbf{E}/4\pi$ and Eq. (II) by $c\mathbf{H}/4\pi$ and summing the results, we get

$$\begin{aligned} \frac{1}{4\pi} \mathbf{E} \frac{\partial \mathbf{D}}{\partial t} + \mathbf{j}\mathbf{E} + \frac{1}{4\pi} \mathbf{H} \frac{\partial \mathbf{B}}{\partial t} &= \\ = \frac{c}{4\pi} (\mathbf{E} \cdot \text{curl } \mathbf{H} - \mathbf{H} \cdot \text{curl } \mathbf{E}) &= \\ = - \frac{c}{4\pi} \text{div } [\mathbf{E}\mathbf{H}] & \end{aligned}$$

Integrating this equation over the volume of the ferromagnetic V and using for all the terms of the equation obtained in this way except for the last one in the left-hand side the same transformations as in Sec. 7.2, we get

$$-\int_S S_n \cdot dS = \frac{1}{8\pi} \frac{\partial}{\partial t} \int \varepsilon E^2 dV + \\ + \frac{1}{4\pi} \int \mathbf{H} \frac{\partial \mathbf{B}}{\partial t} dV + \int \left(\frac{j^2}{\kappa} - \mathbf{j} \mathbf{E}_{\text{ext}} \right) dV$$

Let us assume that extraneous e.m.f.'s are absent in the ferromagnetic ($\mathbf{E}_{\text{ext}} = 0$). We shall limit ourselves, further, to the consideration of an infinitely slow change in its state. For this condition, we may disregard the Joule heat liberated in it. Indeed, the density of the currents \mathbf{j} attending the given change in the state of a body is inversely proportional to the time t during which this change occurs:

$$\mathbf{j} \sim \frac{dq}{t}$$

where dq is the quantity of electricity flowing during the entire process through a unit cross-sectional area of the body. Therefore, the Joule heat $\int \frac{j^2}{\kappa} dt$ liberated in a unit volume of the body during the entire process is inversely proportional to the duration of the process t

$$\left(\int j^2 dt \sim t j^2 \sim \frac{1}{t} \right)$$

Thus, in the assumptions made, we can ignore the last integral in the right-hand side in the expression for the flux of the electromagnetic energy so that Eq. (7.179) becomes

$$dU_{\text{tot}} = Q + d \int \frac{\varepsilon E^2}{8\pi} dV - \frac{1}{4\pi} \int \mathbf{H} d\mathbf{B} dV$$

From the total energy localized in the volume of a ferromagnetic, we can separate the electric energy:

$$dU = d \left(U_{\text{tot}} - \frac{1}{8\pi} \int \varepsilon E^2 dV \right) = \\ = Q + \frac{1}{4\pi} \int \mathbf{H} d\mathbf{B} dV \quad (7.180)$$

In the following, we shall consider only the energy U and not U_{tot} , and shall call it the energy of a ferromagnetic for brevity. The unam-

biguous separation from U of the magnetic energy is impossible, however, because the last term in the right-hand side of Eq. (7.180) for a ferromagnetic is not the total differential of the quantities characterizing its state.

We shall show in the following that if hysteresis is absent in a ferromagnetic, it is possible in a rational way to separate the *free* energy of the magnetic field from its *free* energy* [see Eq. (7.184)].

2. Up to now, our reasoning was of an absolutely general nature. For our following treatment, however, it is necessary to consider separately media having magnetic hysteresis and not having it. Let us first assume that a medium has neither hysteresis nor “permanent” magnetization, i.e. that the magnetic field intensity \mathbf{H} in the medium is an unambiguous function of the magnetic induction \mathbf{B} and the temperature T (with a given volume or density of the medium), and that $\mathbf{H} = 0$ when $\mathbf{B} = 0$.

This occurs not only in dia- and paramagnetics, but also in pure undeformed ferromagnetic monocrystals** and, finally, occurs with a sufficient degree of accuracy in some grades of soft iron and in certain technical alloys (for instance in Permalloy—a nickel-iron composition).

In the absence of hysteresis, the magnetization of a medium (provided that it is sufficiently slow) proceeds *reversibly*. The quantity of heat Q absorbed by a body in a reversible process is known to be

$$Q = T dS$$

where T = absolute temperature

dS = increment of the entropy S of the body.

* The possibility of separating the electric energy from U_{tot} is due to the fact that when deriving Poynting's theorem we disregarded the temperature dependence of the permittivity of the medium and in general any possible changes in v . Only for this reason could we assume in Sec. 7.2 that

$$\mathbf{E} \frac{\partial \mathbf{D}}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t} (\epsilon E^2)$$

In the general case, however, when we take into consideration the dependence of ϵ on the state of the medium, the dependence of the energy of the medium on the electric field is quite similar to its dependence on the magnetic field. The only difference is that in dielectrics (except for ferroelectrics) there is no electric hysteresis. In the general case, the expression

$$\psi_{e1} = \frac{1}{4\pi} \int_{\mathbf{E}=\mathbf{D}=0}^{\mathbf{D}} (\mathbf{E})_T d\mathbf{D}$$

similar to Eq. (7.184) determines the *free* energy of an electric field.

** In anisotropic media deprived of hysteresis, the value of \mathbf{H} depends not only on the value and the direction of the vector \mathbf{B} , but also on the orientation of the crystal axes relative to this vector.

We shall introduce this expression for Q into Eq. (7.180) and assume, in addition, that this equation, also proved for the change in the energy U of the entire ferromagnetic, also remains true for the change in the energy of each of the elements of its volume. As a result, we get the following equation for the change in the *density* u of the energy of a ferromagnetic:

$$du = T ds + \frac{1}{4\pi} \mathbf{H} d\mathbf{B} \quad (7.181)$$

where s stands for the unit entropy of a unit volume of the body.

For many purposes, it is convenient to express the density of the energy u through the free energy of a unit volume ψ that is related to u by the known equation of thermodynamics

$$\psi = u - Ts \quad (7.182)$$

From Eqs. (7.181) and (7.182), we get the fundamental equation for the range of questions we are interested in:

$$d\psi = -s dT + \frac{1}{4\pi} \mathbf{H} d\mathbf{B} \quad (7.183)$$

Let $\psi_0(T)$ signify the density of the free energy of a medium when $\mathbf{H} = \mathbf{B} = 0$. Integrating Eq. (7.183) at constant T from $\mathbf{B} = 0$ to an arbitrary preset value of \mathbf{B} , we get

$$\psi = \psi_0(T) + \frac{1}{4\pi} \int_{\mathbf{H}=\mathbf{B}=0}^{\mathbf{B}} (\mathbf{H})_T d\mathbf{B}$$

Owing to the single-valued dependence of \mathbf{H} on \mathbf{B} and T , the integral in the right-hand side is a single-valued function of the parameters of state \mathbf{B} and T . Thus, in the absence of hysteresis, the density of the free energy of a medium can be resolved into the part $\psi_0(T)$ depending only on the temperature and not interesting us here, and the part

$$\psi_m = \frac{1}{4\pi} \int_{\mathbf{H}=\mathbf{B}=0}^{\mathbf{B}} (\mathbf{H})_T d\mathbf{B} \quad (7.184)$$

which is called the *density of the free energy of a magnetic field*.

In the particular case of para- and diamagnetics, $\mathbf{H} = \frac{1}{\mu} \mathbf{B}$, and Eq. (7.184) is equivalent to the following expression which we are already acquainted with:

$$\psi_m = \frac{1}{8\pi\mu} B^2 = \frac{1}{8\pi} \mathbf{H}\mathbf{B} = \frac{\mu H^2}{8\pi} \quad (7.185)$$

For ferromagnetics, however, even not having an appreciable hysteresis, the dependence of \mathbf{H} on \mathbf{B} and, consequently, of ψ_m on \mathbf{B} is in general very complicated.

Since the work of the ponderomotive forces of a magnetic field done in an isothermal process equals the decrease in the free energy of the magnetic field in this process, then knowing ψ_m , we can also determine the ponderomotive forces of the magnetic field (cf. Sec. 6.8).

3. It must be noted that the quantity

$$\psi' = \psi - \frac{H^2}{8\pi}$$

is often called the density of the free energy of a ferromagnetic instead of the quantity ψ determined by Eqs. (7.182) and (7.183). Since

$$\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$$

then

$$\frac{1}{4\pi} \mathbf{H} d\mathbf{B} = d\left(\frac{H^2}{8\pi}\right) + \mathbf{H} d\mathbf{M}$$

and therefore on the basis of Eq. (7.183) we have

$$d\psi' = d\left(\psi - \frac{H^2}{8\pi}\right) = -s dT + \mathbf{H} d\mathbf{M} \quad (7.186)$$

Still more often, the quantity

$$\psi'' = \psi' - \mathbf{M} \cdot \mathbf{H}$$

is called the density of the free energy of a ferromagnetic. Its total differential equals

$$d\psi'' = -s dT - \mathbf{M} d\mathbf{H} \quad (7.187)$$

It is indeed often the most convenient to use ψ'' in various applications. This confusion in terminology is very deplorable. It is important, however, that the quantities ψ , ψ' , and ψ'' are equivalent characteristic functions of state of a medium as understood in thermodynamics, i.e. are single-valued functions of state of a medium, and their increments in an arbitrary process are total differentials in variables characterizing the state of the medium. In addition to the temperature, we can choose either \mathbf{B} or \mathbf{H} as such a variable. The choice of one of the quantities ψ , ψ' , and ψ'' as the characteristic function of state corresponds to these three possibilities*.

* This is quite similar to the fact that in the conventional theory of equations of state and phase transitions we can choose either the temperature T and volume V of a system or the temperature T and the pressure p as independent variables. The corresponding characteristic functions are the free energy Ψ , for which $d\Psi = -S dT - p dV$, and the thermodynamic potential $\Phi = \Psi + pV$, for which $d\Phi = -S dT + V dp$. Equilibrium is known to correspond to a minimum of Ψ (if T and V are set) or to a minimum of Φ (if T and p are set). In absolutely the same way as these statements are proved, we can prove that the stable state of a magnetic is determined (1) by a minimum of the function ψ if T and the magnetic field induction \mathbf{B} are set, or (2) by a minimum of the function ψ' if T and the magnetization of the magnetic \mathbf{M} are set, or, finally (3) by a minimum of the function ψ'' if T and the magnetic field intensity \mathbf{H} are set.

4. If a medium has hysteresis, then there is no single-valued functional relationship between \mathbf{H} and \mathbf{B} , magnetization proceeds irreversibly, and the state of the medium cannot be unambiguously characterized by such parameters as \mathbf{B} , \mathbf{H} , and T because it appreciably depends not only on the instantaneous value of these parameters, but also on the prehistory of the medium. Therefore, only Eq. (7.180) may be applied to media with hysteresis, and not Eq. (7.181), and the following ones.

We shall limit ourselves to a brief consideration of hysteresis when a given ferromagnetic was already repeatedly subjected to magnetization and demagnetization in the past. Assume that the magnetizing field H reached a certain maximum value H_{\max} in two opposite directions (for example along the x -axis to the right and to the left). In this case, the state of the ferromagnetic upon following changes in H ranging from H_{\max} to $-H_{\max}$ and vice versa (provided that \mathbf{H} remains parallel and antiparallel to the x -axis) can be depicted, as is known, by the diagram in Fig. 88. In the given case, two values of the induction B correspond to each value of the field intensity H' depending on whether this field H' was preceded by a weaker one (point a) or a stronger one (point b). Thus, the point depicting the state of a ferromagnetic in an H - B curve travels along a closed *hysteresis loop* in the direction shown by the arrows in Fig. 88 when the field changes.

If the system completes a cycle, i.e. if the point showing its state after travelling over the entire hysteresis loop returns to its initial position, then the system will return to its initial state (the process is assumed to go on at a constant temperature). Particularly, the

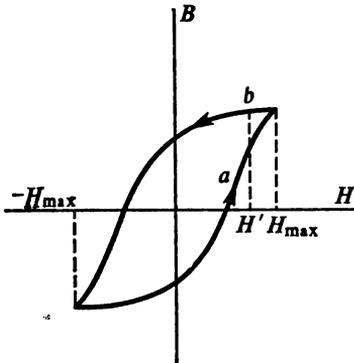


Fig. 88

energy of the ferromagnetic in Eq. (7.180) will take on its initial value in the final state. Therefore, integrating Eq. (7.180) over the

closed hysteresis loop, we get

$$-\sum Q = \frac{1}{4\pi} \int dV \oint \mathbf{H} \cdot d\mathbf{B} \quad (7.188)$$

where $-\sum Q$ stands for the algebraic sum of the quantities of heat given up by the ferromagnetic to the surroundings during the entire cycle. The integral $\oint \mathbf{H} \cdot d\mathbf{B}$ is essentially positive because \mathbf{H} and \mathbf{B} are parallel to each other during the greater part of the cycle. (With a view to the anisotropy of the ferromagnetic, it is more correct to say that in the major part of the cycle \mathbf{H} and \mathbf{B} form an acute angle.) Consequently, $-\sum Q$, the so-called heat of hysteresis, is also positive. If we denote by q the heat of hysteresis related to a unit volume of a ferromagnetic, then Eq. (7.188) can be written as follows:

$$q = \frac{1}{4\pi} \oint \mathbf{H} \cdot d\mathbf{B} = \oint \mathbf{H} \cdot d\mathbf{M} \quad (7.189)$$

Equation (7.189) is written on the basis of the fact that the difference

$$\frac{1}{4\pi} \mathbf{H} \cdot d\mathbf{B} - \mathbf{H} \cdot d\mathbf{M} = \frac{1}{4\pi} \mathbf{H} \cdot d\mathbf{H} = \frac{1}{8\pi} d\mathbf{H}^2$$

is a total differential and, consequently, the integral of this difference over the closed hysteresis loop equals zero.

7.19 General Characteristic of the Theories of Short-Range and Long-Range Interaction

1. In this closing section of the present chapter, we shall try to summarize some of the results following from our treatment of the fundamental differences between the various theories of electricity and of the essence of the ideas relating to the nature of electromagnetic phenomena underlying these theories.

In the theories of *long-range* interaction prevailing up to the middle of the 19th century, the part of the fundamental, primary concept was played by that of an electric *substance* (charges). All electromagnetic phenomena were reduced to the *instantaneous interaction of charges at a distance (actio in distans)*. In other words, it was assumed that the forces of interaction of both stationary and moving charges (*currents*) at each moment of time are determined by the distribution and the state of motion of these charges *at the same moment*. It was assumed that these forces could depend not only on the velocity, but also on the acceleration of the charges, on the derivative of the current with respect to time, etc. It was only important that the forces of interaction be unambiguously determined by

the *instantaneous* value of these physical quantities. Thus, in the theories of long-range interaction, the concept of field plays the part of an auxiliary concept that in general could be left out of consideration if so desired. Typical examples of long-range interaction laws are Coulomb's law (1.3) and the law of mechanical interaction of current elements (4.6):

$$\mathbf{F}_{12} = \frac{q_1 q_2}{R_{12}^3} \mathbf{R}_{12} \quad \text{and} \quad \mathbf{F}_{12} = \frac{I_1 I_2}{c^2 R_{12}^3} [ds_2 [ds_1 \mathbf{R}_{12}]]$$

Conversely, in the classical Faraday-Maxwell theory of *short-range* interaction, the part of the fundamental primary concept is played by the concept of the field, while the charge and current are relegated to the rank of secondary auxiliary concepts characterizing the properties of a field (Sec. 7.1, p. 457). From this viewpoint, all electromagnetic phenomena consist in changes of a field and obey *differential equations in partial derivatives* relating the values of the electromagnetic vectors at *adjacent points of space in consecutive moments of time* (short-range action)*. Accordingly, any change or, as is customarily said, *perturbation* of a field appearing at a given region of space *directly* acts only on adjacent regions of the field. Thus, any electromagnetic perturbation is gradually transmitted from point to point and requires a finite time for its propagation (*a finite velocity of propagation*).

It was exactly this conception of electromagnetic phenomena advanced by M. Faraday and J. Maxwell that underlied the attempts of mechanical interpretation of these phenomena and their reduction to deformations and movements of a hypothetical elastic medium—*ether*.

2. If we digress from the questions of interpretation and from the problem of creating an illustrative picture of the phenomena, however, then *as regards actual content*, the short-range action theory differs from the long-range one in essence in the matter of the velocity of propagation of electromagnetic perturbations (Sec. 7.7). Therefore, the controversy between the theories of short-range and long-range interaction can be solved (and was solved) only by the experimental studying of *fast-varying* fields, the nature of the phenomena in which appreciably depends on the velocity of propagation of electromagnetic perturbations (the velocity of light, the velocity of radio telegraph signals, etc.). The phenomena in stationary fields, as we have repeatedly indicated, can be interpreted equally well from the viewpoint of *both* theories.

For example, the statement sometimes encountered that the theory of long-range interaction cannot take into account the part played

* For example, the values of the spatial derivatives *div* and *curl* at each point of a field depend exclusively on the values of the corresponding vectors at adjacent points of space.

by the medium in electromagnetic phenomena is absolutely wrong, because for this purpose it is actually quite sufficient to postulate the presence of elementary charges and currents in the molecules of the medium. Particularly, the entire theory of dielectrics and magnetics set out in Chapters 2 and 5 can be completely retained in the theory of long-range interaction. Finally, the laws of *quasistationary* currents can also be placed within the confines of the theory of long-range interaction because the induction interaction of quasistationary currents is determined by their mutual arrangement and the instantaneous value of the derivative of the current with respect to time [see Sec. 6.3, particularly Eq. (6.18), and also the second half of Sec. 7.7].

3. Let us now turn to the question of electromagnetic energy.

In the theory of long-range interaction by electromagnetic energy is naturally meant not the energy of the field, but the energy of interaction of charges or currents. The form of our equations (1.100) and (6.26) corresponds quite well to this interpretation of energy, namely,

$$U_{e1} = \frac{1}{2} \sum_{ik} \frac{q_i q_k}{R_{ik}} \quad (i \neq k), \text{ and } U_m = \frac{1}{2c^2} \sum_{ik} L_{ik} J_i J_k$$

The question of the localization of energy in space loses all its meaning here because the total energy is composed of separate terms each of which expresses the *energy of interaction* of a definite *pair of charges or currents* determined by the *instantaneous* state of these charges or currents that are in *different* regions of space*.

Conversely, in the theory of short-range interaction by electromagnetic energy is meant the *energy of a field*, which is considered to be *localized in space* in a quite definite way. This means that the theory has a quite definite answer to the question of the quantity of energy at each given region of space, the volume density of the energy being determined by Eq. (VIa) (see Sec. 7.2):

$$u = \frac{\epsilon E^2}{8\pi} + \frac{\mu H^2}{8\pi}$$

Within the confines of stationary and quasistationary fields, both of these viewpoints agree equally well with the data of experiments because, as we have seen, Eqs. (1.100) and (6.26) for these fields are mathematically equivalent to Eq. (VIa). For fast-varying fields, however, the equivalence of these equations is violated, while the fact that the velocity of propagation of electromagnetic perturbations

* Particularly, by the *proper* energy of a charge (or current) is meant the energy of interaction of the elements of the charge (or the filaments of the current) into which the given charge (or current) can be resolved. (See Secs. 1.16 and 6.6.)

is finite, in connection with the law of conservation of energy, solves the controversy in favour of localization of energy in the field.

Assume, for example, that a radio telegraph or light signal was sent from the station A at the moment t_0 . This means that a certain amount of non-electromagnetic energy has transformed into the energy of electromagnetic radiation. Assume further that the station B received this signal at the moment t_1 . This means that at this moment a certain amount of energy brought in from A by electromagnetic waves was liberated at B . This energy could be used, for instance, for actuating a relay (the reception of a radio signal) or for decomposing silver bromide (the reception of a light signal on a photographic film), etc.

If the distance between A and B is R , then the interval $t_1 - t_0$ between the sending and the reception of the signal should equal R/c . If the duration of the signal is small, then there was such an intermediate moment t' ($t_0 < t' < t_1$) at which the signal was not yet received at B , whereas at A the process of emission of energy had already been terminated and all the parameters reached a state corresponding to a reduced store of energy owing to radiation. Where was the energy emitted by the station A at this moment?

If we do not want to refute the law of conservation of energy, then we can answer this question *only* in the sense that the amount of electromagnetic energy is determined not by the instantaneous distribution of currents and charges, but by the state of the *field*, and that the energy given up at the station A was converted at the moment t_0 into the *energy of an electromagnetic field of radiation*. This energy propagates in space together with the field, and only later, at the moment t_1 , is it partially used by the station B .

4. Thus, the fact of the finite velocity of propagation of electromagnetic perturbations confirms the assumption on the localization of energy in an electromagnetic field and proves the groundlessness of the theories of long-range interaction. It does not at all follow from this circumstance, however, that the views of the classical theory of the field with its negation of the substantial existence of charges are the only correct ones. Indeed, we showed in Secs. 7.4-7.6 that an electromagnetic field is unambiguously determined by the distribution of charges and *conduction* currents, true, provided without fail that this distribution is known not only for the moment of time being considered, but also for the preceding moments of time (*delayed potentials*). It is exactly this circumstance that underlies the modern *electron theory*, which is a sort of synthesis of the theories of long-range interaction and the Faraday-Maxwell theory of the field.

It is related to the classical theories of long-range interaction by the acknowledgement of the primary physical reality of electric substance. A charge is not simply a term characterizing certain specific features of a field at a given point of space; on the contrary, a field can be induced only by charges and by their motion. The most

vivid expression of the notions of the substantial nature of electricity related to the theories of long-range interaction is the postulate of the *atomistic structure of electricity* characteristic of the electron theory. This postulate is absolutely alien to the consistent Faraday-Maxwell conception. From the viewpoint of the electron theory, the primary meaning of the concept of a field consists in that this concept makes it easier to study the main question of the forces acting on charges; studying of a field is only an intermediate step in solving the problem of the interaction of charges. On the other hand, the electron theory has adopted from the Faraday-Maxwell theory of the field the principle of the finite velocity of propagation of a field (i.e. propagation of the interaction of charges). Therefore, from the viewpoint of the electron theory, we can speak only of a sort of *delayed long-range interaction* of charges and currents (delayed potentials), and not of instantaneous long-range interaction, as was assumed by the theories of the 19th century.

We can speak only very conditionally, however, about delayed long-range interaction as applied to the electron theory. The electron theory not only retains for a vacuum the system of Maxwell's equations satisfying the principle of short-range interaction, but also considers these equations for a vacuum (supplemented with terms taking into account the density of the charges and currents induced by elementary electric charges) to also hold for a *microscopic* field in an arbitrary medium. The principle of the finite velocity of propagation of a field follows from all the equations for a field. It follows from this principle in turn, as we have seen, that the electromagnetic field is the carrier of electromagnetic energy. It also follows from this principle, as can easily be shown, that a field is a carrier not only of a definite energy, but also of a definite momentum, which we have treated in Secs. 7.13-7.15. Thus, in the electron theory too, the concept of a field is advanced in the long run from the rank of an auxiliary concept (facilitating the solution of the problem of charge interaction) to that of objective reality (the carrier of energy and momentum).

8

Electromagnetic Phenomena in Slowly Moving Media

8.1 Differential Equations of a Field in Moving Media

1. In the preceding chapter (as, by the way, in the major part of the book), we limited ourselves to considering an electromagnetic field when all the bodies in the field are stationary. Thus, the results of Chapter 7, strictly speaking, can be applied neither to phenomena in generators and motors in which there are rotating parts, nor to the reflection of light from a moving mirror, nor to a great number of other important phenomena and processes. True, the essence of the processes occurring, for example, in an electric motor, can be understood on the basis of the facts set out in Chapter 6, but the development of a consistent theory of electromagnetic phenomena in moving media is naturally absolutely necessary.

A consistent theory of these phenomena can be based only on A. Einstein's theory of relativity. We shall not assume, however, that our reader is sufficiently acquainted with this theory, and therefore shall limit our treatment only to *slowly* moving media. By the latter, we shall understand motion whose velocity u is small in comparison with the velocity of light c :

$$\frac{u}{c} \ll 1 \tag{8.1}$$

In other words, we shall ignore all the effects that are proportional to the square and higher powers of the ratio u/c . The matter is that the theory of slowly moving media can mainly be constructed on the basis of the classical, prerelativistic physical notions, so that we shall have to use only two equations, Eq. (8.21) and (8.22), from the theory of relativity. The accurate theory of the effects proportional to $(u/c)^2$ is based completely on the theory of relativity. At the same time, condition (8.1) for the slow motion of material media is observed in the overwhelming majority of practically interesting phenomena and processes (because $c = 3 \times 10^5$ km/s!).

2. We shall proceed from the *microscopic* equations of an electromagnetic field:

$$\left. \begin{aligned} \text{curl } \mathbf{H}_m - \frac{1}{c} \frac{\partial \mathbf{E}_m}{\partial t} &= \frac{4\pi}{c} \mathbf{j}_m \\ \text{div } \mathbf{E}_m &= 4\pi \rho_m \\ \text{curl } \mathbf{E}_m + \frac{1}{c} \frac{\partial \mathbf{H}_m}{\partial t} &= 0 \\ \text{div } \mathbf{H}_m &= 0 \end{aligned} \right\} \quad (8.2)$$

where the subscript “m” signifies the microscopic value of the corresponding quantity. These equations hold for any motion of elementary electric charges and, consequently, for any motion of the material bodies containing these charges. Our task is to obtain equations for a *macroscopic* field in moving media from Eqs. (8.2).

Averaging Eqs. (8.2) over infinitely small volumes, as we did, for example, in Secs. 2.7 and 5.3, using expressions of the type of (2.45), and introducing the notation [cf. Eq. (5.23)]

$$\bar{\mathbf{E}}_m = \mathbf{E} \quad \text{and} \quad \bar{\mathbf{H}}_m = \mathbf{B} \quad (8.3)$$

we get

$$\text{curl } \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c} \bar{\mathbf{j}}_m \quad (8.4)$$

$$\text{div } \mathbf{E} = 4\pi \bar{\rho}_m \quad (8.5)$$

$$\text{curl } \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0 \quad (8.6)$$

$$\text{div } \mathbf{B} = 0 \quad (8.7)$$

Let us resolve, as we did in Secs. 2.7 and 5.1, the microscopic densities of the charges and currents ρ_m and \mathbf{j}_m into densities corresponding to free and bound charges:

$$\rho_m = \rho_{\text{free}} + \rho_{\text{bound}} \quad \text{and} \quad \mathbf{j}_m = \mathbf{j}_{\text{free}} + \mathbf{j}_{\text{bound}}$$

[see Eqs. (2.47) and (5.1)]; in the last of these equations, we have changed the symbols \mathbf{j}_{cond} and \mathbf{j}_{mol} to \mathbf{j}_{free} and $\mathbf{j}_{\text{bound}}$, respectively. Further, we shall equate the mean densities corresponding to *free* charges to the macroscopic densities of charges and currents [cf. Secs. 2.7 and 5.2, particularly Eq. (5.7)]:

$$\bar{\rho}_{\text{free}} = \rho \quad \text{and} \quad \bar{\mathbf{j}}_{\text{free}} = \mathbf{j} \quad (8.8)$$

Thus, we get

$$\bar{\rho}_m = \rho + \bar{\rho}_{\text{bound}} \quad \text{and} \quad \bar{\mathbf{j}}_m = \mathbf{j} + \bar{\mathbf{j}}_{\text{bound}} \quad (8.9)$$

For media at rest, we showed that

$$\bar{\rho}_{\text{bound}} = -\text{div } \mathbf{P} \quad \text{and} \quad \bar{\mathbf{j}}_{\text{bound}} = c \text{ curl } \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t} \quad (8.10)$$

where \mathbf{P} stands for the polarization, and \mathbf{M} for the magnetization of the medium. Indeed, the first of these equations coincides with Eqs. (2.10) and (2.48). As regards the second of Eqs. (8.10), for a field constant in time it coincides with Eqs. (5.13) for $\bar{\mathbf{j}}_{\text{mol}}$; for a varying field it is necessary to add to $\bar{\mathbf{j}}_{\text{mol}}$ the term

$$\frac{\partial \mathbf{P}}{\partial t} = \frac{\partial}{\partial t} \sum_{\text{bound}} q_i \mathbf{r}_i = \sum_{\text{bound}} q_i \mathbf{v}_i$$

taking into account the displacement of the charges connected with the molecules of a dielectric [cf. (Eq. (6.100))].

For media at rest, the following equations also hold in addition to Eqs. (8.10) [cf. Eqs. (2.6) and (5.2)]:

$$\mathbf{P} = \sum q_i \mathbf{R}_i \quad \text{and} \quad \mathbf{M} = \frac{1}{2c} \sum q_i [\mathbf{R}_i \mathbf{v}_i] \quad (8.11)$$

where summation covers all the bound charges in a unit volume of the medium. Any pair of equations (8.10) or (8.11) can be considered as the *definition* of the vectors \mathbf{P} and \mathbf{M} . Only the statement that the vectors \mathbf{P} and \mathbf{M} determined by one of the pairs of these equations also satisfies the other pair has a physical meaning.

For moving media, however, both pairs of equations (8.10) and (8.11) cannot hold simultaneously. Consequently, in the theory of moving media we can either retain Eqs. (8.10), correspondingly modifying Eqs. (8.11) or, conversely, retain Eqs. (8.11) and modify Eqs. (8.10).

The first derivation of equations for a field in moving media from Eqs. (8.2) of the electron theory was given by H. Lorentz, who selected the second of the ways indicated above, in the nineties of last century. More correctly, Lorentz retained for moving media the definitions (8.11) of the vectors \mathbf{P} and \mathbf{M} with the refinement that by \mathbf{v}_i in these equations we should understand the velocity of bound charges *relative to the medium* which they belong to, and not relative to the observer. As regards Eqs. (8.10), then according to Lorentz, the only change that must be introduced into them for moving media consists in substituting $\mathbf{M} + \frac{1}{c} [\mathbf{P}\mathbf{u}]$ for \mathbf{M} , where \mathbf{u} is the velocity of the medium relative to the observer. Lorentz's theory, however,

could naturally take no account of the fundamental laws of the theory of relativity, which had not yet appeared in the 19th century. It therefore contradicted both the theory of relativity and experiments (even in the interpretation of some effects of the first order of magnitude relative to u/c).

Dellenbach showed in 1919* how it is necessary to modify Lorentz's reasoning so as, while following the path of constructing the electrodynamics of moving media he selected, to take into account the requirements of the theory of relativity. We, however, shall choose a different path of constructing the electrodynamics of moving media directly adhering to the customary interpretation of this problem in the theory of relativity and at the same time having a more phenomenological nature. Namely, first we shall retain for moving media Eqs. (8.10) considering them to be *definitions* of the vectors \mathbf{P} and \mathbf{M} . Second, negating Eqs. (8.11) for moving media, we shall not consider at all the explicit dependence of the polarization \mathbf{P} and magnetization \mathbf{M} on the coordinates and velocities of elementary charges of the type of Eqs. (8.11), but shall directly establish the dependence of \mathbf{P} and \mathbf{M} on the field vectors \mathbf{E} and \mathbf{B} .

3. Thus, we shall consider Eqs. (8.10) as the definition of the vectors \mathbf{P} and \mathbf{M} . To prove that there is no contradiction in this definition, it is sufficient to use the continuity equation (IVa), which as applied to the mean density of bound charges and currents can be written as follows:

$$\frac{\partial}{\partial t} \bar{\rho}_{\text{bound}} + \text{div } \bar{\mathbf{j}}_{\text{bound}} = 0 \tag{8.12}$$

It is quite obvious that with any dependence of ρ_{bound} on the coordinates and time, we can always find an infinite set of vectors \mathbf{P} satisfying the first of Eqs. (8.10). Introducing this equation into (8.12), we get

$$-\frac{\partial}{\partial t} \text{div } \mathbf{P} + \text{div } \bar{\mathbf{j}}_{\text{bound}} = 0, \text{ or}$$

$$\text{div } \left\{ \bar{\mathbf{j}}_{\text{bound}} - \frac{\partial \mathbf{P}}{\partial t} \right\} = 0$$

Hence it follows that the vector $\bar{\mathbf{j}}_{\text{bound}} - \partial \mathbf{P} / \partial t$ can be equated to the curl of the auxiliary vector \mathbf{M} . It is exactly this corollary that is expressed by the second of equations (8.10).

* Dellenbach, *Ann. d. Physik*, **58**: 523 (1919) [see also W. Pauli *The Theory of Relativity* New York, Pergamon Press (1958)].

On the basis of Eqs. (8.8) and (8.10), Eqs. (8.4) and (8.5) become

$$\text{curl } \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c} \mathbf{j} + 4\pi \text{curl } \mathbf{M} + \frac{4\pi}{c} \frac{\partial \mathbf{P}}{\partial t}$$

$$\text{div } \mathbf{E} = 4\pi\rho - 4\pi \text{div } \mathbf{P}$$

Using the conventional symbols

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} \quad \text{and} \quad \mathbf{H} = \mathbf{B} - 4\pi\mathbf{M} \quad (8.13)$$

these equations can be written in the form

$$\text{curl } \mathbf{H} - \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} = \frac{4\pi}{c} \mathbf{j} \quad (\text{I})$$

$$\text{div } \mathbf{D} = 4\pi\rho \quad (\text{IV})$$

coinciding with Eqs. (I) and (IV) of Chapter 7. From Eqs. (I) and (IV) we get the continuity equation

$$\text{div } \mathbf{j} = - \frac{\partial \rho}{\partial t} \quad (\text{IVa})$$

At the same time, Eqs. (8.6) and (8.7) coincide with Eqs. (II) and (III) of Chapter 7:

$$\text{curl } \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0 \quad (\text{II})$$

$$\text{div } \mathbf{B} = 0 \quad (\text{III})$$

Thus, Maxwell's *differential* equations can be applied without any modifications not only to stationary media, but also to moving ones [and regardless of whether or not the condition of the slow nature of the motion (8.1) is observed].

This result has a purely formal nature, however, and is simply due to the fact that we retained the previous definitions (8.10) and (8.13) for the vectors \mathbf{P} , \mathbf{M} , \mathbf{D} , and \mathbf{H} . Maxwell's differential equations (I)-(IV) lose their nature of formal definitions and acquire a concrete physical content only provided that additional relationships such as Eq. (V) (see Sec. 7.1) are joined to them that relate the values of the fundamental vectors of an electromagnetic field. The following section is devoted to the establishment of these relationships.

4. We shall note in conclusion that whereas Maxwell's differential equations (I)-(IV) remain in force for moving media, the boundary conditions for the field vectors following from them are changed

somewhat (cf. Sec. 7.1):

$$\text{Curl } \mathbf{H} = [\mathbf{n} \cdot \mathbf{H}_2 - \mathbf{H}_1] = \frac{4\pi}{c} \mathbf{i} - \frac{v_n}{c} (\mathbf{D}_2 - \mathbf{D}_1) \quad (\text{I}')$$

$$\text{Curl } \mathbf{E} = [\mathbf{n} \cdot \mathbf{E}_2 - \mathbf{E}_1] = \frac{v_n}{c} (\mathbf{B}_2 - \mathbf{B}_1) \quad (\text{II}')$$

$$\text{Div } \mathbf{B} = B_{2n} - B_{1n} = 0 \quad (\text{III}')$$

$$\text{Div } \mathbf{D} = D_{2n} - D_{1n} = 4\pi\sigma \quad (\text{IV}')$$

where v_n is the projection of the velocity of a boundary element onto the normal \mathbf{n} .

8.2 Convection Current. Polarization and Magnetization of Moving Media

1. In this section, we shall consider the dependence of the polarization of a medium, its magnetization, and the macroscopic current density on the field intensity. We shall begin with the current density. According to Eqs. (8.8) and (5.65), we have

$$\mathbf{j} = \overline{\mathbf{j}}_{\text{free}} = (\overline{\rho\mathbf{v}})_{\text{free}}$$

where by \mathbf{v} should obviously be understood the velocity of the free electric charges relative to the observer. Strictly speaking, we assume that all the readings of the position and the motion of charges, material bodies, and various media are made in a definite *inertial* frame of reference. We shall conditionally call the latter stationary and consider it connected with the observer (see Sec. 6.2). This is briefly expressed by the words "velocity relative to the observer".

All inertial frames of reference are naturally equivalent. The results of this section, however, are approximate and hold only for a sufficiently slow nature of the motion of the medium relative to the observer [condition (8.1)]. In other words, they hold only in such inertial frames of reference whose velocity relative to the material bodies in a field are much smaller than that of light.

Let \mathbf{v}' signify the velocity of charges relative to the element of the medium which they are in. Hence,

$$\mathbf{v} = \mathbf{v}' + \mathbf{u} \quad (8.14)$$

where \mathbf{u} is the velocity of an element of the medium relative to the observer. Introducing this into the preceding equation, we get

$$\mathbf{j} = (\overline{\rho\mathbf{v}})_{\text{free}} = \overline{\rho(\mathbf{v}' + \mathbf{u})}_{\text{free}} = (\overline{\rho\mathbf{v}'})_{\text{free}} + \overline{\mathbf{u}\rho}_{\text{free}}$$

because the velocity of the medium \mathbf{u} as a macroscopic quantity should be considered constant at all the points of the infinitely small

volume over which averaging is performed in the preceding equations. Using the symbol (8.8) for $\bar{\rho}_{\text{free}}$ and introducing the symbol

$$\mathbf{j}_{\text{cond}} = \overline{(\rho \mathbf{v}')}_{\text{free}} \quad (8.15)$$

for the density of the conduction current, we get

$$\mathbf{j} = \rho \mathbf{u} + \mathbf{j}_{\text{cond}} \quad (8.16)$$

The first term in this equation, $\rho \mathbf{u}$, is called the density of the *convection current* and takes into account the circumstance that the free charges of the medium having the density ρ are moving together with the medium with its velocity \mathbf{u} . For example, the current due to the motion of a charged dielectric is purely a convection current because the conduction current \mathbf{j}_{cond} is absent in this case. Indeed, according to definition (8.15) of the density of a conduction current, this current is due to the motion of free charges *relative to the medium* (the velocity \mathbf{v}'), and in a dielectric this motion is impossible.

For stationary media, the conduction current, according to Eq. (3.23) or (V), is

$$\mathbf{j}_{\text{cond}} = \kappa(\mathbf{E} + \mathbf{E}_{\text{ext}}) \quad (8.17)$$

where κ is the conductivity of the medium. When generalizing this equation for moving media it must be taken into consideration that charges entrained by the medium with the velocity \mathbf{u} are acted upon not by the force $q\mathbf{E}$, but by the Lorentz force $q\left(\mathbf{E} + \left[\frac{\mathbf{u}}{c} \cdot \mathbf{H}\right]\right)$. Since we are speaking of the *macroscopic* quantity \mathbf{j}_{cond} , then when substituting the vector $\mathbf{E} + \left[\frac{\mathbf{u}}{c} \cdot \mathbf{H}\right]$ for \mathbf{E} in Eq. (8.17), we must understand \mathbf{H} to signify the *mean* value of the microscopic intensity \mathbf{H}_m which according to Eq. (8.3) equals \mathbf{B} . In other words, in moving media, the force acting on the charges is determined not by the electric field \mathbf{E} , but by the effective field

$$\mathbf{E}^* = \mathbf{E} + \left[\frac{\mathbf{u}}{c} \cdot \mathbf{B}\right] \quad (8.18)$$

Thus, in moving media instead of Eq. (8.17), we obtain

$$\mathbf{j}_{\text{cond}} = \kappa(\mathbf{E}^* + \mathbf{E}_{\text{ext}}) = \kappa\left(\mathbf{E} + \left[\frac{\mathbf{u}}{c} \cdot \mathbf{B}\right] + \mathbf{E}_{\text{ext}}\right) \quad (8.19)$$

Strictly speaking, the sum in the parentheses ought to be supplemented with the term $\left[\frac{\mathbf{v}'}{c} \cdot \mathbf{B}\right]$ because the total velocity of the free charges according to Eq. (8.14) equals $\mathbf{v} = \mathbf{u} + \mathbf{v}'$. Taking account of this term corresponds to taking account of the Hall effect (see Sec. 4.4) which owing to its small value we shall disregard.

The combination of Eqs. (8.16) and (8.19) determines the dependence of the macroscopic current on the field vectors, on ρ , on the conductivity and the velocity of the medium. It must be noted that these equations hold only for the slowly moving media [(condition (8.1)].

2. Let us now pass over to the polarization \mathbf{P} and magnetization \mathbf{M} of a medium. For stationary media, these quantities depend on \mathbf{E} and \mathbf{B} as follows [see Eqs. (2.22), (5.32) and (5.34)]:

$$\mathbf{P}_0 = \frac{\varepsilon - 1}{4\pi} \mathbf{E} \quad \text{and} \quad \mathbf{M}_0 = \frac{\mu - 1}{4\pi} \mathbf{H} = \frac{1}{4\pi} \left(1 - \frac{1}{\mu}\right) \mathbf{B} \quad (8.20)$$

(the subscript 0 signifies that the relevant quantities relate to a stationary medium). As in Chapter 7, we shall limit ourselves to the case when the field contains no permanent magnets and ferromagnetics, and shall consider that the material constants ε and μ depend neither on the field intensity nor on the velocity of the medium.

Generalization of Eqs. (8.20) for slowly moving media yields

$$\mathbf{P} = \frac{\varepsilon - 1}{4\pi} \left(\mathbf{E} + \left[\frac{\mathbf{u}}{c} \cdot \mathbf{B} \right] \right) + \frac{1}{4\pi} \left(1 - \frac{1}{\mu}\right) \left[\frac{\mathbf{u}}{c} \cdot \mathbf{B} \right] \quad (8.21)$$

$$\mathbf{M} = \frac{1}{4\pi} \left(1 - \frac{1}{\mu}\right) \left(\mathbf{B} - \left[\frac{\mathbf{u}}{c} \cdot \mathbf{E} \right] \right) - \frac{\varepsilon - 1}{4\pi} \left[\frac{\mathbf{u}}{c} \cdot \mathbf{E} \right] \quad (8.22)$$

These expressions can be obtained from the corresponding formulas of the theory of relativity if we disregard in them the quantity u^2/c^2 in comparison with unity. We shall not give the theoretical substantiation of Eqs. (8.21) and (8.22), however, but shall consider them to be *given by experiments* and shall only indicate the illustrative physical meaning of some of the terms in these expressions.

The first term of Eq. (8.21) is obtained directly from Eq. (8.20) for \mathbf{P}_0 by replacing \mathbf{E} with the intensity of the effective field \mathbf{E}^* [Eq. (8.18)]. This simply means that the Lorentz force acting in a magnetic field on the bound charges of the dielectric moving with the velocity \mathbf{u} is taken into account in addition to the electric force $q\mathbf{E}$.

The last term of Eq. (8.22) also has a simple meaning—it takes into account the magnetic field induced by the motion of a polarized dielectric. To convince ourselves that this is true, it will be sufficient to consider the very simple case experimentally investigated by the Russian scientist A. Eikhenvald. An ebonite disk rotates about its axis between the plates of a capacitor formed by two flat metal rings (Fig. 89). The uniform (in a first approximation) field \mathbf{E} of the capacitor resulted in uniform polarization of the disk \mathbf{P} perpendicular to its plane. Thus, the volume density of the bound charges inside the dielectric equalled zero, whereas, according to Eq. (2.10), bound

surface charges having the density

$$\bar{\sigma}_{\text{bound}} = \pm P = \pm \frac{\varepsilon - 1}{4\pi} E$$

concentrated on the top and the bottom surfaces of the disk. The motion of these charges with the velocity \mathbf{u} is equivalent to surface

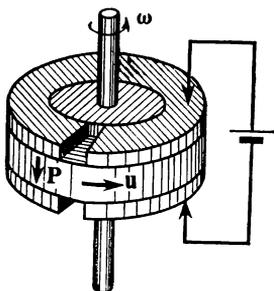


Fig. 89

currents having the density

$$\mathbf{i} = \mathbf{u} \bar{\sigma}_{\text{bound}} = \pm \mathbf{u} P = \pm \frac{\varepsilon - 1}{4\pi} \mathbf{u} E \quad (8.23)$$

circulating through the top and the bottom surfaces of the disk, respectively. These currents are detected according to the deviation of a magnetic pointer brought close to the rotating disk. To make quantitative measurements possible, both flat rings forming the capacitor were designed sectional, and a current was passed in opposite directions through them when the disk was at rest. The current having the same action on the magnetic pointer as rotation of the disk was determined. This experiment confirmed the validity of Eq. (8.23).

In our interpretation of the phenomena in moving media, the motion of the bound charges of the medium, according to Eq. (8.10), is taken into account by its *magnetization* \mathbf{M} (and also by the term $\partial \mathbf{P} / \partial t$, which in Eikhenvald's experiments equalled zero). According to Eq. (8.22), for a medium whose permeability μ equals zero, we have

$$\mathbf{M} = - \frac{\varepsilon - 1}{4\pi c} [\mathbf{u} \mathbf{E}] \quad (8.24)$$

It is easy to see that the system of surface currents (8.23) is indeed equivalent to such magnetization of the medium. Inside the disk, the vector \mathbf{M} is directed along the radius of the disk \mathbf{r} , its numerical value depending only on \mathbf{r} . Therefore, curl \mathbf{M} equals zero, and the volume density of the molecular currents, according to Eq. (8.10), also equals zero. The surface density of the molecular currents

corresponding to the magnetization (8.24), according to Eqs. (5.14) and (5.16), equals zero on the vertical side surface of the disk and numerically equals

$$i_{\text{mol}} = \pm cM = \pm \frac{\epsilon - 1}{4\pi} uE$$

on its flat horizontal surfaces, which coincides with Eq. (8.23). Thus, Eikhenvald's experiments can indeed be interpreted from the viewpoint of the notion of radial magnetization of a dielectric disk rotating in the electric field of a capacitor that corresponds to Eq. (8.22).

Summarizing the above reasoning, we can show in a general form that the last term of Eq. (8.22) takes into account the magnetic field created by the motion of the bound charges of a polarized dielectric. The last term in Eq. (8.21) and the second term in the first vector brackets in Eq. (8.22) cannot be obtained on the basis of similar elementary considerations not taking into account the theory of relativity. It must only be noted that if by analogy with Eq. (8.18) we introduce the concept of the effective magnetic induction \mathbf{B}^* , i.e.

$$\mathbf{B}^* = \mathbf{B} - \left[\frac{\mathbf{u}}{c} \cdot \mathbf{E} \right] \tag{8.25}$$

and if we use expressions (8.20) for the polarization \mathbf{P}_0 and the magnetization \mathbf{M}_0 of a medium *at rest*, then Eqs. (8.21) and (8.22) can be written in the following simple form:

$$\left. \begin{aligned} \mathbf{P} &= \frac{\epsilon - 1}{4\pi} \mathbf{E}^* + \left[\frac{\mathbf{u}}{c} \cdot \mathbf{M}_0 \right] \\ \mathbf{M} &= \frac{1}{4\pi} \left(1 - \frac{1}{\mu} \right) \mathbf{B}^* - \left[\frac{\mathbf{u}}{c} \cdot \mathbf{0} \right] \end{aligned} \right\} \tag{8.26}$$

We can say that the motion of a magnetized medium induces an electric field ($\mathbf{P} \neq 0$ when $\epsilon = 1$ and $\mathbf{M}_0 \neq 0$) similar to how the motion of a polarized medium induces a magnetic field ($\mathbf{M} \neq 0$ when $\mu = 1$ and $\mathbf{P}_0 \neq 0$).

3. Using Eqs. (8.21) and (8.22) in Eqs. (8.26), we get

$$\left. \begin{aligned} \mathbf{D} &= \epsilon \mathbf{E} + \left(\epsilon - \frac{1}{\mu'} \right) \left[\frac{\mathbf{u}}{c} \cdot \mathbf{B} \right] = \epsilon \mathbf{E}^* - \frac{1}{\mu} \left[\frac{\mathbf{u}}{c} \cdot \mathbf{B} \right] \\ \mathbf{H} &= \frac{1}{\mu} \mathbf{B} + \left(\epsilon - \frac{1}{\mu} \right) \left[\frac{\mathbf{u}}{c} \cdot \mathbf{E} \right] = \frac{1}{\mu} \mathbf{B}^* + \epsilon \left[\frac{\mathbf{u}}{c} \cdot \mathbf{E} \right] \end{aligned} \right\} \tag{8.27}$$

Such (with an accuracy up to terms of the order of magnitude u^2/c^2) are the relationships between the basic vectors of an electromagnetic field in a moving medium corresponding to relationships (V) of

Chapter 7 for a medium at rest.* These relationships together with the expressions for the current density (8.16) and (8.19) and Maxwell's differential equations (I)-(IV) are sufficient for constructing a macroscopic theory of the electromagnetic field in slowly moving media. Sections 8.3-8.5 will be devoted to a treatment of this theory.

The qualitative distinction of Eqs. (8.27) from the corresponding relationships for media at rest consists in that the equality to zero of one of the electric field vectors \mathbf{E} or \mathbf{D} does not of necessity cause the other of these vectors to equal zero. The same relates to the magnetic field vectors. The reason is that in moving media, according to Eqs. (8.21) and (8.22), the polarization \mathbf{P} may differ from zero when $\mathbf{E} = 0$, and the magnetization \mathbf{M} may differ from zero when $\mathbf{B} = 0$.

4. Equations (8.27) express \mathbf{D} and \mathbf{H} through \mathbf{E} and \mathbf{B} , i.e. through the mean intensities of a microscopic field. We can naturally solve these equations, for example, relative to \mathbf{E} and \mathbf{H} . From the first equation (8.27), we get directly

$$\mathbf{E} = \frac{1}{\epsilon} \mathbf{D} - \left(1 - \frac{1}{\epsilon\mu}\right) \left[\frac{\mathbf{u}}{c} \cdot \mathbf{B}\right] \quad (8.28)$$

Introducing this into the second equation (8.27) and discarding terms having the order of magnitude u^2/c^2 (it would be inconsistent to retain them because the fundamental relationships of the theory being treated are correct only with an accuracy up to u^2/c^2), we get

$$\mathbf{H} = \frac{1}{\mu} \mathbf{B} + \left(1 - \frac{1}{\epsilon\mu}\right) \left[\frac{\mathbf{u}}{c} \cdot \mathbf{D}\right] \quad (8.29)$$

5. It must be noted in conclusion that the condition for the applicability of the formulas of this and all the following sections, strictly speaking, is not only the sufficiently slow motion of the medium [condition (8.1)], but also the requirement that the medium move

* The accurate relativistic relationships corresponding to Eqs. (8.27) are usually formulated as follows:

$$\mathbf{D} + \left[\frac{\mathbf{u}}{c} \cdot \mathbf{H}\right] = \epsilon \left\{ \mathbf{E} + \left[\frac{\mathbf{u}}{c} \cdot \mathbf{B}\right] \right\}$$

$$\mathbf{B} - \left[\frac{\mathbf{u}}{c} \cdot \mathbf{E}\right] = \mu \left\{ \mathbf{H} - \left[\frac{\mathbf{u}}{c} \cdot \mathbf{D}\right] \right\}$$

Solving these equations relative to \mathbf{D} and \mathbf{H} and discarding terms of the order of magnitude u^2/c^2 , it is easy to obtain Eq. (8.17). [More exactly, in comparison with unity we discard not only terms having the order of magnitude $(u/c)^2$ but also terms having the order $\epsilon\mu(u/c)^2$.]

*translationally and uniformly**: we took no account of the effects connected with acceleration of the motion of the medium. Particularly, the Tolman effect (Sec. 3.6) manifesting itself upon the sudden acceleration of conductors cannot be explained on the basis of the treated theory. If we nevertheless apply this theory to media moving with acceleration, particularly to uniformly rotating disks, cylinders and the like [Eikhenvald's experiment already considered in this section, a unipolar (homopolar) machine, etc.], then this is justified by the vanishingly small influence of acceleration on the phenomena of interest to us in this chapter.

It follows at least from our treatment of the theory of the Tolman effect set out in Sec. 3.6 that the influence of acceleration on motion of a medium is determined by the inertial force $-m(du/dt)$, where m is the mass of an electron. (In Sec. 3.6 the velocity was designated by \mathbf{v} instead of by \mathbf{u} as in this chapter.) By du/dt , we should understand the *total* acceleration of an element of the medium that consists of the local acceleration $\partial\mathbf{u}/\partial t$ and the acceleration of following $(\mathbf{u}\nabla)\mathbf{u}$:

$$\frac{d\mathbf{u}}{dt} = \frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\nabla)\mathbf{u}$$

[see any course of hydrodynamics; cf. also Eq. (2.99)].

Thus, the influence of acceleration, i.e. of the inertial force $-m\{(\partial\mathbf{u}/\partial t) + (\mathbf{u}\nabla)\mathbf{u}\}$, on the electrons of the medium is equivalent to an effective electric field of the intensity

$$\mathbf{E}' = -\frac{m}{e} \left\{ \frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\nabla)\mathbf{u} \right\} \quad (8.30)$$

It is important that the ratio m/e of the mass of an electron to its charge equals 1.9×10^{-18} absolute (electrostatic) cgs units, i.e. is very small.

Let us consider as an example a disk or cylinder uniformly rotating about its axis z with the angular velocity ω :

$$\frac{\partial u}{\partial t} = 0; \quad u_x = -y\omega, \quad u_y = x\omega, \quad u_z = 0$$

$$(\mathbf{u}\nabla)u_x = -x\omega^2, \quad (\mathbf{u}\nabla)u_y = -y\omega^2, \quad (\mathbf{u}\nabla)u_z = 0$$

If the radius of the rotating body is a , then the maximum value of E' is

$$E' = \frac{m}{e} a\omega^2$$

* More accurately, what is needed is a sufficiently small value of the derivatives of the velocity of the medium \mathbf{u} both with respect to time and the coordinates.

(This expression for E' could also be obtained directly from the expression $u^2/a = a\omega^2$ for the centrifugal acceleration in rotation.) Assuming that $a = 5$ cm and $\omega = 100\pi$ s⁻¹, which corresponds to 50 rps, we get $E' \approx 9 \times 10^{-13}$ esu $\approx 3 \times 10^{-10}$ V, i.e. an absolutely negligible value. It must be noted that no account is taken here of the deformation of the disk appearing upon its rotation and leading to the induction of an additional electric field; no account is also taken of the Barnett effect (see p. 347).

8.3 Ohm's Law and Electromagnetic Induction in Moving Conductors. Unipolar Induction

1. Formula (8.19) for the density of the conduction currents in moving conductors, i.e.

$$\mathbf{j}_{\text{cond}} = \kappa(\mathbf{E}^* + \mathbf{E}_{\text{ext}}) = \kappa \left(\mathbf{E} + \left[\frac{\mathbf{u}}{c} \cdot \mathbf{B} \right] + \mathbf{E}_{\text{ext}} \right)$$

differs from the corresponding formula (3.23) for stationary conductors

$$\mathbf{j} = \kappa(\mathbf{E} + \mathbf{E}_{\text{ext}})$$

only in the replacement of \mathbf{E} with the effective electric field intensity \mathbf{E}^* . On this basis, we can immediately transfer the results of the theory of currents in stationary conductors to the case of moving ones. For example, the current I in portion 1, 2 of a moving conductor having the resistance R_{12} , by analogy with Eq. (3.26), is

$$I = \frac{\mathcal{E}_{12}^* + \mathcal{E}_{\text{ext},12}}{R_{12}} \quad (8.31)$$

where $\mathcal{E}_{\text{ext},12}$ stands for the extraneous e.m.f., and \mathcal{E}_{12}^* for the effective voltage applied across this section of conductor:

$$\mathcal{E}_{12}^* = \int_1^2 \mathbf{E}^* ds = \int_1^2 \left(\mathbf{E} + \left[\frac{\mathbf{u}}{c} \cdot \mathbf{B} \right] \right) ds \quad (8.32)$$

For a quasilinear current loop deprived of branches, by analogy with Eq. (3.28), we get

$$IR = \mathcal{E}^* + \mathcal{E}_{\text{ext}} \quad (8.33)$$

where \mathcal{E}^* and \mathcal{E}_{ext} are the total effective voltage and the total extraneous e.m.f. in the circuit of the current:

$$\mathcal{E}^* = \oint_L \mathbf{E}^* ds = \oint_L \left(\mathbf{E} + \left[\frac{\mathbf{u}}{c} \cdot \mathbf{B} \right] \right) ds \quad (8.34)$$

Equation (8.33) coincides with our previous equation (6.14) because the induced e.m.f. \mathcal{E}_{ind} , which we operated with in Sec. 6.2 and in general in Chapter 6, and the total effective voltage \mathcal{E}^* are identical concepts. Indeed, on the basis of Stokes's theorem and Maxwell's equation (II), we have

$$\oint_L \mathbf{E} \, ds = \int_S \text{curl } \mathbf{E} \, d\mathbf{S} = -\frac{1}{c} \int_S \frac{\partial \mathbf{B}}{\partial t} \, d\mathbf{S}$$

where S stands for the surface resting on the contour L . Further substituting in the derivation of Eq. (6.6) \mathbf{u} for \mathbf{v} , \mathbf{B} for \mathbf{H} , and Ψ for Φ , we see that

$$\oint \left[\frac{\mathbf{u} \cdot \mathbf{B}}{c} \right] ds = -\frac{1}{c} \oint \mathbf{B} [\mathbf{u} \, ds] = -\frac{1}{c} \left(\frac{d\Psi}{dt} \right)_{B=\text{const}} \quad (8.35)$$

where $(d\Psi/dt)_{B=\text{const}}$ signifies the rate of change of the magnetic induction flux Ψ through the surface S calculated on the assumption that the induction \mathbf{B} does not change with time, i.e. that part of the rate of change of the flux that is due to the motion of the contour L . Thus,

$$\begin{aligned} \mathcal{E}^* &= -\frac{1}{c} \int_S \frac{d\mathbf{B}}{dt} \, d\mathbf{S} - \frac{1}{c} \left(\frac{d\Psi}{dt} \right)_{B=\text{const}} = \\ &= \frac{1}{c} \left(\frac{d\Psi}{dt} \right)_{u=0} - \frac{1}{c} \left(\frac{d\Psi}{dt} \right)_{B=\text{const}} \end{aligned}$$

because $\frac{\partial \mathbf{B}}{\partial t} \, d\mathbf{S}$ equals the part of the rate of change of the induction flux Ψ through the contour L that is due to the change of the induction in time and in which no account is taken of the motion of the contour. It is obvious that

$$\left(\frac{d\Psi}{dt} \right)_{u=0} + \left(\frac{d\Psi}{dt} \right)_{B=\text{const}} = \frac{d\Psi}{dt} \quad (8.36)$$

where $d\Psi/dt$ without a subscript signifies the total rate of change of the induction flux through the contour L due both to the change in time of the induction \mathbf{B} and to motion of the contour. We thus finally have

$$\mathcal{E}^* = -\frac{1}{c} \frac{d\Psi}{dt} \quad (8.37)$$

which, as we were to prove, coincides with Eq. (6.13) for \mathcal{E}_{ind} .

2. Thus, Eqs. (8.34) and (8.37) are equivalent to the law of induction formulated in Sec. 6.2 according to which the induced e.m.f. \mathcal{E}_{ind}

in an arbitrary closed contour is determined by the rate of change of the magnetic induction flux Ψ through this contour. This law can and must be defined more precisely, however, in the sense that it must be applied to a *material contour*, i.e. to a combination of material points (elements) of the medium forming a closed contour. In other words, when calculating the change in the induction flux through a given contour we must consider that at the moment of time $t + dt$ this contour is formed by the same material points of the medium which formed it at the preceding moment t . This is clear from the above derivation of Eq. (8.37), especially from Eq. (8.35) according to which $(d\Psi)_{B=\text{const}}$ equals the difference between the fluxes through the given contour and through the contour whose points are displaced relative to the given one over the distance $\mathbf{u} dt$.

3. This rule for calculating \mathcal{E}_{ind} needs additional clarification when the material contour which is closed at the moment t becomes open at the following moment $t + dt$ owing to motion of the medium. Let us gain an understanding in this case using the example of the so-called *unipolar induction* and the unipolar (homopolar) machine.

In principle, a unipolar machine (this unsuccessful name is explained by historical reasons) consists of a cylindrical permanent magnet rotating about its axis. If we use sliding contacts A and B to connect a conductor to the axis and to the side surface of the rotating magnet (Fig. 90), then a current will flow through the conductor AVB . Upon uniform rotation of the cylindrical magnet, the intensity of the electromagnetic field and the density of the current at each point of space will be constant in time.*

Let us apply the law of induction to a contour passing along the external conductor AVB and along the magnet, for example to the contour $COAVBC$. At the moment $t + dt$, the material points that were on this contour at the moment t will be displaced over the distance $\mathbf{u} dt$ and will occupy the position $C'OAVBC$. Hence, to determine the induced e.m.f. \mathcal{E}_{ind} in the contour $COAVBC$, it is necessary to calculate the difference $d\Psi$ between the induction fluxes through this contour and through the contour $C'OAVBC$. Let us denote these fluxes by Ψ and Ψ' , respectively, so that $d\Psi = \Psi - \Psi'$.

* An animated discussion went on for a long time in the 19th century on the question of unipolar induction. The discussion was connected with attempts to interpret this phenomenon in the sense that the lines of force of the magnetic field induced by the magnet rotate together with the latter about its axis. It was considered that the motion of the lines of force intersecting the stationary conductor AVB resulted in the appearance of induced e.m.f.'s in it. There is no need to say that such an interpretation cannot withstand any criticism: lines of force are only an auxiliary concept used to describe a field, and not material formations whose separate elements could be individualized, related to definite sources of a field (the impossibility of doing which is especially clear, for instance, for the superposition of the fields of two magnets — a movable and a fixed ones), have their motion in space watched, etc.

The contour $C'OAVBC$, however, unlike the contour $COAVBC'$, is not closed, so that, strictly speaking, the concept of the flux Ψ' through this unclosed contour is not definite.

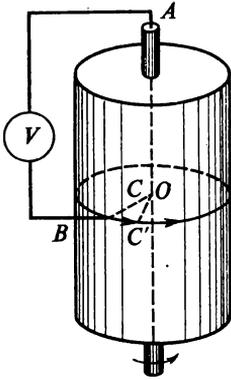


Fig. 90

Let us therefore turn to the derivation of the law of induction (8.37) given above. It follows from Eqs. (8.36) and (8.35) that in the case being considered by $d\Psi/dt$ we should understand the quantity

$$\frac{d\Psi}{dt} = \oint \mathbf{B}[\mathbf{u} ds]$$

it being necessary to take this integral over the closed contour $COAVBC$. Since $[\mathbf{u} ds] \neq 0$ only on the portion CO of this contour, then

$$d\Psi = \int_c^o \mathbf{B}[\mathbf{u} dt \cdot ds]$$

It is easy to see that this expression for $d\Psi$ with an accuracy up to values of the second order of magnitude relative to dt equals the induction flux through the infinitely small round sector COC' . Consequently, when calculating $d\Psi$ from the relationship $d\Psi = \Psi' - \Psi$ by Ψ' we can understand the flux through the *closed* contour $C'OAVBCC'$. The latter is obtained by *closing the contour $C'OAVBC$ deformed by motion with the portion CC' of the path covered by the point C' of discontinuity of the contour*. This is exactly the more precise definition of the law of induction (8.37) which can be applied, as we can readily see, to any opening movement of the contour.

4. Let us return to our unipolar machine. As we have seen, the induced e.m.f. \mathcal{E}_{ind} in the contour $COAVBC$ is

$$\mathcal{E}_{\text{ind}} = -\frac{1}{c} \frac{d\Psi}{dt} = -\frac{1}{c} \int_c^o \mathbf{B}[\mathbf{u} ds]$$

In a similar way, we can calculate \mathcal{E}_{ind} for any other closed contour. For each contour fixed in space, the quantity \mathcal{E}_{ind} has a constant value that does not change with time. When calculating the currents excited in the magnet and in the external conductor by these induced e.m.f.'s, it is no longer necessary to take rotation of the magnet into account; the influence of this rotation is completely accounted for by the value of \mathcal{E}_{ind} .

When considering a unipolar machine, however, it is simpler to proceed not from the integral form (8.37) of the law of induction, but directly from Ohm's law for moving media [Eq. (8.19)]. It follows from Eq. (8.19) that the current in the external portion of the circuit AB and the distribution of the currents over the volume of the magnet will be the same as for a magnet at rest throughout whose volume extraneous e.m.f.'s are distributed of the intensity

$$\mathbf{E}_{\text{ext}} = \left[\frac{\mathbf{u}}{c} \cdot \mathbf{B} \right] \quad (8.38)$$

The intensity of the electric field inside and outside the magnet will correspond to the same case of a magnet at rest with the distributed e.m.f. \mathbf{E}_{ext} .

Indeed, from the continuity equation* $\text{div } \mathbf{j}_{\text{cond}} = 0$ and Ohm's law (8.19) it follows that $\text{div } (\kappa \mathbf{E}) = -\text{div } \kappa \left[\frac{\mathbf{u}}{c} \cdot \mathbf{B} \right]$. Since a stationary electric field has the potential ($\mathbf{E} = -\nabla \phi$), then the vector \mathbf{E} is determined unambiguously by this equation (if we consider the induction \mathbf{B} inside the magnet to be given). The same equation is also obtained for $\text{div } \kappa \mathbf{E}$ from Eq. (3.23) for a stationary magnet over whose volume there are distributed extraneous e.m.f.'s of intensity (8.38).

Let us consider as an example a uniformly rotating cylindrical magnet to which no conductors are connected and in which therefore no currents circulate. The absence of currents means that the Lorentz force $q \left[\frac{\mathbf{u}}{c} \cdot \mathbf{B} \right]$ directed along the radius r of the cylinder is compensated inside the magnet by the radial electric field \mathbf{E} , i.e. that inside the magnet

$$\mathbf{E} = - \left[\frac{\mathbf{u}}{c} \cdot \mathbf{B} \right] \quad (8.39)$$

* On the basis of continuity equation (IVa) and Eq. (8.16), we have $\text{div } \mathbf{j}_{\text{cond}} = \text{div } \rho \mathbf{u} = 0$. It is easy to see, however, that the divergence of the convection current $\rho \mathbf{u}$ equals zero. Indeed, in the cylindrical system of coordinates z, r and α the vector $\rho \mathbf{u}$ has only one component ρu_α differing from zero. The absolute value of this component owing to the axial symmetry of the problem does not depend on the angle α . Hence on the basis of Eq. (A.22) it follows that $\text{div } \rho \mathbf{u} = 0$.

Assuming for simplicity that the vector \mathbf{B} in the magnet has a constant value and is directed along the axis of rotation, we get

$$E_r = -\frac{u}{c} B = -\frac{\omega r}{c} B$$

where ω stands for the angular velocity of rotation of the magnet. Thus, the following potential difference sets in between the cylindrical surface of the magnet and its axis:

$$\varphi_{\text{axis}} - \varphi_{\text{surf}} = \int_{r=0}^{r=a} E_r dr = -\frac{\omega a^2}{2c} B$$

where a is the radius of the magnet. Knowing the potential on the surface of the magnet, we can also determine the field in the external space. On the other hand, if we connect one end A of the conductor AVB to the axis and its other end B to the surface of the cylinder (see Fig. 90), then owing to the potential difference between the points A and B a current will flow through the conductor. Naturally, when the circuit AVB is closed the potential difference between A and B will diminish in the same way as the potential difference between the electrodes of an accumulator diminishes when the external circuit connecting these electrodes is closed.

5. What is the reason for the appearance of the radial electric field (8.39) in the insulated rotating magnet? This field is partly due to the redistribution of the conduction electrons in the magnet under the action of the Lorentz force $q \left[\frac{\mathbf{u}}{c} \cdot \mathbf{B} \right]$. The main part of the electric field appearing upon the motion of the magnet, however, has a purely relativistic origin and is connected with the circumstance noted when discussing formula (8.36) that, according to the theory of relativity, the motion of a magnetized medium induces an electric field. This relativistic effect manifests itself in the pure form not when the magnet rotates, but when it performs uniform translational motion with the velocity \mathbf{u} perpendicular to the axis of the magnet. In this case, the system of coordinates (frame of reference) S' connected with the magnet will be an inertial one. In this system, the electric field \mathbf{E} will evidently equal zero if there are no other bodies in the field of the magnet, for instance an external conductor sliding along it. Applying to this case the relativistic formulas of field transformation (8.58), it is not difficult to see that in a "stationary" (laboratory) system of coordinates S the intensity of the electric field (up to values of the order of magnitude u^2/c^2) will be

$$\mathbf{E} = \mathbf{E}' - \left[\frac{\mathbf{u}}{c} \cdot \mathbf{B}' \right] = - \left[\frac{\mathbf{u}}{c} \cdot \mathbf{B}' \right]$$

where \mathbf{B}' is the induction of the magnetic field measured in the system S' . In this expression, we can replace \mathbf{B}' with the value \mathbf{B} of the induction in the "stationary" system of coordinates S with an accuracy up to a value of the order of magnitude u^2/c^2 :

$$\mathbf{E} = - \left[\frac{\mathbf{u} \cdot \mathbf{B}}{c} \right]$$

which coincides with Eq. (8.39). Thus, for translational motion of a magnet, Eq. (8.39) may be applied to all the points of the field both inside and outside the magnet and without any regard to whether the magnet conducts electricity or is an insulator. Thus, the appearance of an electric field in the uniform translational motion of a magnet is explained by the fact that, as will be shown in Sec. 8.6, the division of an electromagnetic field into an electric and a magnetic fields has a relative nature and depends on the frame of reference.

Problem 39. An iron magnetized sphere with the radius a in a conducting medium (for example in an electrolyte) rotates about its centre with a constant angular velocity ω . The magnetic induction \mathbf{B}_0 of the field induced by the permanent magnetization of the sphere is constant throughout its volume and is directed parallel to the axis of rotation. Show that the potential of the electric field induced by rotation of the sphere is: when $R \leq a$

$$\varphi = \varphi_1 = - \frac{\omega}{3c} B_0 \left\{ \frac{\kappa_1 R^2}{2\kappa_1 + 3\kappa_2} (3 \cos^2 \theta - 1) + a^2 - R^2 \right\}$$

and when $R \geq a$

$$\varphi = \varphi_2 = - \frac{\omega}{3c} B_0 \frac{\kappa_1 a^5}{(2\kappa_1 + 3\kappa_2) R^3} (3 \cos^2 \theta - 1)$$

where κ_1 and κ_2 = conductivities of iron and the electrolyte, respectively

\mathbf{R} = radius-vector conducted from the centre of the sphere to the given point of space

θ = polar angle between \mathbf{R} and the axis of rotation.

Show that the lines of the electric current in the electrolyte begin in the equatorial region of the sphere surface and terminate in the polar regions of this surface; these regions are separated by parallel circles corresponding to the angles

$$\theta = \arccos \frac{1}{\sqrt{3}} \approx 55^\circ \text{ and } \theta = \pi - \arccos \frac{1}{\sqrt{3}} \approx 125^\circ$$

8.4 A Dielectric Moving in an Electromagnetic Field

1. The first consistent theory of electromagnetic phenomena in moving media was constructed by H. Hertz in about 1890. As regards the induction of currents in slowly moving conductors, Hertz's theory leads to the same results confirmed experimentally as the modern theory. This is exactly the explanation why Hertz's very simple theory, although inconsistent in principle, is widely employed in electrical engineering to date. The application of Hertz's theory to the motion of dielectrics and non-conducting (electricity) magnetics in an electromagnetic field, however, leads to incorrect results. This was proved, particularly, by the experiments of W. Roentgen, A. Eikhenvald, and C. Wilson that greatly affected the development of the modern theory. We shall consider some of these experiments schematically in this section.

2. A plane capacitor whose plates are connected to each other by a conductor is placed in a homogeneous magnetic field \mathbf{H} parallel to the capacitor plates. The space between the capacitor plates is filled with a dielectric having the permittivity ϵ . If we impart to the dielectric the velocity \mathbf{u} perpendicular to the direction of the magnetic field \mathbf{H} (Fig. 91), then the capacitor is charged (the arrangement of Wilson's experiment in which, however, the dielectric rotated between the capacitor plates instead of performing translational motion).

Let us consider a stationary case of a uniformly moving dielectric. A constant electric field according to Maxwell's equation (II) has the potential φ . The capacitor plates connected to each other have an identical potential. Therefore, the electric field will equal zero not only in the metal of the plates, but also in the dielectric between them. The displacement \mathbf{D} will evidently also equal zero in the metal of the plates, but in the moving dielectric, according to Eq. (8.27₁), \mathbf{D} will be (with $\mathbf{E} = 0$ and $\mu = 1$)

$$\mathbf{D} = (\epsilon - 1) \left[\frac{\mathbf{u}}{c} \cdot \mathbf{B} \right]$$

At the same time from Eq. (8.27₂) with $\mathbf{E} = 0$ and $\mu = 1$, we have

$$\mathbf{H} = \mathbf{B}$$

Consequently, the magnetic field in a moving dielectric will be the same as if it were at rest. The displacement in it will be (see Fig. 91)

$$D_x = D_y = 0 \text{ and } D_z = - \frac{(\epsilon - 1) u}{c} H$$

Thus, the normal component D_z of the vector \mathbf{D} undergoes a jump on the boundaries between the dielectric and the capacitor plates.

On the basis of the boundary condition (IV'), this signifies that there are surface free charges having the density

$$\sigma = \pm \frac{1}{4\pi} D_z = \pm \frac{(\epsilon - 1)u}{4\pi c} H \quad (8.40)$$

on the capacitor plates. This conclusion was experimentally confirmed by Wilson. Hertz's theory, however, resulted in an improper expression for σ obtained from Eq. (8.40) by substituting ϵ for $\epsilon - 1$.

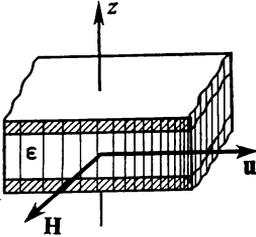


Fig. 91

3. We have already considered Eikhenvald's experiment in Sec. 8.2. We shall consider it again here as an example of application of the general equations of the theory.

The scheme of Eikhenvald's experiment differs from that of Wilson's experiment which we have just considered in that there is no external magnetic field, the capacitor plates are insulated from each other, and the capacitor is charged. The motion of the dielectric in the charged capacitor creates a magnetic field.

Since in the metal of the plates, we have $\mathbf{E} = \mathbf{D} = 0$, then it follows from the boundary conditions (II') and (IV') that in the dielectric

$$E_x = E_y = 0 \text{ and } D_z = 4\pi\sigma$$

where σ is the charge density on the bottom plate of the capacitor (the charge density on the top plate is $-\sigma$; we choose the coordinate axes in the same way as in Fig. 91).

The induction \mathbf{B} of the magnetic field appearing owing to motion of the dielectric will evidently be proportional to its velocity; therefore it follows from Eq. (8.27₁) with an accuracy up to terms of the order of magnitude of u^2/c^2 that in the dielectric $\mathbf{D} = \epsilon\mathbf{E}$ and, consequently,

$$E_x = E_y = 0 \text{ and } E_z = \frac{4\pi\sigma}{\epsilon}$$

Introducing this into (8.27₂) and assuming that $\mu = 1$, we get that inside the dielectric

$$H_x = B_x + \frac{4\pi(\epsilon - 1)u\sigma}{\epsilon c}, \quad H_y = B_y, \quad H_z = B_z$$

Let us denote the intensity and the induction of the magnetic field in the plates of the capacitor by \mathbf{H}' and \mathbf{B}' , it being obvious that $\mathbf{H}' = \mathbf{B}'$. It follows from the continuity of the tangential components of the intensity \mathbf{H} and the normal component of \mathbf{B} that

$$\begin{aligned} B'_x = H'_x = H_x = B_x + \frac{4\pi(\varepsilon - 1)u\sigma}{\varepsilon c}, \quad B'_y = H'_y = \\ = H_y = B_y, \quad B'_z = B_z \end{aligned} \quad (8.41)$$

Thus, the tangential component B_x of the magnetic induction undergoes on the dielectric-metal interfaces a jump of $\pm \frac{4\pi(\varepsilon - 1)u\sigma}{\varepsilon c}$, whereas the components B_y and B_z are continuous. At the same time, throughout the entire space

$$\operatorname{div} \mathbf{B} = 0 \quad \text{and} \quad \operatorname{curl} \mathbf{B} = 0 \quad (8.42)$$

The first of these equations is Maxwell's equation (III). The second one follows from the fact that owing to the absence of currents, $\operatorname{curl} \mathbf{H} = 0$ throughout the entire space, and that outside the dielectric \mathbf{B} coincides with \mathbf{H} . Although B_x differs from H_x inside the dielectric, the difference is a constant quantity.

The combination of the differential equations (8.42) and the boundary conditions (8.41) determines the field of the vector \mathbf{B} throughout the entire space. It is identical with the combination of equations and conditions for the intensity of the magnetic field \mathbf{H} induced in a stationary medium having the permeability $\mu = 1$ by surface currents with the density

$$i_y = \pm \frac{(\varepsilon - 1)u\sigma}{\varepsilon} \quad (8.43)$$

flowing in opposite directions through the top and bottom interfaces between the dielectric and the metal. Since $\mathbf{B} = \mathbf{H}$ outside the dielectric, then, consequently, the magnetic field outside the dielectric coincides with the field of this system of surface currents. This conclusion coincides with the results which we have obtained in a different way in Sec. 8.2 [see Eq. (8.23)].

8.5 Propagation of Light in Moving Dielectrics.

Fresnel Drag Coefficient.

Reflection from a Moving Mirror

1. Let us consider a plane light wave having the frequency ω in a homogeneous isotropic non-magnetic ($\mu = 1$) dielectric moving with the velocity \mathbf{u} . Let \mathbf{E}_0 , \mathbf{H}_0 , \mathbf{D}_0 , and \mathbf{B}_0 be the constant amplitudes of

the wave field vectors. Hence, for instance, the intensity of the electric field of the wave will be expressed by Eq. (7.96₁):

$$\mathbf{E} = \mathbf{E}_0 \exp [i(\omega t - k\mathbf{nR})]$$

where \mathbf{n} = unit vector in the direction of wave propagation
 k = wave number.

Similar expressions will also be obtained for the remaining vectors \mathbf{H} , \mathbf{B} , and \mathbf{D} .

According to Eq. (7.89), the action of the differential operator nabla (∇) on the vectors of a wave field consists in multiplication of these vectors by $-ik\mathbf{n}$, so that, for example,

$$\operatorname{div} \mathbf{E} = -ik (\mathbf{nE})$$

$$\operatorname{curl} \mathbf{E} = -ik [\mathbf{nE}]$$

Therefore Maxwell's equations (I) and (II) after cancelling $\exp [i(\omega t - k\mathbf{nR})]$ become

$$\left. \begin{aligned} \frac{\omega}{c} \mathbf{D}_0 &= -k [\mathbf{nH}_0] \\ \frac{\omega}{c} \mathbf{B}_0 &= k [\mathbf{nE}_0] \end{aligned} \right\} \quad (8.44)$$

Introducing these expressions into Eqs. (8.28) and (8.29), we get after cancelling the same factor

$$\left. \begin{aligned} \mathbf{E}_0 &= -\frac{ck}{\varepsilon\omega} [\mathbf{nH}_0] - \frac{ck}{\omega} \left(1 - \frac{1}{\varepsilon}\right) \left[\frac{\mathbf{u}}{c} [\mathbf{nE}_0]\right] \\ \mathbf{H}_0 &= \frac{ck}{\omega} [\mathbf{nE}_0] - \frac{ck}{\omega} \left(1 - \frac{1}{\varepsilon}\right) \left[\frac{\mathbf{u}}{c} [\mathbf{nH}_0]\right] \end{aligned} \right\} \quad (8.45)$$

Thus, the induction vectors \mathbf{D} and \mathbf{B} in a wave field are perpendicular to its direction \mathbf{n} . The intensity vectors \mathbf{E} and \mathbf{H} , however, in general have components differing from zero along \mathbf{n} (if only the direction of the velocity of the dielectric does not coincide with the direction of the wave \mathbf{n} or with the directly opposite one). The propagation of light in a moving *isotropic* dielectric, generally speaking, is quite similar to the propagation of light in a stationary *anisotropic* dielectric (more exactly, in an optically uniaxial crystal whose principal axis coincides with the direction of motion of the dielectric).

Let us choose the z -axis in the direction of propagation of the wave so that Eq. (7.96₁) becomes

$$\mathbf{E} = \mathbf{E}_0 \exp [i(\omega t - kz)]$$

and let us assume for simplicity that the direction of the velocity of the dielectric \mathbf{u} coincides with the direction of the wave or is directly opposite to it:

$$u_x = u_y = 0 \text{ and } u_z = \pm u$$

In this case

$$[\mathbf{nH}_0] = -\mathbf{i}H_{0y} + \mathbf{j}H_{0x}$$

$$\left[\frac{\mathbf{u}}{c} [\mathbf{nE}_0] \right] = \mathbf{n} \left(\frac{\mathbf{u}}{c} \mathbf{E}_0 \right) - \mathbf{E}_0 \left(\frac{\mathbf{un}}{c} \right) = -\mathbf{i} \frac{u_z}{c} E_{0x} - \mathbf{j} \frac{u_z}{c} E_{0y}$$

where \mathbf{i} and \mathbf{j} are unit vectors directed along the x - and y -axes. Taking into account, in addition, that according to Eq. (7.87) the ratio ω/k equals the wave velocity v :

$$v = \frac{\omega}{k}$$

we get from Eqs. (8.45) after multiplying these equations by $\omega/k = v$

$$\left. \begin{aligned} vE_{0x} &= \frac{c}{\varepsilon} H_{0y} + u_z \left(1 - \frac{1}{\varepsilon} \right) E_{0x} \\ vE_{0y} &= -\frac{c}{\varepsilon} H_{0x} + u_z \left(1 - \frac{1}{\varepsilon} \right) E_{0y} \\ E_{0z} &= 0 \\ vH_{0x} &= -cE_{0y} + u_z \left(1 - \frac{1}{\varepsilon} \right) H_{0x} \\ vH_{0y} &= cE_{0x} + u_z \left(1 - \frac{1}{\varepsilon} \right) H_{0y} \\ H_{0z} &= 0 \end{aligned} \right\} \quad (8.46)$$

As we have already noted, in the case of parallel vectors \mathbf{u} and \mathbf{n} which we are considering, not only \mathbf{D} and \mathbf{B} , but also \mathbf{E} and \mathbf{H} are perpendicular to \mathbf{n} , i.e. the light wave is a transverse one. Two of the equations (8.46) include only the components E_{0x} and H_{0y} ; two others include only E_{0y} and H_{0x} . Let us consider, for example, the equations for E_{0x} and H_{0y} :

$$\begin{aligned} \left\{ v - u_z \left(1 - \frac{1}{\varepsilon} \right) \right\} E_{0x} - \frac{c}{\varepsilon} H_{0y} &= 0 \\ -cE_{0x} + \left\{ v - u_z \left(1 - \frac{1}{\varepsilon} \right) \right\} H_{0y} &= 0 \end{aligned}$$

It follows from these equations, if E_{0x} and H_{0y} differ from zero, that

$$\left\{ v - u_z \left(1 - \frac{1}{\varepsilon} \right) \right\}^2 = \frac{c^2}{\varepsilon}$$

or

$$v - u_z \left(1 - \frac{1}{\varepsilon} \right) = \frac{c}{\sqrt{\varepsilon}}$$

[the choice of the sign when extracting the root is determined by the fact that, in accordance with Eq. (7.87), when $u = 0$ we must have $v = c/\sqrt{\varepsilon}$]. Taking into account that according to Eq. (7.108) $\sqrt{\varepsilon}$ equals the index of refraction n of the medium, we get the final expression for the velocity of light in a moving medium:

$$v = \frac{c}{n} + \left(1 - \frac{1}{n^2} \right) u_z \quad (8.47)$$

Conducting calculations for an arbitrary angle between the velocity of the medium \mathbf{u} and the direction of the wave \mathbf{n} , we can see that Eq. (8.47) remains correct in this general case too if by u_z in it we understand the projection of the velocity of the medium on the direction of propagation of the wave.

2. Formula (8.47) was first obtained by A. Fresnel in 1818 on the basis of groundless, from the modern viewpoint, notions on the motion of light ether, i.e. the hypothetic medium in which light waves propagate. If the light ether piercing a moving dielectric remained at rest, then according to these notions, the velocity of light v in the moving dielectric ought to equal the velocity of light c/n in a stationary dielectric [see Eq. (7.87)]. Conversely, if the ether were completely carried along by the motion of the dielectric, then the resultant velocity of the light ought to equal the sum of the velocity of the light c/n in the ether and the velocity u of the ether itself:

$$v = \frac{c}{n} \pm u$$

if \mathbf{u} is parallel and antiparallel to the direction of the wave. Fresnel, however, assuming that the ether is only partly carried along by the motion of the medium, obtained Eq. (8.47). The factor $1 - 1/n^2$ in it is called the *Fresnel drag coefficient*.

H. Lorentz showed in 1895 that Fresnel's formula must be supplemented with a certain correction taking into account the dispersion of the medium, i.e. the dependence of the refractive index on the wavelength. Fresnel's formula was confirmed experimentally by A. Fizeau in 1851 and with special accuracy by P. Zeeman in 1914. The latter also succeeded in confirming the necessity of Lorentz's correction.

3. Let us also briefly consider the reflection and refraction of light in a moving dielectric. Assume that a plane wave impinges from a

vacuum onto a dielectric moving along the direction of the z -axis, and that the wave also propagates in the direction of this axis:

$$\mathbf{E} = \mathbf{E}_0 \exp [i (\omega t - kz)] \quad \text{and} \quad \mathbf{H} = \mathbf{H}_0 \exp [i (\omega t - kz)]$$

Let, further, the surface of the dielectric coincide with the plane $z = ut$.

We shall denote the quantities relating to the wave reflected from the dielectric and to the refracted wave in the dielectric by the subscripts "r" and "g", respectively, as we did in Sec. 7.11. For example

$$\mathbf{E}_r = \mathbf{E}_{0,r} \exp [i(\omega_r t + k_r z)]$$

$$\mathbf{E}_g = \mathbf{E}_{0,g} \exp [i(\omega_g t - k_g z)], \text{ etc.}$$

The expression for the intensity \mathbf{E}_r of the reflected wave has a plus sign instead of a minus sign in the exponent because the direction of this wave is opposite to that of the z -axis.

Let us consider a boundary condition on the surface of the dielectric, for example condition (II') of the continuity of the tangential components of the vector \mathbf{E} :

$$\begin{aligned} E_{0r} \exp [i (\omega t - kz)] + E_{0t,r} \exp [i (\omega_r t + k_r z)] = \\ = E_{0t,g} \exp [i (\omega_g t - k_g z)] \end{aligned}$$

with $z = ut$.

For this condition to be observed at any value of the time t , after substitution of ut for z , the exponents of all three terms must be identical:

$$\omega - ku = \omega_r + k_r u = \omega_g - k_g u \tag{8.48}$$

[cf. Eq. (7.98)].

Let us express the wave vectors through the frequencies. For the incident and reflected waves in a vacuum, we have

$$k = \frac{\omega}{c} \quad \text{and} \quad k_r = \frac{\omega_r}{c}$$

As regards the refracted wave in the dielectric, then k_g in Eq. (8.48) is multiplied by u . Hence, with an accuracy up to values of the order of magnitude u^2/c^2 , we can introduce the value of k_g for the stationary dielectric into this formula:

$$k_g = \frac{\omega_g}{v} = \frac{n\omega_g}{c}$$

Using these values in Eq. (8.48), we get

$$\omega \left(1 - \frac{u}{c} \right) = \omega_r \left(1 + \frac{u}{c} \right) = \omega_g \left(1 - \frac{nu}{c} \right)$$

or, with an accuracy up to u^2/c^2 ,

$$\omega_r = \omega \left(1 - \frac{2u}{c} \right) \quad (8.49)$$

$$\omega_g = \omega \left\{ 1 + \frac{u}{c} (n - 1) \right\} \quad (8.50)$$

Thus, when light is reflected and refracted in moving media, the frequency of the light changes. The frequency ω_r of the reflected wave unlike the frequency ω_g of the refracted one does not depend on the refractive index of the medium n and in general on the properties of the medium, so that Eq. (8.49), for example, can also be applied to metals.

The expression for ω_r permits the following simple interpretation. The light of a source S reflected, for instance, from a mirror seems to travel from the images S' of this source in the mirror. If the mirror is perpendicular to the incident beam of light and travels in the direction of this beam with the velocity u , then the image S' of the source in the mirror travels in the same direction with the double velocity $2u$. Consequently, if we replace the image S' of the source S with a real source of light having the same natural frequency ω as our source S , then owing to the Doppler effect, the frequency of the light ω' emitted by this moving source in the direction of the reflected wave (i.e. in a direction opposite to that of the source) would equal

$$\omega' = \omega \left(1 - \frac{2u}{c} \right) \quad (8.51)$$

which coincides with Eq. (8.49) for ω_r .

8.6 Transformations of Frame of Reference. Relative Nature of Difference Between Electric and Magnetic Fields

1. In everything said in the preceding material (except for Sec. 6.2), we assumed that when studying electromagnetic phenomena all the readings of the position and motion of charges and material bodies, and also the readings of all physical quantities in general are reduced to a definite inertial frame of reference that we conditionally call stationary (see Secs. 6.2 and 8.2). The laws of electrodynamics which we have set out may be applied only to observations and measurements that were made relative to an *inertial* frame of reference. At the same time, we know that all inertial frames of reference are equivalent

so that the laws of electromagnetic as well as all physical phenomena in general do not change when passing over from one inertial frame of reference S to another frame S' moving rectilinearly and uniformly relative to S with an arbitrary velocity \mathbf{v} .

The absolute values of concrete physical quantities, however, change when passing over from one frame of reference S to another one S' : the results of measuring the *same phenomenon* in two different frames, generally speaking, differ from each other. For example, if the velocity and acceleration of a body relative to the frame S equal \mathbf{u} and \mathbf{a} , respectively, then the velocity and acceleration \mathbf{u}' and \mathbf{a}' of the same body relative to the frame S' in a non-relativistic approximation (i.e. with an accuracy up to terms of the order of magnitude v^2/c^2 and uv/c^2) are

$$\mathbf{u}' = \mathbf{u} - \mathbf{v} \text{ and } \mathbf{a}' = \mathbf{a} \tag{8.52}$$

Therefore, the theory of electromagnetic phenomena must, first, answer the question as to how the absolute values of electromagnetic quantities (the field vectors \mathbf{E} , \mathbf{B} , \mathbf{D} , \mathbf{H} , the charge and current densities ρ and \mathbf{j} , etc.) vary when the frame of reference is changed, and, second, must show that the invariance of the laws of electrodynamics when passing over from the frame of reference S to the frame S' uniformly moving relative to S follows from the established way of recalculating physical quantities from one frame of reference to another.

2. It is assumed in modern electrodynamics that the absolute value of the arbitrary electric charge q does not depend on the frame of reference. As regards the field vectors, it can easily be seen that their values appreciably depend on the frame of reference.

Let us assume, for example, that measurements in the frame S have established that in the given region of space V the electric field $\mathbf{E} = 0$, whereas the magnetic field $\mathbf{H} \neq 0$ (we shall presume for simplicity that there is a vacuum in the region of space being considered: $\epsilon = \mu = 1$). This means that if the charge q placed in V is at rest relative to the frame S , then no forces act on it. If it is moving relative to S with the velocity \mathbf{u} , however, then it is acted upon by the force

$$\mathbf{f} = q \left[\frac{\mathbf{u}}{c} \cdot \mathbf{H} \right]$$

Since the velocity \mathbf{u}' of the charge relative to the frame S' is

$$\mathbf{u}' = \mathbf{u} - \mathbf{v}$$

then

$$\mathbf{f} = q \left[\frac{\mathbf{u}' + \mathbf{v}}{c} \cdot \mathbf{H} \right] = q \left[\frac{\mathbf{v}}{c} \cdot \mathbf{H} \right] + q \left[\frac{\mathbf{u}'}{c} \cdot \mathbf{H} \right] \tag{8.53}$$

Hence, the following force acts on a charge that is at rest relative to the frame S' (i.e. when $\mathbf{u}' = 0$):

$$\mathbf{f} = q \left[\frac{\mathbf{v}}{c} \cdot \mathbf{H} \right]$$

Since, by definition (see Sec. 4.4), the electric field intensity \mathbf{E} equals the force acting on a *stationary* unit positive charge, then from observations performed relative to the frame of reference S' it will follow that an electric field having the intensity

$$\mathbf{E}' = \left[\frac{\mathbf{v}}{c} \cdot \mathbf{H} \right] \quad (8.54)$$

exists in the space V . It follows at the same time from Eq. (8.53) that the magnetic field

$$\mathbf{H}' = \mathbf{H} \quad (8.55)$$

exists relative to the frame of reference S' because the force \mathbf{f} should be expressed in the frame S' , which is quite equivalent to the frame of reference S , by the Lorentz formula*

$$\mathbf{f} = q \left(\mathbf{E}' + \left[\frac{\mathbf{u}'}{c} \cdot \mathbf{H}' \right] \right)$$

We thus arrive at the conclusion that *the division of an electromagnetic field into an electric field and a magnetic field has a relative nature*: a field which in the frame of reference S is only a magnetic one ($\mathbf{E} = 0$, $\mathbf{H} \neq 0$) is from the viewpoint of the equivalent frame of reference S' an electromagnetic field in the narrow meaning of this word ($\mathbf{E}' \neq 0$, $\mathbf{H}' \neq 0$). Similarly, when $\mathbf{E} \neq 0$ in the frame of reference S , then in the frame S' , generally speaking, both \mathbf{E}' and \mathbf{H}' differ from zero. This follows from the general formulas of the transformation of the vectors \mathbf{E} and \mathbf{H} :

$$\mathbf{E}' = \mathbf{E} + \left[\frac{\mathbf{v}}{c} \cdot \mathbf{H} \right] \quad \text{and} \quad \mathbf{H}' = \mathbf{H} - \left[\frac{\mathbf{v}}{c} \cdot \mathbf{E} \right] \quad (8.56)$$

particular cases of which for $\mathbf{E} = 0$ are the equations (8.54) and (8.55) which we have derived. Terms of the order of magnitude v^2/c^2 have been discarded in Eqs. (8.56), as in Eqs. (8.54) and (8.55). The derivation of Eqs. (8.56) can be found in textbooks on the theory of relativity. An illustrative substantiation of them without directly

* In this reasoning, we proceed from the assumption that with an accuracy up to v^2/c^2 the magnitude of the force does not depend on the frame of reference. Indeed, in this approximation, the force is proportional to the acceleration \mathbf{a} it causes, while the magnitude of the acceleration does not depend on the frame of reference [Eq. (8.52)].

reverting to the notions of the theory of relativity is possible only in the particular case of $\mathbf{E} = 0$ considered above.

Since the values of the electric field intensity depend on the frame of reference, then the question appears of invariant, i.e. not depending on the frame of reference, quantitative characteristics of an electromagnetic field. There are two such invariants: it can easily be seen from Eqs. (8.56) that with an accuracy up to terms of the order of magnitude v^2/c^2 , we have

$$\left. \begin{aligned} \mathbf{E}'\mathbf{H}' &= \mathbf{E}\mathbf{H} \\ \text{and} \\ E'^2 - H'^2 &= E^2 - H^2 \end{aligned} \right\} \quad (8.57)$$

If we proceed not from the approximate equations (8.56), but from the corresponding accurate ones taking into account terms of the order of magnitude v^2/c^2 , then we can prove that Eqs. (8.57) strictly hold at all possible values of the relative velocity \mathbf{v} of the frames of reference.

We must also note that generalization of the transformation formulas (8.56) for the case of a material medium gives

$$\left. \begin{aligned} \mathbf{E}' &= \mathbf{E} + \left[\frac{\mathbf{v}}{c} \cdot \mathbf{B} \right] \\ \mathbf{B}' &= \mathbf{B} - \left[\frac{\mathbf{v}}{c} \cdot \mathbf{E} \right] \\ \mathbf{D}' &= \mathbf{D} + \left[\frac{\mathbf{v}}{c} \cdot \mathbf{H} \right] \\ \mathbf{H}' &= \mathbf{H} - \left[\frac{\mathbf{v}}{c} \cdot \mathbf{D} \right] \end{aligned} \right\} \quad (8.58)$$

Thus, the effective values of the electric field intensity and magnetic induction in a moving medium which we introduced in Sec. 8.2, namely,

$$\begin{aligned} \mathbf{E}^* &= \mathbf{E} + \left[\frac{\mathbf{u}}{c} \cdot \mathbf{B} \right] \\ \mathbf{B}^* &= \mathbf{B} - \left[\frac{\mathbf{u}}{c} \cdot \mathbf{E} \right] \end{aligned}$$

[see Eqs. (8.18) and (8.25)] are nothing but the true values of these quantities in the frame of reference S' whose velocity \mathbf{v} equals the velocity \mathbf{u} of the medium, i.e. in the frame of reference in which the medium is at rest.

Formulas (8.58) break up into two groups. The first includes only the values of the vectors \mathbf{E} and \mathbf{B} in the frames S and S' , and the second only the values of the vectors \mathbf{D} and \mathbf{H} . This corresponds to the fact that in its physical meaning the analogue of the electric field intensity \mathbf{E} is the magnetic induction \mathbf{B} , and not the magnetic field intensity \mathbf{H} (see, for example, Sec. 5.3).

3. It remains for us to show that the invariance of the laws of electrodynamics upon a change in the frame of reference follows from the transformation formulas of the field vectors (8.56) and (8.58). Naturally, when the frame of reference is changed, not only the electromagnetic quantities must be changed, but also the space coordinates and time.

Prerelativistic physics was based on the assumption that the readings of time intervals have an absolute nature (when "correct" time-pieces are used) and do not depend on the motion of the frame of reference:

$$t' = t \quad (8.59)$$

so that upon a change in the frame of reference only the space coordinates change:

$$\mathbf{R}' = \mathbf{R} - \mathbf{v}t \quad (8.60)$$

These transformation formulas of the coordinates and time, however, are incompatible with the invariance of the laws of electrodynamics. This can be proved in the simplest way as follows. It can be seen from the laws of electrodynamics that the velocity of propagation of light in a vacuum equals the electromagnetic constant c . If these laws remain invariant upon the transformation of the coordinates, then in any inertial frame of reference the velocity of light in a vacuum must be the same and equal c . From the classical transformation formulas of the coordinates and time (8.59) and (8.60), however, there follows the transformation of the velocity (8.52):

$$\mathbf{u}' = \mathbf{u} - \mathbf{v}$$

which, particularly, must be applicable to the velocity of light too. If in the frame of reference S the velocity of light is c , then in the equivalent frame S' moving relative to S with the velocity \mathbf{v} , for example, along a ray of light, the velocity of this ray ought to be $c' = c - v \neq c$. Thus, the classical transformation formulas of the coordinates and time (8.59) and (8.60) are incompatible with the invariance of the laws of electrodynamics.

It was exactly this circumstance that caused the long and strenuous experimental and theoretical investigations that terminated in the creation of the theory of relativity. A. Einstein, who profoundly analysed the concept of simultaneity, proved the relativity of this

concept and the groundlessness of the assumption expressed by Eq. (8.59) that the interval of time between two events does not depend on the frame of reference.* The constancy of the velocity of light in a vacuum was raised by Einstein to the rank of one of the main postulates of the theory of relativity, so that we can say that the transformation formulas of the coordinates and time are derived in the theory of relativity from the requirement of the invariance of the laws of electrodynamics.

A treatment of the theory of relativity, however, is beyond the scope of this book.

* In the theory of relativity, Eq. (8.59) is replaced by the following one:

$$t' = \frac{t - \frac{\mathbf{v}\mathbf{R}}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

or, with an accuracy up to terms of the order of magnitude v^2/c^2 ,

$$t' = t - \frac{\mathbf{v}\mathbf{R}}{c^2} \quad (8.61)$$

This relationship has the following meaning: if according to measurements in the frame of reference S the time interval between two events A and B is t , while the vector distance between the points where these events occurred is \mathbf{R} , then according to measurements in the frame of reference S' the time interval between these events will be $t' = t - \mathbf{v}\mathbf{R}/c^2$. Particularly, if according to measurements in the frame of reference S events are simultaneous ($t = 0$), but do not occur at the same place ($\mathbf{R} \neq 0$), then in the frame S' they will, in general, not be simultaneous ($t' \neq 0$): the concept of the simultaneity of events occurring at different places is relative.

9

Appendix

Vector Analysis

Vector analysis is set out in this appendix in the volume needed for reading the present book. We did not try to achieve completeness or mathematical strictness in our treatment.

A.1 Vector Algebra

We assume that our reader has a knowledge of vector algebra, and here we shall only remind him of a few of its fundamental definitions and formulas.

The *scalar product* of the vectors

$$\mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k}$$

$$\mathbf{b} = b_x \mathbf{i} + b_y \mathbf{j} + b_z \mathbf{k}$$

where \mathbf{i} , \mathbf{j} , and \mathbf{k} are unit vectors along the coordinate axes x , y , and z , equals

$$\mathbf{a} \cdot \mathbf{b} = (\mathbf{a} \cdot \mathbf{b}) = \mathbf{b} \cdot \mathbf{a} = ab \cos(\mathbf{a}, \mathbf{b}) = a_x b_x + a_y b_y + a_z b_z$$

The *vector product* $[\mathbf{a} \mathbf{b}]$ of the vectors \mathbf{a} and \mathbf{b} is a vector perpendicular to \mathbf{a} and \mathbf{b} and in absolute value equal to the area of a parallelogram constructed on these vectors:

$$|[\mathbf{a} \mathbf{b}]| = ab \sin(\mathbf{a}, \mathbf{b})$$

$$[\mathbf{a} \mathbf{b}] = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = (a_y b_z - a_z b_y) \mathbf{i} + (a_z b_x - a_x b_z) \mathbf{j} + (a_x b_y - a_y b_x) \mathbf{k}$$

$$[\mathbf{a} \mathbf{b}] = -[\mathbf{b} \mathbf{a}]$$

The direction of the vector $[\mathbf{ab}]$ is determined from the requirement that the vectors \mathbf{a} , \mathbf{b} , and $[\mathbf{ab}]$ form a *right-handed* system (Fig. 92).*

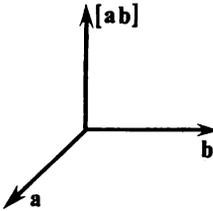


Fig. 92

A *scalar triple product* of the three vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} is a scalar and numerically equals the volume of a parallelepiped constructed on these vectors:

$$\mathbf{a}[\mathbf{bc}] = \mathbf{b}[\mathbf{ca}] = \mathbf{c}[\mathbf{ab}] = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

$$\mathbf{a}[\mathbf{bc}] = -\mathbf{b}[\mathbf{ac}] = -\mathbf{a}[\mathbf{cb}]$$

The *vector triple product* of the vectors \mathbf{a} , \mathbf{b} and \mathbf{c} is

$$[\mathbf{a}[\mathbf{bc}]] = \mathbf{b}(\mathbf{ac}) - \mathbf{c}(\mathbf{ab}) = -[[\mathbf{bc}]\mathbf{a}]$$

* Let us call a combination of three mutually orthogonal unit vectors \mathbf{a} , \mathbf{b} and \mathbf{c} conducted from a common origin an orthogonal triplet. All such triplets are divided into two classes called *right-handed* and *left-handed* triplets. Triplets of the same class can be made to coincide by rotation (so that \mathbf{a} will coincide with \mathbf{a}' , \mathbf{b} with \mathbf{b}' , etc.). Right-handed triplets transform into left-handed ones by *mirror reflection*, i.e. by inverting the direction of all three vectors of the triplet (and also by inverting the direction of one, but not two of these vectors).

It is very significant that there is no invariant geometrical definition of these two classes of triplets: to define, for instance, right-handed triplets, it is necessary to concretely indicate a triplet of this class (by referring to the fingers of a hand, to a screw with a definite thread, etc.). It is obvious that all relationships having a geometrical and physical meaning cannot depend on which of the classes of triplets we agree to call right-handed. This statement is generally formulated as follows: all relationships must be invariant with respect to their mirror reflection or, simply, mirror invariant.

The direction of the vector product of two vectors is determined by the requirement that $[\mathbf{ab}]$ should form a right-handed triplet with \mathbf{a} and \mathbf{b} , therefore the direction of $[\mathbf{ab}]$ is reversed upon mirror reflection. Vectors of this class are called *axial* unlike vectors in the direct meaning of the word whose direction is given regardless of the choice of the coordinates or right-handed triplets and that are called *polar*. The equalities between two polar vectors or between two axial vectors are mirror invariant; the equality between one polar and one axial vector is not invariant and cannot have a physical meaning.

It is easy to see that the gradient of a scalar is a polar vector; the curl of a polar vector is an axial vector. Particularly, the intensity of an electric field is a polar vector, while the intensity of a magnetic field is an axial vector.

If vectors are functions of a scalar variable t , then we can differentiate the vectors with respect to this variable while observing the usual conditions. Here the following relationships hold:

$$\frac{d}{dt}(\mathbf{a} + \mathbf{b}) = \frac{d\mathbf{a}}{dt} + \frac{d\mathbf{b}}{dt}$$

$$\frac{d}{dt}(\varphi\mathbf{a}) = \varphi \frac{d\mathbf{a}}{dt} + \frac{d\varphi}{dt} \mathbf{a}$$

$$\frac{d(\mathbf{a}\mathbf{b})}{dt} = \left(\frac{d\mathbf{a}}{dt} \mathbf{b} \right) + \left(\mathbf{a} \frac{d\mathbf{b}}{dt} \right), \text{ etc.}$$

A.2 Vector and Scalar Fields. Gradient

1. By a *vector* or a *scalar field* is meant a region of space to each point of which there is related the value of a certain vector or scalar. Since each point of a field is determined by its radius-vector \mathbf{R} , the setting of a vector or scalar field is equivalent to setting a vector function $\mathbf{a}(\mathbf{R})$ or, correspondingly, a scalar function $\varphi(\mathbf{R})$. The functions $\mathbf{a}(\mathbf{R})$ and $\varphi(\mathbf{R})$ can naturally also depend, apart from \mathbf{R} , on scalar arguments such as time. We shall consider that the functions $\mathbf{a}(\mathbf{R})$ and $\varphi(\mathbf{R})$ are continuous and differentiable with respect to all their arguments.

Let us consider the scalar field of the function $\varphi(\mathbf{R}) = \varphi(x, y, z)$. Such fields include, for example, the field of the temperature of a non-uniformly heated body ($\varphi = T$), the field of the density of a non-homogeneous body ($\varphi = \tau$), and the field of the electrostatic potential.

2. Assume that the scalar φ has the value φ_0 at the point P_0 and that upon the displacement $d\mathbf{s}$ in the direction of the vector \mathbf{s} we arrive from the point P_0 at the point P where the scalar φ has the value φ_s . The increment of φ upon this displacement equals $d\varphi = \varphi_s - \varphi_0$. The limit of the ratio of this increment $d\varphi$ to the absolute value of the displacement ds is designated by $\partial\varphi/\partial s$ and is called *the derivative of the scalar φ at the point P_0 in the direction of \mathbf{s}* :

$$\frac{\partial\varphi}{\partial s} = \lim_{ds \rightarrow 0} \frac{\varphi_s - \varphi_0}{ds} \quad (\text{A.1})$$

It is evident that the value of this derivative appreciably depends on the choice of the direction of \mathbf{s} and that it should never be confused with the conventional partial derivative with respect to the scalar parameter s .

3. To study how the derivative $\partial\varphi/\partial s$ depends on the direction of differentiation of \mathbf{s} , let us consider the points of the field at which φ

has an identical value, equal, for instance, to φ_0 . The combination of these points, generally speaking, forms a surface called a *level* or an *equipotential surface*. (Apart from the regions of space in which

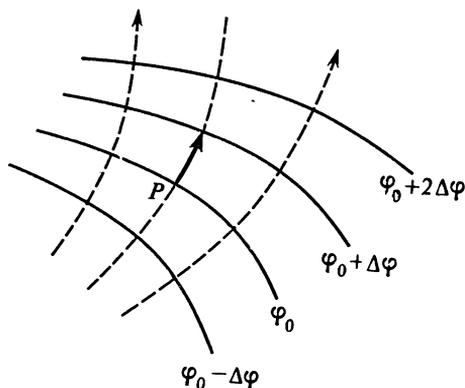


Fig. 93

$\varphi = \text{const}$, an exception here may be the isolated points of the field where the value of φ reaches a maximum or a minimum.) Analytically, such a surface is characterized by the equation

$$\varphi(x, y, z) = \varphi_0$$

Figure 93 shows a section by the plane of the drawing of a number of level or equipotential surfaces corresponding to values of the scalar φ equal to $\varphi_0, \varphi_0 \pm \Delta\varphi, \varphi_0 \pm 2\Delta\varphi$, etc. In the field of a point charge or a charged sphere the level surfaces of the electrostatic potential are concentric spheres, in the field of a charged infinite cylinder—coaxial cylinders, etc. In general, however, in more complicated cases, consecutive, equipotential surfaces differ not only in their position and dimensions, but also in their shape. At any rate, however, *the surface of every conductor is an equipotential surface* because the potential of a conductor in an electrostatic field is constant over its entire length (Sec. 1.9).

Let \mathbf{n} stand for a normal to the level surface $\varphi = \varphi_0$ running in the direction of *increasing* φ and let us show that knowing the derivative $\partial\varphi/\partial n$ in the direction of this normal, we can determine the value of the derivative of the scalar φ in any direction \mathbf{s} .

Let us assume that the level surface passing through the point P_s lying in the direction of \mathbf{s} intersects the normal \mathbf{n} (or its continuation in the opposite direction) at the point P_n (Fig. 94). The value of φ at the point P_n equals the value of φ at the point P_s (i.e. $\varphi_n = \varphi_s$), and

$$P_0 P_s = \frac{P_0 P_n}{\cos(\mathbf{s}, \mathbf{n})}$$

Consequently,

$$\begin{aligned} \left(\frac{\partial \varphi}{\partial s}\right)_0 &= \lim_{P_0 P_s \rightarrow 0} \frac{\varphi_s - \varphi_0}{P_0 P_s} = \cos(\mathbf{s}, \mathbf{n}) \lim_{P_0 P_n \rightarrow 0} \frac{\varphi_n - \varphi_0}{P_0 P_n} = \\ &= \left(\frac{\partial \varphi}{\partial n}\right)_0 \cos(\mathbf{s}, \mathbf{n}) \end{aligned}$$

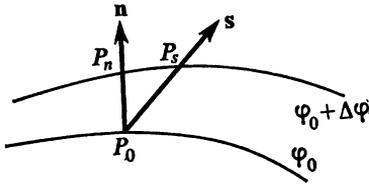


Fig. 94

Thus,

$$\frac{\partial \varphi}{\partial s} = \frac{\partial \varphi}{\partial n} \cos(\mathbf{s}, \mathbf{n}) \quad (\text{A.2})$$

The vector numerically equal to $\partial \varphi / \partial n$ and directed along a normal to the level surface in the direction of increasing φ is called the *gradient of the scalar φ* :

$$\text{grad } \varphi = \frac{\partial \varphi}{\partial n} \mathbf{n} \quad (\text{A.3})$$

Hence, Eq. (A.2) can be written as follows:

$$\frac{\partial \varphi}{\partial s} = |\text{grad } \varphi| \cdot \cos(\mathbf{s}, \mathbf{n}) = \text{grad}_s \varphi \quad (\text{A.4})$$

Therefore, the derivative of φ with respect to the direction of \mathbf{s} equals the projection of the vector of the gradient of φ onto the direction of \mathbf{s} . If, particularly, we introduce the system of Cartesian coordinates x , y , and z whose axes are parallel to the unit vectors \mathbf{i} , \mathbf{j} , and \mathbf{k} , then according to Eq. (A.4), we get

$$\text{grad}_x \varphi = \frac{\partial \varphi}{\partial x}, \quad \text{grad}_y \varphi = \frac{\partial \varphi}{\partial y}, \quad \text{grad}_z \varphi = \frac{\partial \varphi}{\partial z} \quad (\text{A.5})$$

i.e.

$$\left. \begin{aligned} \text{grad } \varphi &= \mathbf{i} \frac{\partial \varphi}{\partial x} + \mathbf{j} \frac{\partial \varphi}{\partial y} + \mathbf{k} \frac{\partial \varphi}{\partial z} \\ |\text{grad } \varphi| &= \sqrt{\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 + \left(\frac{\partial \varphi}{\partial z}\right)^2} \end{aligned} \right\} \quad (\text{A.6})$$

It follows from Eq. (A.4), and also directly from Fig. 94 that the direction of the gradient \mathbf{n} is that of the most rapid growth of the scalar φ , while the direction ($-\mathbf{n}$) is that of the most rapid diminishing of φ . In directions perpendicular to \mathbf{n} , i.e. tangent to the level surface, the value of φ does not change at all ($\partial\varphi/\partial s = 0$).

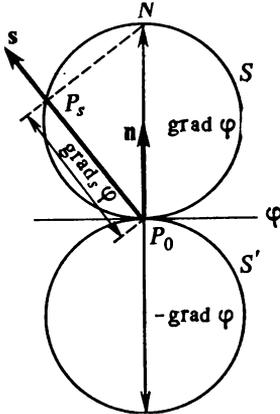


Fig. 95

To illustrate the dependence of the value of the derivatives of φ on the direction, let us conduct from the given point P_0 two equal and opposite vectors $\text{grad } \varphi$ and $-\text{grad } \varphi$ and circumscribe about each of them like about a diameter the spherical surfaces S and S' (Fig. 95). Hence, the absolute value of the derivative $\partial\varphi/\partial s$ at the point P_0 with respect to the arbitrary direction \mathbf{s} will be depicted by the portion P_0P_s of the ray conducted from P_0 in the direction of \mathbf{s} because the angle P_0P_sN equals 90 degrees, and

$$P_0P_s = P_0N \cos (\mathbf{s}, \mathbf{n}) = \text{grad } \varphi \cdot \cos (\mathbf{s}, \mathbf{n})$$

A similar relationship also holds for the case when \mathbf{s} runs in the direction of the spherical surface S' . The surface tangent to the spheres S and S' at the point P_0 is evidently a level surface.

4. Thus, if we know the field of the scalar φ , then at each point of this field we can determine the vector $\text{grad } \varphi$ perpendicular to the level surfaces of this scalar. If we conduct a system of *orthogonal trajectories* of the level surface, i.e. a system of lines perpendicular to these surfaces (in Fig. 93 these lines are dashed), then at each point of the field the *direction* of the gradient will coincide with that of these lines. Hence, the orthogonal trajectories of level surfaces are called *gradient lines*.

If we conduct the level surfaces so that the value of φ on consecutive surfaces grows according to an arithmetic progression, i.e. is $\varphi_0, \varphi_0 \pm \Delta\varphi, \varphi_0 \pm 2\Delta\varphi$, etc. (see Fig. 93), then the distances between adjacent level surfaces with a sufficiently small value of $\Delta\varphi$ will be inversely proportional to the absolute values of the gradient. Indeed, if

Δn stands for the distance between adjacent level surfaces measured along a normal, then from the approximate relationship

$$\Delta\varphi = \frac{\partial\varphi}{\partial n} \Delta n = \text{grad } \varphi \Delta n$$

at constant $\Delta\varphi$ it follows that

$$\text{grad } \varphi = \frac{\text{const}}{\Delta n}$$

Therefore, when level surfaces are drawn as indicated above, the density of their arrangement gives an approximate idea of the absolute value of the gradient.

It should also be noted that if the scalar φ is expressed as a function of a different scalar ψ that is a position function [$\varphi = f(\psi)$], then upon any choice of the direction of differentiation of s we have

$$\text{grad}_s \varphi = \frac{\partial\varphi}{\partial s} = \frac{\partial\varphi}{\partial\psi} \frac{\partial\psi}{\partial s} = \frac{\partial\varphi}{\partial\psi} = \text{grad}_s \psi$$

because

$$\text{grad } \varphi = \frac{\partial\varphi(\psi)}{\partial\psi} \text{grad } \psi \quad (\text{A.7})$$

which follows from the formula for the conventional differentiation of a function of a function.

Example 1. *The gradient of the absolute value of the radius-vector \mathbf{R} .* We must note first of all that the absolute value of the radius-vector \mathbf{R} is a scalar function of the position of *two points*: the origin O of the radius-vector and its end point P (Fig. 96). We

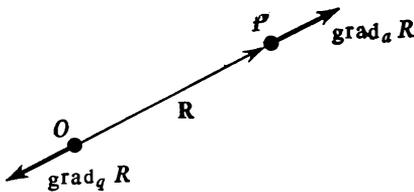


Fig. 96

shall call the first of these points the *source point*, and the second the *observation point* because we often have to do with radius-vectors conducted from the sources of a field (for instance electric charges) to the “observation point” at which the value of the field potential or intensity is being determined.

When determining the value of $\text{grad } R$, two cases must be distinguished depending on the conditions of the problem: (1) the source

point O is fixed, and \mathbf{R} is considered as the position function of the observation point P , and (2) the point P is fixed, and \mathbf{R} is considered as the position function of the source point O . We shall denote the value of $\text{grad } R$ corresponding to the first case by $\text{grad}_a R$, and to the second one by $\text{grad}_q R$.

Let us first determine $\text{grad}_a R$, i.e. assume that the source point O is fixed. The direction of $\text{grad}_a R$, i.e. the direction of the most rapid growth in the distance R upon possible displacements of the point P , obviously coincides with the direction of the radius-vector \mathbf{R} from O to P . The absolute value of the derivatives of R with respect to this direction evidently equals unity because upon the displacement of the point P in the direction of \mathbf{R} over the distance ds , the value of R grows by the same amount ds . Hence, $\text{grad}_a R$ is a unit vector directed along \mathbf{R} , i.e.

$$\text{grad}_a R = \frac{\mathbf{R}}{R}$$

As regards $\text{grad}_q R$, it should be directed oppositely to \mathbf{R} because the distance R grows the most rapidly when the point O moves in the direction opposite to P (Fig. 96). The absolute value of $\text{grad}_q R$, however, also evidently equals unity so that

$$\text{grad}_q R = -\frac{\mathbf{R}}{R} = -\text{grad}_a R \quad (\text{A.8})$$

Having thus determined $\text{grad } R$, we can use Eq. (A.7) to determine the gradient of any scalar function $f(R)$ of the numerical value of R :

$$\text{grad } f(R) = \frac{\partial f(R)}{\partial R} \text{grad } R \quad (\text{A.9})$$

The absolute value of this vector is

$$|\text{grad } f(R)| = \left| \frac{\partial f(R)}{\partial R} \right|$$

Particularly,

$$\text{grad}_q \left(\frac{1}{R} \right) = \frac{\mathbf{R}}{R^3} = -\text{grad}_a \left(\frac{1}{R} \right) \quad (\text{A.10})$$

We shall let our reader prove formula (A.8) as an exercise by directly calculating the components of $\text{grad } R$ in Cartesian coordinates after first expressing R as a function of the coordinates x, y, z and x', y', z' of the points O and P .

Example 2. Prove that if \mathbf{b} is a vector constant in value and direction, then

$$\text{grad}_a(\mathbf{bR}) = \mathbf{b} \quad (\text{A.11})$$

The vector \mathbf{R} has the components $x' - x$, $y' - y$, and $z' - z$. Hence, $\mathbf{bR} = b_x(x' - x) + b_y(y' - y) + b_z(z' - z)$. The component of the vector $\text{grad}_a(\mathbf{bR})$ along the x -axis is $\partial(\mathbf{bR})/\partial x' = b_x$. The two other components are b_y and b_z , respectively, whence we get Eq. (A.11).

A.3 Vector Flux Through a Surface

If the field of an arbitrary, but differentiable scalar $\varphi(\mathbf{R})$ is given, then the field of the derivatives of this scalar with respect to an arbitrary direction is thus also given. We have seen that the field of the vector $\text{grad } \varphi$ is an invariant (not depending on the choice of the system of coordinates) characteristic of this field of derivatives. Now our task is to determine the invariant characteristics of the field of spatial derivatives of the arbitrary vector $\mathbf{a}(\mathbf{R})$. We naturally arrive at these characteristics by considering the surface and curvilinear integrals of the vector \mathbf{a} . We shall begin with studying the surface integrals.

Let us mentally separate in the field of an arbitrary vector an infinitely small flat area element dS , i.e. an area element that is so small that at all its points the vector \mathbf{a} with a given degree of accuracy remains constant in magnitude and direction. Let us conduct a normal to this element and consider one of the directions of this normal \mathbf{n} to be positive, or outward, and the other negative, or inward. If we are given the direction of circumvention of the contour of the element, then we shall select the direction of the positive normal so that

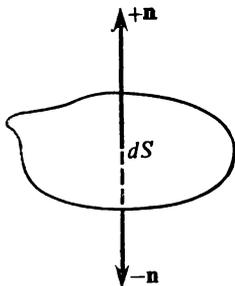


Fig. 97

it forms a *right-handed* system together with the contour. This means that if we turn a screw with a right-handed thread in the direction given for the circumvention of the contour, then the point of the screw will move along the *positive* normal (Fig. 97). Conversely, if we are

given the direction of the outward normal, then we shall correspondingly choose the direction of positive circumvention of the contour of the area element.

Finally, if the direction of circumvention of the contour and the direction of a normal to its plane are given independently of each other, then we shall say for brevity that the direction of circumvention and that of the normal form a *right-handed* system if they comply with the above-mentioned condition, and a *left-handed* system if they do not comply with it.

We shall characterize the direction of the normal by the unit vector \mathbf{n} ($n = 1$) coinciding with it.

By the *flux of the vector* \mathbf{a} through an infinitely small area element dS is meant the quantity

$$dN = \mathbf{a}\mathbf{n} dS = a \cos(\mathbf{a}, \mathbf{n}) dS = a_n dS \quad (\text{A.12})$$

where \mathbf{a} = value of the vector on the area element dS

a_n = its component along the direction of \mathbf{n} .

We have chosen an infinitely small area element dS so as to ensure that the vector \mathbf{a} has one definite value on this element.

To determine the vector flux through a surface having finite dimensions, we must divide it into infinitely small area elements dS so that not only the vector \mathbf{a} would remain constant on each element, but also the area elements themselves could be considered flat (Fig. 98).

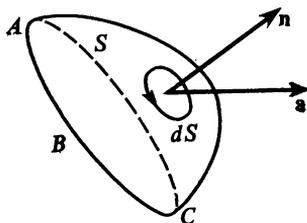


Fig. 98

Let us call one of the sides of the surface S the internal one and the other the external one, and correspondingly choose the direction of the outward normals to each of the elements dS . The flux N of the vector \mathbf{a} through the surface S is the algebraic sum of the fluxes $a_n dS$ through the separate elements of this surface. This summation is identical with the operation of finding the definite integral

$$N = \iint_S a_n dS$$

and is called *integration over the surface* S . It is designated by a double integral because a surface has two dimensions. For simplification, however, in this book we have designated double integrals, like single

ones, with a single integral sign:

$$N = \int_s a_n dS \quad (\text{A.13})$$

We remind our reader that in all surface (and only in surface) integrals we have designated the element of integration by dS .

The name vector flux given to the quantity N has been taken from hydrodynamics. The latter studies the vector field of the velocity of a liquid: at each given moment a definite value of the velocity vector \mathbf{v} is connected with each point of the space filled with the liquid, namely, the value of this velocity which the element of the liquid at this point has. The flux of the liquid velocity vector through the element of surface area dS , i.e.

$$dN = v_n dS$$

is nothing but the volume of the liquid flowing through this element during a unit time in the direction of the outward normal to dS . Indeed, during a unit time, each element of the liquid moves over the distance v ; hence, all those and only those particles of the liquid which at the beginning of the time interval being considered occupied the cylindrical volume with the base dS and the generatrices \mathbf{v} will pass through the area element dS (Fig. 99). The volume of this cylinder

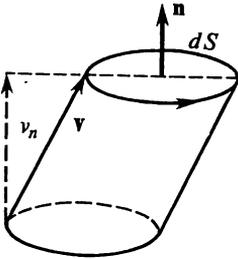


Fig. 99

is $v_n dS$ if $v_n > 0$. When the vectors \mathbf{v} and \mathbf{n} form an obtuse angle, then $v_n < 0$, and the liquid flux is negative. This means that the liquid flows through dS in a direction opposite to the outward normal \mathbf{n} .

The flow of a liquid through a finite surface S obviously equals the flux of the velocity vector \mathbf{v} through this surface:

$$N = \int_s v_n dS$$

* We shall assume for simplicity that the velocity \mathbf{v} does not depend on the time and is constant over the entire length of the cylinder. Otherwise the above reasoning must be applied not to a unit time, but to the element of time dt , when the flux through the area element dS will be $v_n dS dt$.

It often becomes necessary to calculate the vector flux through closed surfaces (the surface of a sphere, cube, etc.). When integrating over a *closed* surface we shall note this circumstance by using a circle on the integral sign so that, for example, the flow of a liquid through the closed surface S will be written as follows:

$$N = \oint_S v_n dS$$

This flux is evidently equal to the quantity of liquid flowing out in a unit time from the volume confined by the closed surface S . If $N < 0$, this means that more liquid flows into the surface than out of it.

A.4 Gauss's Theorem. Divergence

1. The surface integral $\oint a_n dS$ can be transformed into a volume one.

This is the content of one of the most important theorems of vector analysis—*Gauss's theorem*.

Let us first consider the flux dN of an arbitrary, but differentiable vector \mathbf{a} through the surface of an infinitely small parallelepiped and for convenience of calculations so choose the direction of the coordinate axes x , y , and z that they coincide with the edges dx , dy , and dz of this parallelepiped (Fig. 100). The integral

$$dN = \oint a_n dS$$

consists in this case of the sum of six integrals over each of the faces of the parallelepiped. Using the mean value theorem known from integral calculus, we can represent each of these six integrals as the product of the area of the relevant face and a certain mean value of the normal component of the vector \mathbf{a} on the given face.

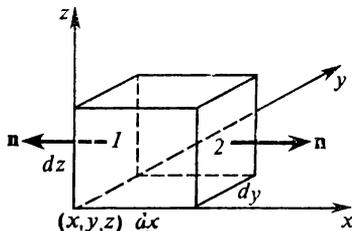


Fig. 100

Let us first consider the flux of the vector \mathbf{a} through the two parallel faces 1 and 2 perpendicular to the x -axis. The flux through the front face 2 is

$$a_{2n}(x + dx, \bar{y}, \bar{z}) dS = a_{2x}(x + dx, \bar{y}, \bar{z}) dy dz$$

where \bar{y} and \bar{z} are certain mean values of the coordinates y and z on the face 2, and \mathbf{a}_2 is the value of the vector \mathbf{a} on the face 2. The flux through the rear face 1 is

$$a_{1n} dS = -a_{1x} dy dz \quad [\text{here } a_{1n} = a_{1n}(x, \bar{y}, \bar{z})]$$

where \mathbf{a}_1 is the value of the vector \mathbf{a} on the face 1 because the outward normal to this face is directly opposite to the x -axis. Hence, the total flux through the faces 1 and 2 is

$$(a_{2x} - a_{1x}) dy dz$$

The difference $a_{2x} - a_{1x}$ is the increment of the vector component a_x when the coordinate x changes by the distance dx between the faces 1 and 2. With an accuracy up to infinitesimals of the second order, this increment is

$$a_{2x}(x + dx, y, z) - a_{1x}(x, y, z) = \frac{\partial a_x}{\partial x} dx$$

where owing to the infinitely small size of the parallelepiped by $\partial a_x / \partial x$ we can understand the value of this derivative at any point of the parallelepiped. Thus, the total flux through both faces perpendicular to the x -axis is

$$\frac{\partial a_x}{\partial x} dx dy dz$$

For the fluxes through the pairs of faces perpendicular to the y - and z -axes, we get, similarly,

$$\frac{\partial a_y}{\partial y} dy dx dz \quad \text{and} \quad \frac{\partial a_z}{\partial z} dz dx dy$$

Summating the obtained expressions, we get the total flux of the vector \mathbf{a} through all the six faces of an elementary parallelepiped:

$$dN = \oint a_n dS = \left(\frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} \right) dx dy dz \quad (\text{A.14})$$

It is customary practice to denote the sum of the derivatives of the vector \mathbf{a} with respect to the coordinate axes by the symbol $\text{div } \mathbf{a}$ for brevity:

$$\text{div } \mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} \quad (\text{A.15})$$

(read “divergence of \mathbf{a} ”, for its meaning see below). If, in addition, we introduce for the infinitely small element of volume the designation dV :

$$dV = dx dy dz$$

then the expression for the flux dN acquires the form

$$dN = \operatorname{div} \mathbf{a} dV \quad (\text{A.16})$$

2. It is not difficult to generalize this formula expressing the flux of the vector \mathbf{a} through the surface of an *infinitely small* parallelepiped for a surface of an arbitrary shape and dimensions. Let us consider an arbitrary closed surface S . We shall divide the volume V it confines by a system of mutually perpendicular planes into a set of infinitely small cubic elements. Naturally, the extreme elements of the volume adjacent to the surface S will, generally speaking, not have a cubic shape. By further division, however, the faces of the extreme cubes can be made to coincide with the given surface S with any degree of accuracy. Let us use Eq. (A.16) to calculate the flux of the vector \mathbf{a} through the surface of each cube inside S and summate the expressions obtained:

$$\sum dN = \sum \operatorname{div} \mathbf{a} dV = \iiint_V \operatorname{div} \mathbf{a} dV$$

In this equation, the triple integral signifies that summation of the integrand must be conducted over all the elements of the three-dimensional volume V confined within the surface S . Throughout this book, however, we have designated *integrals of any multiplicity with a single integral sign*. Different integrals were distinguished by using different symbols for the elements of integration:

dV stands for a volume element (triple integral);

dS stands for a surface area element (double integral);

ds stands for the element of a line (single integral).

The faces of all the elementary cubes whose combination forms the volume V can be divided into two classes—external faces coinciding with elements of area of the surface S , and internal faces forming the boundaries between adjacent cubes. It is clear that the flux of the vector \mathbf{a} through each *internal face* will enter twice in the sum $\sum dN$: when counting the flux through the surface of the cube at one side of this face and when counting the flux through the surface of the cube at the other side of it. Since the normal to the face that is outward relative to the first cube is opposite to the normal to the same face that is outward relative to the second cube, then the two fluxes through this face will have opposite signs. Consequently, all the terms of the sum $\sum dN$ relating to the internal faces will cancel, and this sum will consist of the sum of the fluxes of the vector \mathbf{a} through only the external faces of the cubes that coincide with the elements of area of the surface S . (This statement requires, in essence, a stricter mathematical proof which we shall not stop to consider

here.) Thus, $\sum dN$ equals the flux N of the vector \mathbf{a} through the given surface area S and, consequently,

$$N = \oint_S \mathbf{a}_n dS = \int_V \operatorname{div} \mathbf{a} dV \quad (\text{A.17})$$

This expression is the mathematical formulation of *Gauss's theorem*: the flux of the vector \mathbf{a} , which is a continuous function of a point, through an arbitrary closed surface S equals the integral of the divergence of this vector over the volume V confined by this surface.

3. If the surface S is so small that $\operatorname{div} \mathbf{a}$ can be considered constant at all the points inside it, then in Eq. (A.17) $\operatorname{div} \mathbf{a}$ can be put outside the integral. Hence, the flux dN through an infinitely small closed surface S of an *arbitrary* shape is expressed by the same formula (A.16):

$$dN = \oint \mathbf{a}_n dS = \operatorname{div} \mathbf{a} dV$$

as the flux through the surface of an elementary parallelepiped. Since this formula holds only in the limiting case of an infinitely small surface, then the following form will be more correct for it

$$\operatorname{div} \mathbf{a} = \lim_{\Delta V \rightarrow 0} \frac{\oint \mathbf{a}_n dS}{\Delta V} \quad (\text{A.18})$$

It is most correct to consider this formula as the *definition of divergence*: the divergence of the vector \mathbf{a} at a given point of a field is the limit to which the ratio between the flux of the vector \mathbf{a} through an arbitrary surface surrounding this point and the volume ΔV confined by this surface tends (with $\Delta V \rightarrow 0$). It follows from this definition of divergence that its value does not at all depend on the choice of the system of coordinates, i.e. that the divergence of a vector is a true scalar. On the basis of Eq. (A.18) and using Eq. (A.16), for the particular case of Cartesian coordinates, we shall obviously again arrive at Eq. (A.14).

We must note in conclusion that in hydrodynamics the divergence of the velocity \mathbf{v} of a liquid has a direct physical meaning. Indeed, at each point of a liquid

$$\operatorname{div} \mathbf{v} = \lim_{\Delta V \rightarrow 0} \frac{\oint v_n dS}{\Delta V}$$

equals the amount of liquid flowing out of the volume element dV surrounding the point being considered calculated per unit volume.

The name “divergence” was chosen for this quantity exactly because the liquid spreads out or diverges from those and only from those points or portions of the space it occupies at which $\text{div } \mathbf{v} > 0$. It is evident that the sources of the liquid should be arranged at these points. By analogy, the points of the field of an arbitrary vector \mathbf{a} at which $\text{div } \mathbf{a} \neq 0$ are generally called the *sources* of this field. The numerical value of $\text{div } \mathbf{a}$ is called the *source strength* of the field; depending on the sign of the divergence, the source strength can be either positive or negative. Sometimes the negative sources of a field are called the *sinks* of the field. Vector fields for which $\text{div } \mathbf{a} = 0$ are called source-free or *solenoidal*.

Example 1. Determine the divergence of the vector \mathbf{a} which at each point of a field is directed parallel or antiparallel to the radius-vector \mathbf{R} conducted to this point from the point O .

Let us apply for this purpose Eq. (A.18) to the volume element dV cut out from the spherical layer between the spheres having the radii R and $R + dR$ by a cone with its centre at O that intersects these spheres along the arcs of meridians α and $\alpha + d\alpha$ and the arcs of the parallel circles θ and $\theta + d\theta$ (Fig. 101). Since we are

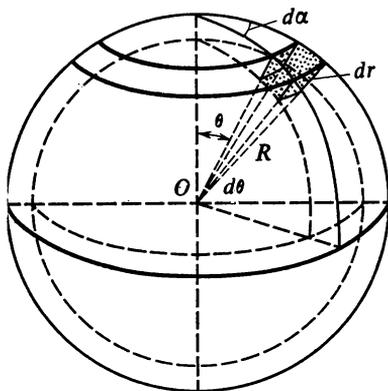


Fig. 101

given that the vector \mathbf{a} is parallel to \mathbf{R} , then its flux through the side (formed by the cone) surface of the volume dV equals zero. Further, since the area element dS of the surface of the sphere having the radius R cut out by the cone is

$$dS = R^2 \sin \theta \, d\theta \, d\alpha$$

then the flux of the vector \mathbf{a} through it is

$$a_n dS = -a_R R^2 \sin \theta \, d\theta \, d\alpha$$

where a_R is the component of \mathbf{a} in the direction of \mathbf{R} ($a_R = \pm a$) because the outward normal to dS is directed oppositely to the radius-vector \mathbf{R} . The flux through an area element of the surface

of the sphere having the radius $R + dR$, up to a second-order infinitesimal, evidently equals

$$a_R R^2 \sin \theta \, d\theta \, d\alpha + \frac{\partial}{\partial R} (a_R R^2 \sin \theta \, d\theta \, d\alpha) \, dR$$

Thus, the total flux is

$$\oint a_n \, dS = \frac{\partial}{\partial R} (a_R R^2) \sin \theta \, d\theta \, d\alpha \, dR$$

On the other hand,

$$dV = dS \cdot dR = R^2 \sin \theta \, d\theta \, d\alpha \, dR$$

so that

$$\begin{aligned} \operatorname{div} \mathbf{a} &= \lim_{dV \rightarrow 0} \frac{\oint a_n \, dS}{dV} = \frac{1}{R^2} \frac{\partial}{\partial R} (R^2 a_R) \\ &= \frac{\partial a_R}{\partial R} + \frac{2}{R} a_R \end{aligned} \quad (\text{A.19})$$

We invite our reader to show that for an arbitrary vector \mathbf{a} the expression for the divergence in spherical coordinates acquires the form

$$\begin{aligned} \operatorname{div} \mathbf{a} &= \frac{1}{R^2} \frac{\partial}{\partial R} (R^2 a_R) + \frac{1}{R \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta a_\theta) + \\ &+ \frac{1}{R \sin \theta} \frac{\partial a_\alpha}{\partial \alpha} \end{aligned} \quad (\text{A.20})$$

where a_R , a_θ , and a_α are the components of the vector in the direction of growth of the coordinates R , θ , and α .

Example 2. Determine the divergence of the gradient of an arbitrary function $f(R)$.

We shall consider the radius-vector \mathbf{R} as a function of the observation point (see p. 615). Denoting the vector $\operatorname{grad}_a f(R)$ by the letter \mathbf{a} , we get on the basis of Eqs. (A.8) and (A.9)

$$\mathbf{a} = \operatorname{grad}_a f(R) = \frac{\partial f}{\partial R} \frac{\mathbf{R}}{R} \quad \text{and} \quad a_R = \frac{\partial f}{\partial R}$$

Since the remaining components of \mathbf{a} equal zero, then on the basis of Eq. (A.19) we have

$$\operatorname{div}_a \operatorname{grad}_a f(R) = \frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 \frac{\partial f}{\partial R} \right) = \frac{\partial^2 f}{\partial R^2} + \frac{2}{R} \frac{\partial f}{\partial R} \quad (\text{A.21})$$

The divergence of $\text{grad}_q f(R)$ has the same value because as we can see by calculations in Cartesian coordinates,

$$\text{div}_q \text{grad}_q f(R) = -\text{div}_a [-\text{grad}_a f(R)] = \text{div}_a \text{grad}_a f(R)$$

Example 3. Determine the expression for the divergence of an arbitrary vector \mathbf{a} in the cylindrical system of coordinates $z, r,$ and α (Fig. 102).

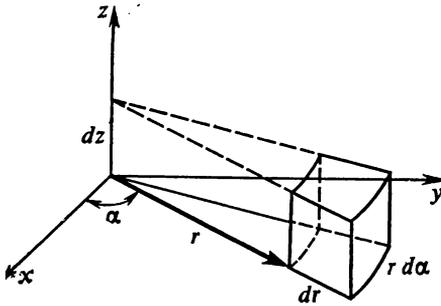


Fig. 102

Let the components of the vector \mathbf{a} in the direction of the growth of the coordinates $z, r,$ and α be $a_z, a_r,$ and $a_\alpha,$ respectively. Let us apply Eq. (A.18) to the volume dV confined between two cylindrical surfaces having the radii r and $r + dr,$ two meridian planes $\alpha = \alpha_1$ and $\alpha = \alpha_1 + d\alpha,$ and two planes perpendicular to the z -axis: $z = z_1$ and $z = z_1 + dz$ (see Fig. 102). The flux of the vector \mathbf{a} through an area element of the cylindrical surface having the radius r is $-a_r r d\alpha dz$; for a cylindrical surface having the radius $r + dr$ it is

$$a_r r d\alpha dz + \frac{\partial}{\partial r} (a_r r d\alpha dz) dr$$

(with an accuracy up to second-order infinitesimals), while the sum of the fluxes through both cylindrical surfaces is

$$\frac{\partial}{\partial r} (ra_r) d\alpha dz dr$$

Calculating in a similar way the flux of the vector through the remaining area elements of the surface of the volume $dV,$ we get

$$\oint a_n dS = \left\{ \frac{\partial}{\partial r} (ra_r) + r \frac{\partial a_z}{\partial z} + \frac{\partial a_\alpha}{\partial \alpha} \right\} dr dz d\alpha$$

Since $dV = r d\alpha dz dr,$ then Eq. (A.18) leads to the result

$$\text{div } \mathbf{a} = \frac{\partial a_z}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} (ra_r) + \frac{1}{r} \cdot \frac{\partial a_\alpha}{\partial \alpha} \tag{A.22}$$

A.5 Circulation of a Vector. Curl of a Vector. Stokes's Theorem

The transformation of the integral of a vector over a closed surface into an integral over a volume led us to the concept of the divergence of a vector. Let us now consider the integral of a vector over a closed curve.

Assume that a curve L is given in the field of the vector $\mathbf{a}(\mathbf{R})$ and that of the two possible directions of motion along this curve the one to be considered positive is also given. We divide the curve L into infinitely small elements ds whose direction coincides with that of positive motion along the curve, and multiply each element ds scalarly by the value of the vector \mathbf{a} at the corresponding point of the field. The limit of the sum of these products $\mathbf{a} ds = a_s ds$ when $ds \rightarrow 0$ covering all the elements of the curve is called the *line integral* of the vector \mathbf{a} along the curve L :

$$\int_L \mathbf{a} ds = \int_L a_s ds$$

If the curve L is *closed*, which is indicated by a circle on the integral sign, then the line integral of the vector \mathbf{a} along it is called the *rotation* or *circulation* of \mathbf{a} along L :

$$C(\mathbf{a}) = \oint_L \mathbf{a} ds = \oint_L a_s ds \quad (\text{A.23})$$

Let us assume that the contour L is one of a plane rectangle $ABCD$ and choose the x - and y -axes of Cartesian coordinates so that they are parallel to the sides of the rectangle and intersect at its centre

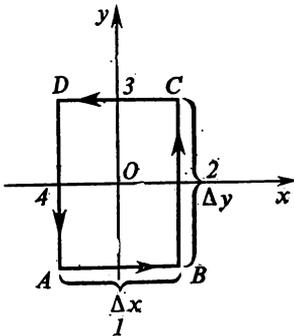


Fig. 103

(Fig. 103). Let the sides of the rectangle be Δx and Δy , respectively. If we select the positive direction around the contour so that the

corresponding positive normal to the area of the rectangle is directed along the z -axis (Fig. 103), then

$$C = \oint a_s ds = \int_A^B a_x dx + \int_B^C a_y dy + \int_C^D a_x dx + \int_D^A a_y dy$$

Using the mean value theorem known from integral calculus, we get (when \mathbf{n} is parallel to the z -axis)

$$C = \oint a_s ds = a'_x \Delta x + a''_y \Delta y - a'''_x \Delta x - a''''_y \Delta y$$

where a'_x, a''_y , etc. are the mean values of the components a_x and a_y on the first, second, etc. sides of the rectangle. The negative sign, for example, of the last term of the sum is explained by the fact that integration over the side AD is performed in the direction of diminishing of the coordinate y .

Let us now make the length of the sides of a rectangle tend to zero. The mean value of the component a_y over the segment BC at a distance of Δx from the segment AD in the direction of the x -axis will therefore differ from the value of a_y over the segment AD by the amount $(\partial a_y / \partial x) \Delta x$ with an accuracy up to second-order infinitesimals:

$$a''_y = a''''_y + \frac{\partial a_y}{\partial x} \Delta x$$

Correspondingly,

$$a'''_x = a'_x + \frac{\partial a_x}{\partial y} \Delta y$$

because CD is at the distance Δy from AB in the direction of the y -axis. In the limit with infinitely small dimensions of the rectangle, we can understand $\partial a_y / \partial x$ and $\partial a_x / \partial y$ to be the values of these quantities at the centre O of the rectangle. Introducing these expressions into the preceding equation, we get (when \mathbf{n} is parallel to the z -axis)

$$\begin{aligned} dC &= \oint a_s ds = (a''_y - a''''_y) \Delta y - (a'''_x - a'_x) \Delta x = \\ &= \frac{\partial a_y}{\partial x} \Delta x \Delta y - \frac{\partial a_x}{\partial y} \Delta x \Delta y \end{aligned}$$

where we have replaced C with dC to note that this relationship holds only for an infinitely small rectangle. (Naturally, dC is not at

all the total differential of C .) Denoting, finally, the area of the rectangle $\Delta x \Delta y$ by dS , we get (when \mathbf{n} is parallel to the z -axis)

$$dC = \oint a_s dS = \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y} \right) dS \quad (\text{A.24})$$

Since the x -, y -, and z -axes form a right-handed system, then, by rotating the subscripts x , y , and z , we evidently get the circulation of the vector \mathbf{a} along the contour of an infinitely small rectangle, the positive normal to which is directed along the x - or y -axis: when \mathbf{n} is parallel to the x -axis

$$dC = \oint a_s ds = \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z} \right) dS \quad (\text{A.24a})$$

when \mathbf{n} is parallel to the y -axis

$$dC = \oint a_s dS = \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x} \right) dS \quad (\text{A.24b})$$

We shall show that the combinations of the derivatives of the components of the vector \mathbf{a} in Eqs. (A.24), (A.24a), and (A.24b) are components of a certain vector which is customarily designated by $\text{curl } \mathbf{a}^*$:

$$\left. \begin{aligned} \text{curl}_x \mathbf{a} &= \frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z} \\ \text{curl}_y \mathbf{a} &= \frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x} \\ \text{curl}_z \mathbf{a} &= \frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y} \end{aligned} \right\} \quad (\text{A.25})$$

The vector $\text{curl } \mathbf{a}$ can be called the vector spatial derivative of the vector \mathbf{a} (unlike its scalar spatial derivative $\text{div } \mathbf{a}$).

* It must be noted that $\text{curl } \mathbf{a}$ can be expressed by means of the following symbolic determinant:

$$\text{curl } \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}$$

where \mathbf{i} , \mathbf{j} , and \mathbf{k} are unit vectors along the coordinate axes x , y and z . When calculating this determinant it is naturally necessary to understand that products such as $(\partial/\partial x)a_y$ signify the partial derivative of a_y with respect to x , i.e. $\partial a_y/\partial x$.

Using the symbols (A.25), expressions (A.24) can be written as follows:

$$dC = \oint_L a_s ds = \text{curl}_n \mathbf{a} dS \tag{A.26}$$

Here by \mathbf{n} we must understand a positive normal to the area element dS forming a *right-handed* system with the positive direction around the contour of this element. Assuming consecutively that \mathbf{n} is parallel to the x -, y - and z -axes, we get Eqs. (A.24), (A.24a), and (A.24b) from Eqs. (A.26) and (A.25).

Since the coordinate axes can always be chosen so that one of them is perpendicular to the area element dS , then Eq. (A.26) obviously remains correct for the circulation of the vector \mathbf{a} along the contour of an *arbitrarily* arranged infinitely small rectangle*.

Let us now pass over to considering the circulation of a vector over a contour having an arbitrary shape and dimensions. Let us arrange the surface S so that it rests on the contour L , i.e. so that this contour is the boundary one of the surface S . Let us next divide this surface by two mutually perpendicular systems of parallel lines into a set of infinitely small elements (Fig. 104) that can be considered

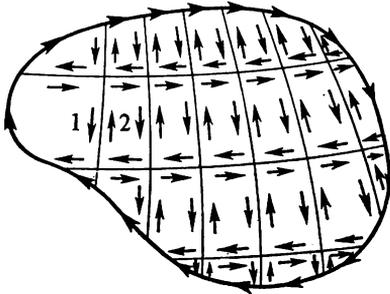


Fig. 104

flat owing to their small size. Applying Eq. (A.26) to each of these elements and summing the expressions obtained, we find that

$$\sum dC = \sum \oint a_s ds = \sum \text{curl}_n \mathbf{a} dS = \int_S \text{curl}_n \mathbf{a} dS$$

where \mathbf{n} is an outward normal to dS . The outer side of the surface S must be selected in accordance with the positive direction around its contour (*a right-handed system*).

Upon integration over the contours of the elementary area elements, each boundary AB between two adjacent elements will be passed

* The latter conclusion holds only if $\text{curl } \mathbf{a}$ is a true vector which is invariant relative to a transformation of the coordinates. This is indeed true (see below).

twice, and in opposite directions. Consequently, the sum $\sum \oint a_s ds$ will contain both terms $\int_A^B a_s ds$ and $\int_B^A a_s ds$, which when taken together produce zero. Thus $\sum \oint a_s ds$ consists of the sum of the terms relating only to the outer boundaries of the area elements, i.e. to the integral of the vector \mathbf{a} over the external contour L of the area S , whence

$$\sum dC = \sum \oint a_s ds = \oint_L a_s ds = C$$

where C stands for the circulation of the vector \mathbf{a} over the contour L . Using this expression in the preceding equation, we get

$$C = \oint_L a_s ds = \int_S \text{curl}_n \mathbf{a} dS \quad (\text{A.27})$$

In deriving this formula, we did not take into account that the external (adjoining the contour L) elementary area elements, generally speaking, will not be rectangular, whereas we have proved the correctness of Eq. (A.26) only for rectangular elements. Upon an unlimited reduction of the size of the rectangles, however, the broken line formed from the outer sides of the extreme rectangles will coincide as close as desired with the contour L of the area S (although, for instance, the limit of the sum of the lengths of the outer sides of the rectangular area elements may not equal the length of the contour L). On this basis, we can give the derivation of Eq. (A.27) an absolutely accurate form. We shall not give the corresponding reasoning here.

Thus, the only condition for the correctness of Eq. (A.27) consists in the requirement of the continuity and differentiability of the vector \mathbf{a} at all the points of the surface S .

This equation expresses the so-called *Stokes's theorem* according to which *the circulation of an arbitrary vector \mathbf{a} around a closed curve L equals the flux of the curl of this vector through the surface S resting on the curve L .*

The shape of the surface S here remains absolutely ambiguous. Hence, the *identical* flux of the curl of any continuous vector \mathbf{a} passes through any two surfaces S_1 and S_2 if only they have the same contour L . This flux equals the circulation of the vector around the common contour of the two surfaces.

It immediately follows from Eq. (A.27), by the way, that

$$\oint \text{curl}_n \mathbf{a} dS = 0 \quad (\text{A.28})$$

because when the surface S is closed, the contour L contracts into a point, and $C = 0^*$.

Passing back from Eq. (A.27) to such a small element of area dS of the surface S that it can be considered as a flat one at all the points of which $\text{curl } \mathbf{a}$ retains a constant value, we can put $\text{curl}_n \mathbf{a}$ outside the integral and write

$$dC = \oint a_s ds = \text{curl}_n \mathbf{a} dS$$

which coincides with Eq. (A.26). Since Eq. (A.27) can be applied to a surface having *any* shape, then Eq. (A.26) can also be applied to infinitely small area elements of any shape. Seeing that this equation holds only in the limiting case of an infinitely small surface, then it will be more correct to write it as follows:

$$\text{curl}_n \mathbf{a} = \lim_{dS \rightarrow 0} \frac{\oint a_s ds}{dS} \tag{A.29}$$

Thus, the component of the vector $\text{curl } \mathbf{a}$ at the given point of a field P in the given direction \mathbf{n} equals the limit of the ratio between the circulation of the vector \mathbf{a} around the contour of an arbitrary element of area dS passing through P and perpendicular to \mathbf{n} and the value of the area of this element dS .

Hence it follows that the value of the component of $\text{curl } \mathbf{a}$ does not depend at all on the selection of the coordinate systems, i.e. that $\text{curl } \mathbf{a}$ is indeed a true vector. We can thus consider that the invariance of the vector $\text{curl } \mathbf{a}$ is proved.

This statement is not quite true in essence because, first, in deriving Eqs. (A.26) and (A.27) we have already used the property of invariance of the vector $\text{curl } \mathbf{a}$ relative to transformation of the coordinates which we want to prove. It is exactly this invariance of $\text{curl } \mathbf{a}$ which we referred to when we stated that Eq. (A.26) can be applied to an area

* Owing to the importance of Eq. (A.28), we shall give another proof of it.

Let us draw a closed curve L on the surface S that divides the latter into two parts S_1 and S_2 . Applying to each of these parts Stokes's formula (A.27), we get

$$\oint_L a_s ds = \pm \int_{S_1} \text{curl}_n \mathbf{a} dS = \pm \int_{S_2} \text{curl}_n \mathbf{a} dS$$

where \mathbf{n} signifies an outward normal to the closed surface S . The sign of the last two terms is determined by whether the direction of the normal \mathbf{n} forms a right- or a left-handed system with the direction of circumvention of the contour L . With any choice of the direction around the contour, these two integrals, as can easily be seen, will have opposite signs, which leads to the required result.

having an *arbitrary* direction (see the footnote on p. 629. Second, we have omitted the *strict* proof of the applicability of Eq. (A.27) [and, consequently, of Eq. (A.26)] to a contour having an arbitrary shape. The correctness of these statements can be proved by directly calculating the circulation of a vector around the contour of an arbitrary surface in Cartesian coordinates. It is simpler and more correct, however, to consider relationship (A.29), which is *invariant* relative to the transformation of the coordinates, as the *definition* of the concept “curl of the vector \mathbf{a} ”. It is not difficult to prove all the formulas derived above on the basis of this definition by reversing the sequence of our reasoning.

In conclusion, to explain the geometrical meaning of a curl, let us consider the rotation of a solid body with the angular velocity ω . As usual, we shall consider the vector ω to be directed along the axis of rotation so that the direction of rotation forms a right-handed system with the vector ω (the right-hand screw rule). Let us choose the z -axis so that it coincides with the axis of rotation and is directed along ω . Hence, the linear velocity \mathbf{v} of a point (x, y, z) of the body will numerically equal

$$v = r\omega = \omega \sqrt{x^2 + y^2}$$

and its components along the coordinate axes will be (Fig. 105)

$$v_x = -\frac{vy}{\sqrt{x^2 + y^2}} = -y\omega,$$

$$v_y = \frac{vx}{\sqrt{x^2 + y^2}} = x\omega, \text{ and } v_z = 0$$

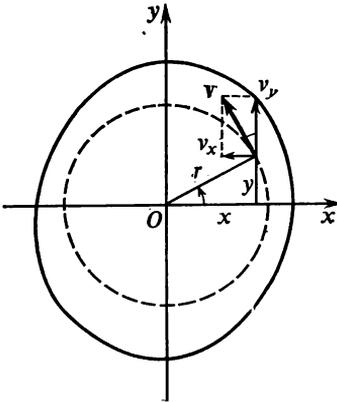


Fig. 105

The components of the vector $\text{curl } \mathbf{v}$, according to Eq. (A.25), are

$$\text{curl}_x \mathbf{v} = \text{curl}_y \mathbf{v} = 0, \text{ and } \text{curl}_z \mathbf{v} = \frac{dv_y}{dx} - \frac{dv_x}{dy} = 2\omega$$

whence

$$\text{curl } \mathbf{v} = 2\omega \tag{A.30}$$

Thus, the curl of the linear velocity of points of a solid body has an identical value at all the points of the body and equals the double angular velocity of its rotation. Equation (A.30) also holds if a body apart from rotational motion is also performing translational motion (because the velocity of the translational motion is the same at all the points of the solid body, and its curl therefore equals zero). Finally, it is proved in the theory of elasticity that Eq. (A.30) holds not only for a solid, but also for an arbitrarily deforming body (for instance a liquid), and in this case by ω we must understand the angular velocity of rotation of an infinitely small element of liquid that is at the point of space being considered.

Thus, $\text{curl } \mathbf{v} \neq 0$ only at the points of a body that belong to its *rotating* elements.

Example. Show that the components of the curl of an arbitrary vector \mathbf{a} in the spherical system of coordinates $R, \theta,$ and α are expressed as follows:

$$\left. \begin{aligned} \text{curl}_R \mathbf{a} &= \frac{1}{R \sin \theta} \left\{ \frac{\partial}{\partial \theta} (\sin \theta a_\alpha) - \frac{\partial a_\alpha}{\partial \alpha} \right\} \\ \text{curl}_\theta \mathbf{a} &= \frac{1}{R \sin \theta} \left\{ \frac{\partial a_R}{\partial \alpha} - \frac{\partial}{\partial R} (R \sin \theta a_\alpha) \right\} \\ \text{curl}_\alpha \mathbf{a} &= \frac{1}{R} \left\{ \frac{\partial}{\partial R} (R a_\theta) - \frac{\partial a_R}{\partial \theta} \right\} \end{aligned} \right\} \tag{A.31}$$

To find, for instance, the value of $\text{curl}_R \mathbf{a}$, we can apply Eq. (A.29) to an elementary element of area cut out from an arbitrary spherical surface $R = \text{const}$ by two meridians $\alpha = \alpha_0$ and $\alpha = \alpha_0 + d\alpha$ and by two parallel circles $\theta = \theta_0$ and $\theta = \theta_0 + d\theta$ (see Fig. 101).

Upon calculating $\oint a_s ds$ in a circumvention of the contour of this element of area forming a right-handed system with the direction of \mathbf{R} , we get

$$\begin{aligned} \oint a_s ds &= \{a_\alpha R \sin \theta d\alpha\}_{\theta=\theta_0+d\theta} - \{a_\theta R d\theta\}_{\alpha=\alpha_0+d\alpha} - \\ &- \{a_\alpha R \sin \theta d\alpha\}_{\theta=\theta_0} + \{a_\theta R d\theta\}_{\alpha=\alpha_0} \end{aligned}$$

or

$$\begin{aligned} \oint a_s ds &= \frac{\partial}{\partial \theta} \{a_\alpha R \sin \theta d\alpha\} d\theta - \frac{\partial}{\partial \alpha} \{a_\theta R d\theta\} d\alpha = \\ &= \left\{ \frac{\partial}{\partial \theta} (a_\alpha \sin \theta) - \frac{\partial a_\theta}{\partial \alpha} \right\} R d\theta d\alpha \end{aligned}$$

Since the value of the element of area dS being considered is $R d\theta \cdot R \sin \theta d\alpha$, then it follows from Eq. (A.29) that

$$\text{curl}_R \mathbf{a} = \frac{1}{R \sin \theta} \left\{ \frac{\partial}{\partial \theta} (a_\alpha \sin \theta) - \frac{\partial a_\theta}{\partial \alpha} \right\}$$

In a similar way, we find the remaining components of the curl of \mathbf{a} .

We invite our reader to prove as an exercise that the components of the curl of an arbitrary vector \mathbf{a} in the cylindrical system of coordinates r , α , and z (see Fig. 102) are expressed as follows:

$$\left. \begin{aligned} \text{curl}_r \mathbf{a} &= \frac{1}{r} \frac{\partial a_z}{\partial \alpha} - \frac{\partial a_\alpha}{\partial z} \\ \text{curl}_\alpha \mathbf{a} &= \frac{\partial a_r}{\partial z} - \frac{\partial a_z}{\partial r} \\ \text{curl}_z \mathbf{a} &= \frac{1}{r} \left\{ \frac{\partial}{\partial r} (r a_\alpha) - \frac{\partial a_r}{\partial \alpha} \right\} \end{aligned} \right\} \quad (\text{A.32})$$

A.6 Derivative of a Vector with Respect to Direction

The scalar $\text{div } \mathbf{a}$ and the vector $\text{curl } \mathbf{a}$, as we have already indicated, can be called the scalar and the vector spatial derivatives of the vector \mathbf{a} , respectively. They have a direct geometrical meaning, which follows from Eqs. (A.18) and (A.29), and together with the gradient of a scalar are the fundamental concepts of vector analysis.

The setting of the values of the scalar $\text{div } \mathbf{a}$ and the vector $\text{curl } \mathbf{a}$ at a given point, however, is not sufficient for determining at this point the arbitrary directional derivative of the vector \mathbf{a} (whereas the arbitrary directional derivative of the scalar φ is unambiguously determined by setting the vector $\text{grad } \varphi$).

Indeed, the derivative of the vector \mathbf{a} with respect to the arbitrary direction \mathbf{c} can be determined by the following geometrical construction. Let us assume that the values of the vector \mathbf{a} at two close points P and P' equal \mathbf{a} and \mathbf{a}' , respectively, the direction of the segment $PP' = \Delta \mathbf{c}$ coinciding with that of \mathbf{c} (Fig. 106). If the difference between \mathbf{a} and \mathbf{a}' equals $\Delta \mathbf{a}$, then the derivative $\partial \mathbf{a} / \partial c$ will be

$$\frac{\partial \mathbf{a}}{\partial c} = \lim_{PP' \rightarrow 0} \frac{\mathbf{a}' - \mathbf{a}}{PP'} = \lim_{\Delta c \rightarrow 0} \frac{\Delta \mathbf{a}}{\Delta c} \quad (\text{A.33})$$

Thus, the direction of the vector $\partial \mathbf{a} / \partial c$ coincides with the limiting direction of the vector $\Delta \mathbf{a}$, but in general differs from the direction of the vectors \mathbf{a} and \mathbf{c} .

Further, if the coordinates of the points P and P' differ from one another by Δx , Δy , and Δz , then with an accuracy up to second-order infinitesimals we have

$$\mathbf{a}' - \mathbf{a} = \frac{\partial \mathbf{a}}{\partial x} \Delta x + \frac{\partial \mathbf{a}}{\partial y} \Delta y + \frac{\partial \mathbf{a}}{\partial z} \Delta z$$

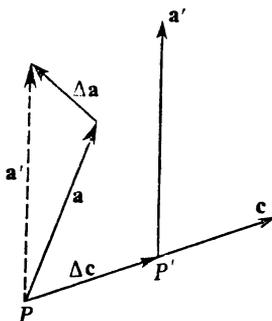


Fig. 106

Introducing this into the preceding equation and taking into account that

$$\frac{\Delta x}{PP'} = \cos(x, \mathbf{c}), \quad \frac{\Delta y}{PP'} = \cos(y, \mathbf{c}), \quad \frac{\Delta z}{PP'} = \cos(z, \mathbf{c})$$

we get

$$\begin{aligned} \frac{\partial \mathbf{a}}{\partial c} = \lim_{PP' \rightarrow 0} \frac{\mathbf{a}' - \mathbf{a}}{PP'} &= \cos(x, \mathbf{c}) \frac{\partial \mathbf{a}}{\partial x} + \\ &+ \cos(y, \mathbf{c}) \frac{\partial \mathbf{a}}{\partial y} + \cos(z, \mathbf{c}) \frac{\partial \mathbf{a}}{\partial z} \end{aligned} \tag{A.34}$$

Thus, to determine the arbitrary directional derivative of the vector \mathbf{a} at a given point, we must set *nine* quantities: the three components $\partial a_x / \partial x$, $\partial a_y / \partial x$, and $\partial a_z / \partial x$ of the quantity $\partial \mathbf{a} / \partial x$ and, correspondingly, three components each of the quantities $\partial \mathbf{a} / \partial y$ and $\partial \mathbf{a} / \partial z$. The combination of these nine quantities forms the components of a *tensor* by setting which we determine both the arbitrary directional derivatives of the vector \mathbf{a} and the values of the quantities $\text{div } \mathbf{a}$ and $\text{curl } \mathbf{a}$. We shall not have to use this tensor in the present course, however.

A.7 The Nabla. Second Derivatives. Derivatives of a Product

1. Above we acquainted ourselves with a number of differential operations on vectors and scalars: the formation of the gradient of a scalar [Eq. (A.6)], the divergence of a vector [Eq. (A.18)], the curl

of a vector [Eq. (A.29)], etc. When employing vector analysis we also encounter many other differential expressions.

Work with these expressions can be simplified and arranged in a simple and harmonious scheme by introducing the *symbolic differential Hamiltonian operator*. This operator is designed by the symbol ∇ (the *nabla* or *del operator*). In the Cartesian system of coordinates, it has the form

$$\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \quad (\text{A.35})$$

where \mathbf{i} , \mathbf{j} , and \mathbf{k} are unit vectors along the axes x , y , and z . In other words, ∇ is a *vector operator* whose components along the coordinate axes are

$$\nabla_x = \frac{\partial}{\partial x}, \quad \nabla_y = \frac{\partial}{\partial y} \quad \text{and} \quad \nabla_z = \frac{\partial}{\partial z} \quad (\text{A.36})$$

This vector operator corresponds in vector analysis to the derivative sign in conventional analysis. In conventional analysis the differential of a function can be considered as the product of the operator of differentiation d and the differentiable function. Similarly, by multiplying scalars and vectors that are position functions by the operator ∇ , we get *spatial derivatives* of these quantities.

For example, the product of ∇ and the scalar φ should obviously be assumed to equal

$$\nabla \varphi = \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \varphi = \mathbf{i} \frac{\partial \varphi}{\partial x} + \mathbf{j} \frac{\partial \varphi}{\partial y} + \mathbf{k} \frac{\partial \varphi}{\partial z}$$

Hence, according to Eq. (A.6), we have

$$\nabla \varphi = \text{grad } \varphi \quad (\text{A.37})$$

Thus, $\nabla \varphi$ can indeed be called the spatial derivative of φ because the vector $\text{grad } \varphi$ completely characterizes the changes which the scalar φ undergoes upon displacement of the “observation point” (i.e. upon a change in the coordinates x , y , and z). Similarly, other expressions including the operator ∇ also characterize various relationships between the values of scalar and vector functions at adjacent points of space.

With certain restrictions which will be treated below, we can form products of ∇ with other vectors and scalars as if ∇ were a true and not a symbolic vector. As when using the differential sign, it is assumed that the operator ∇ “acts” only on the quantities to its right.

For example, the scalar product of the symbolic vector ∇ and the arbitrary vector \mathbf{a} is

$$\nabla \mathbf{a} = \nabla_x a_x + \nabla_y a_y + \nabla_z a_z = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$

i.e. according to Eq. (A.15)

$$\nabla \mathbf{a} = \text{div } \mathbf{a} \tag{A.38}$$

Apart from the scalar product of the symbolic vector ∇ and the vector \mathbf{a} , it is also possible to form the vector product of these vectors which, as can easily be seen, is the curl of the vector \mathbf{a} (see the footnote on p. 628):

$$[\nabla \mathbf{a}] = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix} = \text{curl } \mathbf{a} \tag{A.39}$$

Thus, the component of the vector $[\nabla \mathbf{a}]$ along the x -axis is

$$[\nabla \mathbf{a}]_x = \nabla_y a_z - \nabla_z a_y = \frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z} = \text{curl}_x \mathbf{a}$$

2. The use of the operator ∇ greatly simplifies the finding of the second and higher derivatives of scalar and vector quantities. For instance, the square of the vector ∇ is

$$\begin{aligned} \nabla^2 &= \nabla_x^2 + \nabla_y^2 + \nabla_z^2 = \frac{\partial}{\partial x} \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \frac{\partial}{\partial z} = \\ &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \end{aligned}$$

Therefore, upon disclosing the meaning of the product $\nabla(\nabla\varphi)$ according to the rules of vector algebra

$$\mathbf{b}(\mathbf{b}\varphi) = \mathbf{b}^2\varphi$$

we get

$$\text{div grad } \varphi = \nabla(\nabla\varphi) = \nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} \tag{A.40}$$

(The operator ∇^2 is often designated by Δ and is called the *Laplacian operator*.)

We can convince ourselves of the validity of this equation by direct calculations with the aid of Eqs. (A.5) and (A.15):

$$\begin{aligned} \text{div grad } \varphi &= \frac{\partial \text{grad}_x \varphi}{\partial x} + \frac{\partial \text{grad}_y \varphi}{\partial y} + \frac{\partial \text{grad}_z \varphi}{\partial z} = \\ &= \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} \end{aligned}$$

The expression for grad div \mathbf{a} has an absolutely different meaning:

$$\text{grad div } \mathbf{a} = \nabla(\nabla \mathbf{a}) = \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \times \\ \times \left(\frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} \right)$$

It does not at all equal $\nabla^2 \mathbf{a}$, just like when operating with conventional vectors

$$\mathbf{b}(\mathbf{b}\mathbf{a}) \neq b^2 \mathbf{a}$$

The expression $\nabla^2 \mathbf{a}$ obviously has the following meaning:

$$\nabla^2 \mathbf{a} = (\nabla \nabla) \mathbf{a} = \frac{\partial^2 \mathbf{a}}{\partial x^2} + \frac{\partial^2 \mathbf{a}}{\partial y^2} + \frac{\partial^2 \mathbf{a}}{\partial z^2} \quad (\text{A.41})$$

i.e. is a vector whose component along the x -axis, for example, is

$$(\nabla^2 \mathbf{a})_x = \nabla^2 a_x = \frac{\partial^2 a_x}{\partial x^2} + \frac{\partial^2 a_x}{\partial y^2} + \frac{\partial^2 a_x}{\partial z^2} \quad (\text{A.41a})$$

Naturally, $\nabla^2 \varphi$ and $\nabla^2 \mathbf{a}$ must not be confused with $(\nabla \varphi)^2$ and $(\nabla \mathbf{a})^2$; for example

$$(\nabla \varphi)^2 = (\text{grad } \varphi)^2 = \left(\frac{\partial \varphi}{\partial x} \right)^2 + \left(\frac{\partial \varphi}{\partial y} \right)^2 + \left(\frac{\partial \varphi}{\partial z} \right)^2$$

The known formulas of vector algebra

$$[\mathbf{b}(\mathbf{b}\varphi)] = 0, \quad \mathbf{b}[\mathbf{b}\mathbf{a}] = 0, \quad \text{and} \quad [\mathbf{b}[\mathbf{b}\mathbf{a}]] = \mathbf{b}(\mathbf{b}\mathbf{a}) - (\mathbf{b}\mathbf{b})\mathbf{a}^*$$

remain correct when the vector \mathbf{b} is replaced with the symbolic vector ∇ (at any values of \mathbf{a} and φ):

$$\left. \begin{aligned} [\nabla(\nabla \varphi)] &= [\nabla \text{grad } \varphi] = \text{curl grad } \varphi = 0 \\ \nabla[\nabla \mathbf{a}] &= \nabla \text{curl } \mathbf{a} = \text{div curl } \mathbf{a} = 0 \\ [\nabla[\nabla \mathbf{a}]] &= \nabla(\nabla \mathbf{a}) - \nabla^2 \mathbf{a}, \text{ or } \text{curl curl } \mathbf{a} = \\ &= \text{grad div } \mathbf{a} - \nabla^2 \mathbf{a} \end{aligned} \right\} \quad (\text{A.42})$$

* In the right-hand side of the last equation, we can naturally change the order of the multipliers, for example $[\mathbf{b}[\mathbf{b}\mathbf{a}]] = (\mathbf{a}\mathbf{b})\mathbf{b} - \mathbf{a}(\mathbf{b}\mathbf{b})$. When replacing \mathbf{b} with ∇ , however, we must write this equation so that all the differential operators ∇ are *before* the vector \mathbf{a} being differentiated.

We can easily convince ourselves that these relationships are correct by direct calculation in Cartesian coordinates.* For example,

$$\begin{aligned} \operatorname{div} \operatorname{curl} \mathbf{a} &= \frac{\partial \operatorname{curl}_x \mathbf{a}}{\partial x} + \frac{\partial \operatorname{curl}_y \mathbf{a}}{\partial y} + \frac{\partial \operatorname{curl}_z \mathbf{a}}{\partial z} = \\ &= \frac{\partial}{\partial x} \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z} \right) + \frac{\partial}{\partial y} \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x} \right) + \\ &+ \frac{\partial}{\partial z} \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y} \right) = 0 \end{aligned}$$

3. Thus, since the operator ∇ is a multiplier in products containing only a *single* true scalar or vector, then these products can be transformed according to the conventional rules of vector algebra. Should a product contain two or more true scalars or vectors, however, then these rules can no longer be applied and require modifications. Absolutely the same also occurs in conventional analysis when algebraic quantities are symbolically multiplied by the differential sign d . Similar to how

$$d(\varphi\psi) = \psi d\varphi + \varphi d\psi$$

when a product of scalars or vectors is multiplied by ∇ , the operation of differentiation must be done on each of the multipliers separately. For example, upon the differentiation of a product of two scalars or a scalar and a vector, we get

$$\left. \begin{aligned} \nabla(\varphi\psi) &= \psi(\nabla\varphi) + \varphi(\nabla\psi) \text{ or } \operatorname{grad}(\varphi\psi) = \\ &= \psi \operatorname{grad} \varphi + \varphi \operatorname{grad} \psi \\ \nabla(\varphi\mathbf{a}) &= \varphi(\nabla\mathbf{a}) + \mathbf{a}(\nabla\varphi) \text{ or } \operatorname{div}(\varphi\mathbf{a}) = \\ &= \varphi \operatorname{div} \mathbf{a} + \mathbf{a} \operatorname{grad} \varphi \\ [\nabla(\varphi\mathbf{a})] &= \varphi[\nabla\mathbf{a}] + [(\nabla\varphi)\mathbf{a}] \text{ or } \operatorname{curl}(\varphi\mathbf{a}) = \\ &= \varphi \operatorname{curl} \mathbf{a} + [\operatorname{grad} \varphi \cdot \mathbf{a}] \end{aligned} \right\} \quad (\text{A.43})$$

* The relationships $\operatorname{curl} \operatorname{grad} \varphi = 0$ and $\operatorname{div} \operatorname{curl} \mathbf{a} = 0$ are also obtained directly from Stokes's and Gauss's formulas (A.27) and (A.17) if we assume in them that $\mathbf{a} = \operatorname{grad} \varphi$ or correspondingly $\mathbf{a} = \operatorname{curl} \mathbf{b}$. Using, in addition, Eq. (A.28), we get

$$\int \operatorname{div} \operatorname{curl} \mathbf{b} \cdot dV = \oint \operatorname{curl}_n \mathbf{b} \cdot dS = 0$$

and

$$\int \operatorname{curl}_n \operatorname{grad} \varphi \cdot dS = \oint \operatorname{grad} \varphi \cdot ds = \oint \frac{\partial \varphi}{\partial s} ds = 0$$

whence the above-mentioned relationships follow owing to the arbitrary nature of the domain of integration.

The correctness of these relationships can be verified by direct calculation. For example,

$$\begin{aligned}\nabla(\varphi\psi) &= \mathbf{i} \frac{\partial}{\partial x}(\varphi\psi) + \mathbf{j} \frac{\partial}{\partial y}(\varphi\psi) + \mathbf{k} \frac{\partial}{\partial z}(\varphi\psi) = \\ &= \mathbf{i} \left(\varphi \frac{\partial \psi}{\partial x} + \psi \frac{\partial \varphi}{\partial x} \right) + \mathbf{j} \left(\varphi \frac{\partial \psi}{\partial y} + \psi \frac{\partial \varphi}{\partial y} \right) + \\ &+ \mathbf{k} \left(\varphi \frac{\partial \psi}{\partial z} + \psi \frac{\partial \varphi}{\partial z} \right) = \psi(\nabla\varphi) + \varphi(\nabla\psi)\end{aligned}$$

Matters are somewhat more complicated upon the scalar differentiation of the product of two vectors.

Let us turn, first of all, to the expression

$$\nabla[\mathbf{ab}] = \text{div}[\mathbf{ab}]$$

For conventional vectors, the following relationships hold

$$\mathbf{c}[\mathbf{ab}] = \mathbf{b}[\mathbf{ca}] = -\mathbf{a}[\mathbf{cb}]$$

When a vector is replaced with the differential operator ∇ , we can assume that $\nabla[\mathbf{ab}]$ should be equated to the sum of the expressions

$$\mathbf{b}[\nabla\mathbf{a}] \text{ and } -\mathbf{a}[\nabla\mathbf{b}]$$

because in conventional analysis the derivative of a product equals the sum of two terms in each of which only one of the multipliers is differentiated. Indeed, we can show by direct calculations, which we shall let our reader do, that

$$\nabla[\mathbf{ab}] = \mathbf{b}[\nabla\mathbf{a}] - \mathbf{a}[\nabla\mathbf{b}]$$

i.e. that

$$\text{div}[\mathbf{ab}] = \mathbf{b} \text{ curl } \mathbf{a} - \mathbf{a} \text{ curl } \mathbf{b} \quad (\text{A.44})$$

It is known that when calculating the product $\mathbf{c}(\mathbf{ab})$ of three vectors we must first perform scalar multiplication of the vectors \mathbf{a} and \mathbf{b} before multiplying them by \mathbf{c} . Accordingly, the expression

$$\nabla(\mathbf{ab}) = \text{grad}(\mathbf{ab})$$

cannot be written in the form of the sum of *two* terms in each of which only one of the multipliers is differentiated. We can show further that such transformation also cannot be performed with respect to the expression

$$[\nabla[\mathbf{ab}]] = \text{curl}[\mathbf{ab}]$$

Both of these expressions, however, can be written in the form of the sum of *four* terms in each of which only one of the vectors \mathbf{a} and \mathbf{b} is differentiated.

The relevant formulas are

$$\text{grad}(\mathbf{ab}) = (\mathbf{b}\nabla)\mathbf{a} + (\mathbf{a}\nabla)\mathbf{b} + [\mathbf{b}\text{curl}\mathbf{a}] + [\mathbf{a}\text{curl}\mathbf{b}] \quad (\text{A.45})$$

$$\text{curl}[\mathbf{ab}] = (\mathbf{b}\nabla)\mathbf{a} - (\mathbf{a}\nabla)\mathbf{b} + \mathbf{a}\text{div}\mathbf{b} - \mathbf{b}\text{div}\mathbf{a} \quad (\text{A.46})$$

(Their proofs are given in courses of vector analysis; for special cases they are also proved in Secs. 1.18 and 4.16.)

In the particular case when $\mathbf{b} = \text{const}$ and $\mathbf{a} = \mathbf{R}$, where \mathbf{R} is a radius-vector, it is not difficult to show that Eq. (A.45) becomes the same as Eq. (A.11):

$$\nabla(\mathbf{bR}) = \text{grad}_a(\mathbf{bR}) = \mathbf{b}$$

If, further, we assume in Eq. (A.45) that $\mathbf{a} = \mathbf{b}$, we obtain

$$\frac{1}{2}\nabla a^2 = (\mathbf{a}\nabla)\mathbf{a} + [\mathbf{a}\text{curl}\mathbf{a}] \quad (\text{A.47})$$

4. It remains for us to consider the scalar operator $\mathbf{a}\nabla$ obtained by the scalar multiplication of an arbitrary vector \mathbf{a} by the Hamiltonian operator ∇ at the *right* of \mathbf{a} (unlike $\nabla\mathbf{a} = \text{div}\mathbf{a}$):

$$\mathbf{a}\nabla = a_x\frac{\partial}{\partial x} + a_y\frac{\partial}{\partial y} + a_z\frac{\partial}{\partial z} \quad (\text{A.48})$$

In the particular case when $a = 1$, the operator $\mathbf{a}\nabla$ is obviously equivalent to finding the derivative $\partial/\partial a$ with respect to the direction of the unit vector \mathbf{a} . In general, the performing of the operator $\mathbf{a}\nabla$ on an arbitrary position function is equivalent to multiplying the derivative of this function taken with respect to the direction of the vector \mathbf{a} by the absolute value of the vector \mathbf{a} . In other words,

$$\mathbf{a}\nabla = a\frac{\partial}{\partial a} \quad (\text{A.49})$$

Indeed, upon performing the operation $\mathbf{a}\nabla$ on an arbitrary scalar φ , we get the scalar

$$\mathbf{a}\nabla \cdot \varphi = a_x\frac{\partial \varphi}{\partial x} + a_y\frac{\partial \varphi}{\partial y} + a_z\frac{\partial \varphi}{\partial z} = \mathbf{a} \cdot \nabla \varphi$$

or on the basis of Eq. (A.4) we have $\mathbf{a}\nabla \cdot \varphi = \mathbf{a}\text{grad}\varphi = a\frac{\partial \varphi}{\partial a}$ in accordance with Eq. (A.49).

Upon performing the operation $\mathbf{a}\nabla$ on the arbitrary vector \mathbf{b} , however, we get the vector

$$(\mathbf{a}\nabla)\mathbf{b} = \mathbf{a}\nabla \cdot \mathbf{b} = a_x\frac{\partial \mathbf{b}}{\partial x} + a_y\frac{\partial \mathbf{b}}{\partial y} + a_z\frac{\partial \mathbf{b}}{\partial z} \quad (\text{A.50a})$$

whose component, for example, along the x -axis is

$$\bullet (\mathbf{a} \nabla \cdot \mathbf{b})_x = a_x \frac{\partial b_x}{\partial x} + a_y \frac{\partial b_x}{\partial y} + a_z \frac{\partial b_x}{\partial z} = (\mathbf{a} \nabla) b_x \quad (\text{A.50b})$$

On the other hand, the derivative of the vector \mathbf{b} with respect to the direction \mathbf{a} , according to Eq. (A.34), is

$$\frac{\partial \mathbf{b}}{\partial a} = \cos(x, \mathbf{a}) \frac{\partial \mathbf{b}}{\partial x} + \cos(y, \mathbf{a}) \frac{\partial \mathbf{b}}{\partial y} + \cos(z, \mathbf{a}) \frac{\partial \mathbf{b}}{\partial z}$$

Multiplying this equation by a and comparing the result with Eq. (A.50a), we see that indeed

$$\mathbf{a} \nabla \cdot \mathbf{b} = a \frac{\partial \mathbf{b}}{\partial a} \quad (\text{A.51})$$

Q.E.D. Thus, if the vector \mathbf{a} is sufficiently small, then with an accuracy up to second-order infinitesimals $\mathbf{a} \nabla \cdot \varphi$ and $\mathbf{a} \nabla \cdot \mathbf{b}$, respectively, equal the increment of the scalar φ and the vector \mathbf{b} when the "observation point" is displaced over a distance equal in magnitude and direction to the vector \mathbf{a} .

5. The elementary operations of spatial differentiation consist in the formation of a gradient, a divergence, a curl, and the derivative $\partial \mathbf{b} / \partial a$. All these operations, as we have seen, for example, in Eqs. (A.3), (A.18), (A.29), and (A.33), have a definite geometrical meaning and are therefore *invariant* with respect to transformation of the coordinate system. In other words, the value of the expressions $\text{grad } \varphi$, $\text{div } \mathbf{a}$, $\text{curl } \mathbf{a}$, and $\partial \mathbf{b} / \partial a$ does not depend on the choice of the coordinate system. All the relationships between the differential expressions which we have derived above are also of an invariant nature because although when proving them we always used a definite (Cartesian) coordinate system, the relationships themselves include only *invariant* expressions such as $\text{grad } \varphi$, $\text{div } \mathbf{a}$, and $\text{curl } \mathbf{a}$. Consequently, the form of these relationships cannot change when passing over to other coordinate systems.

A.8 Integral Relationships. Green's Theorem

Gauss's and Stokes's formulas (A.17) and (A.27) are the fundamental integral relationships of vector analysis. We can also get a number of other important relationships between the space (volume, surface and line) integrals of scalar and vector quantities on their basis.

1. Gauss's formula (A.17) permits us to prove *Green's theorem*, which is important for vector analysis and its applications, without

any difficulty. For this purpose, let us assume in Gauss's formula (A.17)

$$\int \operatorname{div} \mathbf{a} \, dV = \oint a_n \, dS$$

that $\mathbf{a} = \psi \operatorname{grad} \varphi = \psi \nabla \varphi$, where ψ and φ are two arbitrary scalars. According to Eqs. (A.43₂) and (A.40), we have

$$\begin{aligned} \operatorname{div} \mathbf{a} &= \psi \operatorname{div} \operatorname{grad} \varphi + \operatorname{grad} \psi \cdot \operatorname{grad} \varphi = \psi \nabla^2 \varphi + \\ &+ (\nabla \varphi) (\nabla \psi) \end{aligned}$$

Further, $a_n = \psi \operatorname{grad}_n \varphi = \psi (\partial \varphi / \partial n)$. It therefore follows from Eq. (A.17) that

$$\int \{ \psi \nabla^2 \varphi + (\nabla \varphi) (\nabla \psi) \} \, dV = \oint \psi \frac{\partial \varphi}{\partial n} \, dS \tag{A.52}$$

where the integral in the right-hand side must be taken over the closed surface S confining the region of integration V . It is exactly this formula that expresses *Green's theorem*.

For some purposes, it is convenient to transform Eq. (A.52) by substituting φ for ψ in it, and vice versa. Subtracting the equation obtained in this way from Eq. (A.52), we get

$$\int (\psi \nabla^2 \varphi - \varphi \nabla^2 \psi) \, dV = \oint \left(\psi \frac{\partial \varphi}{\partial n} - \varphi \frac{\partial \psi}{\partial n} \right) \, dS \tag{A.53}$$

It has already been indicated that the application of Gauss's theorem is restricted by the requirement of the continuity of the vector \mathbf{a} and the finiteness of its first derivatives in the region of integration V . Consequently, Green's theorem can be applied directly only to finite and continuous scalar position functions ψ and φ having derivatives of the first and second orders in the region of integration V .

2. Let us consider the line integral $\oint \varphi \, ds$ of an arbitrary scalar φ having finite derivatives over an arbitrary closed contour L . This integral is a *vector* because by ds we understand the *vector* value of an element of length of the contour.

To transform this integral, we shall multiply it scalarly by an arbitrary vector \mathbf{c} that is constant in magnitude and direction:

$$\mathbf{c} \oint_L \varphi \, ds = \oint_L \varphi \mathbf{c} \, ds = \oint_L \varphi c_s \, ds$$

We can use Stokes's formula (A.27) to transform the last integral into one over the arbitrary surface S resting on the contour L . To do

this, it is sufficient to assume in Eq. (A.27) that $\mathbf{a} = \varphi \mathbf{c}$:

$$\mathbf{c} \oint_L \varphi \, ds = \int_S \operatorname{curl}_n (\varphi \mathbf{c}) \, dS$$

Owing to the constancy of the vector \mathbf{c} , from Eq. (A.43₃) we get

$$\operatorname{curl} (\varphi \mathbf{c}) = \varphi \operatorname{curl} \mathbf{c} + [\operatorname{grad} \varphi \cdot \mathbf{c}] = [\operatorname{grad} \varphi \cdot \mathbf{c}]$$

and, therefore,

$$\operatorname{curl}_n (\varphi \mathbf{c}) = \mathbf{n} [\operatorname{grad} \varphi \cdot \mathbf{c}] = \mathbf{c} [\mathbf{n} \cdot \operatorname{grad} \varphi]$$

Using this expression in the last integral relationship and putting the constant vector \mathbf{c} outside the integral sign, we get

$$\mathbf{c} \oint_L \varphi \, ds = \mathbf{c} \int_S [\mathbf{n} \cdot \operatorname{grad} \varphi] \, dS$$

In view of the arbitrary nature of the vector \mathbf{c} , this equation can hold only provided that both integrals are equal. We thus arrive at the required formula

$$\oint_L \varphi \, ds = \int_S [\mathbf{n} \cdot \operatorname{grad} \varphi] \, dS \quad (\text{A.54})$$

where, as follows from the derivation, \mathbf{n} is normal to the surface S forming a right-handed system with the positive direction around the contour L .

If we presume that the surface area element dS is a *vector* quantity whose direction coincides with that of a positive normal to this element dS , then Eq. (A.54) can be written as follows:

$$\oint_L \varphi \, ds = \int_S [d\mathbf{S} \cdot \operatorname{grad} \varphi] \quad (\text{A.55})$$

3. Let us prove the relationship

$$\int_V \operatorname{curl} \mathbf{a} \, dV = \oint_S [\mathbf{n}\mathbf{a}] \, dS = \oint_S [d\mathbf{S} \cdot \mathbf{a}] \quad (\text{A.56})$$

that makes it possible to transform the integral of the curl of the arbitrary vector \mathbf{a} over the arbitrary volume V into the integral of the tangential components of this vector over the closed surface S confining the volume V .

Let us multiply the volume integral to be transformed scalarly by the arbitrary vector \mathbf{c} that is constant in magnitude and direction.

According to Eq. (A.44), we have

$$\mathbf{c} \operatorname{curl} \mathbf{a} = \mathbf{a} \operatorname{curl} \mathbf{c} + \operatorname{div} [\mathbf{ac}] = \operatorname{div} [\mathbf{ac}]$$

because the curl of the constant vector \mathbf{c} equals zero. Hence,

$$\mathbf{c} \int_V \operatorname{curl} \mathbf{a} \, dV = \int_V \operatorname{div} [\mathbf{ac}] \, dV = \oint_S [\mathbf{ac}]_n \, dS$$

where we have taken advantage of Gauss's theorem (A.17).

Finally, $[\mathbf{ac}]_n = [\mathbf{ac}]\mathbf{n} = \mathbf{c}[\mathbf{na}]$, and, therefore,

$$\mathbf{c} \int_V \operatorname{curl} \mathbf{a} \, dV = \mathbf{c} \oint_S [\mathbf{na}] \, dS$$

Owing to the vector \mathbf{c} being arbitrary, Eq. (A.56) follows from this equation.

In terminating our treatment of vector analysis, we must note that in the present appendix we have not considered some of the limit theorems leading to the concept of the surface curl and the surface divergence. These theorems, as well as some tensor relationships, are derived in the text of the book.

A.9 The Most Important Formulas of Vector Analysis

$$\frac{\partial \varphi}{\partial s} = \lim_{ds \rightarrow 0} \frac{\varphi_s - \varphi_0}{ds} \quad (\text{the derivative of the scalar } \varphi \text{ with respect to the direction } s) \quad (\text{A.1})$$

$$\operatorname{grad} \varphi = \frac{\partial \varphi}{\partial n} \mathbf{n} \quad (\text{A.3})$$

$$\nabla \varphi = \operatorname{grad} \varphi = \mathbf{i} \frac{\partial \varphi}{\partial x} + \mathbf{j} \frac{\partial \varphi}{\partial y} + \mathbf{k} \frac{\partial \varphi}{\partial z} \quad (\text{A.6})$$

$$\operatorname{grad}_s \varphi = \frac{\partial \varphi}{\partial s} \quad (\text{A.4})$$

$$\operatorname{grad} \varphi = \frac{\partial \varphi(\psi)}{\partial \psi} \operatorname{grad} \psi \quad (\text{A.7})$$

$$\operatorname{grad}_q R = -\frac{\mathbf{R}}{R} = -\operatorname{grad}_a R \quad (\text{A.8})$$

$$\text{grad}_q \left(\frac{1}{R} \right) = \frac{\mathbf{R}}{R^3} = - \text{grad}_a \left(\frac{1}{R} \right) \quad (\text{A.10})$$

$$\text{grad}_a(\mathbf{bR}) = \mathbf{b} \quad (\mathbf{b} = \text{const}) \quad (\text{A.11})$$

$$\oint_S \mathbf{a}_n dS = \int_V \text{div } \mathbf{a} dV \quad (\text{Gauss's theorem}) \quad (\text{A.17})$$

$$\text{div } \mathbf{a} = \lim_{\Delta V \rightarrow 0} \frac{\oint a_n dS}{\Delta V} \quad (\text{A.18})$$

$$\nabla \mathbf{a} = \text{div } \mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} \quad (\text{A.15})$$

$$\text{Div } \mathbf{a} = a_{2n} - a_{1n} \quad (\text{1.29})$$

$$\oint_L \mathbf{a}_s ds = \int_S \text{curl}_n \mathbf{a} dS \quad (\text{Stokes's theorem}) \quad (\text{A.27})$$

$$\text{curl}_n \mathbf{a} = \lim_{dS \rightarrow 0} \frac{\oint a_s ds}{dS} \quad (\text{A.29})$$

$$[\nabla \mathbf{a}] = \text{curl } \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix} \quad (\text{A.25})$$

$$\text{Curl } \mathbf{a} = [\mathbf{n} \cdot (\mathbf{a}_2 - \mathbf{a}_1)] \quad (\text{4.55})$$

$$\oint_S \text{curl}_n \mathbf{a} dS = 0 \quad (\text{A.28})$$

$$\frac{\partial \mathbf{b}}{\partial a} = \lim_{\Delta a \rightarrow 0} \frac{\mathbf{b}' - \mathbf{b}}{\Delta a} \quad (\text{the derivative of a vector with respect to the direction } \mathbf{a}) \quad (\text{A.33})$$

$$\mathbf{a} \nabla \cdot \mathbf{b} = \mathbf{a} \frac{\partial \mathbf{b}}{\partial a} = a_x \frac{\partial \mathbf{b}}{\partial x} + a_y \frac{\partial \mathbf{b}}{\partial y} + a_z \frac{\partial \mathbf{b}}{\partial z} \quad (\text{A.51})$$

Second derivatives:

$$\text{div grad } \varphi = \nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} \quad (\text{A.40})$$

$$\nabla^2 f(R) = \frac{\partial^2 f}{\partial R^2} + \frac{2}{R} \frac{\partial f}{\partial R} \tag{A.21}$$

$$\nabla^2 \left(\frac{1}{R} \right) = 0$$

$$\nabla^2 \mathbf{a} = \frac{\partial^2 \mathbf{a}}{\partial x^2} + \frac{\partial^2 \mathbf{a}}{\partial y^2} + \frac{\partial^2 \mathbf{a}}{\partial z^2} \tag{A.41}$$

$$\left. \begin{aligned} \text{curl grad } \varphi &= 0 \\ \text{div curl } \mathbf{a} &= 0 \\ \text{curl curl } \mathbf{a} &= \text{grad div } \mathbf{a} - \nabla^2 \mathbf{a} \end{aligned} \right\} \tag{A.42}$$

Derivatives of products:

$$\left. \begin{aligned} \text{grad } (\varphi\psi) &= \psi \text{ grad } \varphi + \varphi \text{ grad } \psi \\ \text{div } (\varphi\mathbf{a}) &= \varphi \text{ div } \mathbf{a} + \mathbf{a} \text{ grad } \varphi \\ \text{curl } (\varphi\mathbf{a}) &= \varphi \text{ curl } \mathbf{a} + [\text{grad } \varphi \cdot \mathbf{a}] \end{aligned} \right\} \tag{A.43}$$

$$\text{div } [\mathbf{ab}] = \mathbf{b} \text{ curl } \mathbf{a} - \mathbf{a} \text{ curl } \mathbf{b} \tag{A.44}$$

$$\text{grad } (\mathbf{ab}) = (\mathbf{b} \nabla) \mathbf{a} + (\mathbf{a} \nabla) \mathbf{b} + [\mathbf{b} \text{ curl } \mathbf{a}] + [\mathbf{a} \text{ curl } \mathbf{b}] \tag{A.45}$$

$$\text{curl } [\mathbf{ab}] = (\mathbf{b} \nabla) \mathbf{a} - (\mathbf{a} \nabla) \mathbf{b} + \mathbf{a} \text{ div } \mathbf{b} - \mathbf{b} \text{ div } \mathbf{a} \tag{A.46}$$

$$\frac{1}{2} \nabla a^2 = (\mathbf{a} \nabla) \mathbf{a} + [\mathbf{a} \text{ curl } \mathbf{a}] \tag{A.47}$$

Green's theorem:

$$\int \{ \psi \nabla^2 \varphi + (\nabla \varphi) \cdot (\nabla \psi) \} dV = \oint \psi \frac{\partial \varphi}{\partial n} dS \tag{A.52}$$

$$\int (\psi \nabla^2 \varphi - \varphi \nabla^2 \psi) dV = \oint \left(\psi \frac{\partial \varphi}{\partial n} - \varphi \frac{\partial \psi}{\partial n} \right) dS \tag{A.53}$$

$$\oint_L \varphi ds = \int_S [\mathbf{n} \cdot \text{grad } \varphi] dS \tag{A.54}$$

$$\int_V \text{curl } \mathbf{a} dV = \oint_S [\mathbf{na}] dS = \oint_S [d\mathbf{S} \cdot \mathbf{a}] \tag{A.56}$$

Fundamental Formulas in the SI and Gaussian Systems of Units

As is general practice in courses of theoretical physics, the present book uses the Gaussian system of units. Owing to recent insistent recommendations on the preferable use of the SI system, however, the publishers of the Russian edition found it necessary to give a table of the main formulas in both systems.

	SI	Gaussian system
Coulomb's law	$F = \frac{1}{4\pi} \frac{q_1 q_2}{\epsilon R^2}$	$F = \frac{q_1 q_2}{\epsilon R^2}$
Field intensity of a point charge	$E = \frac{1}{4\pi} \frac{q}{\epsilon R^2}$	$E = \frac{q}{\epsilon R^2}$
Intensity of field between charged planes and near the surface of a charged conductor	$E = \frac{\sigma}{\epsilon}$	$E = \frac{4\pi\sigma}{\epsilon}$
Potential of a point charge	$\varphi = \frac{1}{4\pi} \frac{q}{\epsilon R}$	$\varphi = \frac{q}{\epsilon R}$
Relationship between P and E	$P = \alpha\epsilon_0 E$	$P = \alpha E$
Relationship between P and surface density of bound charges on the boundary with a vacuum	$\sigma_{\text{bound}} = \alpha\epsilon_0 P_n$	$\sigma_{\text{bound}} = \alpha P_n$
Electric displacement (definition)	$D = \epsilon_0 E + P$	$D = E + 4\pi P$
Relationship between D and E	$D = \epsilon E$	$D = \epsilon E$
Relationship between D and E in a vacuum	$D = \epsilon_0 E$	$D = E$
<i>D</i> of the field of a point charge	$D = \frac{1}{4\pi} \frac{q}{R^2}$	$D = \frac{q}{R^2}$

	SI	Gaussian system
Gauss's theorem for \mathbf{D}	$\oint D_n dS = \Sigma q$	$\oint D_n dS = 4\pi \Sigma q$
Capacitance of a plane capacitor	$C = \frac{\epsilon S}{d}$	$C = \frac{\epsilon S}{4\pi d}$
Density of energy of electric field	$u = \frac{\epsilon E^2}{2}$	$u = \frac{\epsilon E^2}{8\pi}$
Magnetic field intensity (definition)	$\mathbf{H} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M}$	$\mathbf{H} = \mathbf{B} - 4\pi \mathbf{M}$
Relationship between \mathbf{M} and \mathbf{H}		$\mathbf{M} = \chi \mathbf{H}$
Relationship between \mathbf{B} and \mathbf{H}		$\mathbf{B} = \mu \mathbf{H}$
Relationship between \mathbf{B} and \mathbf{H} in a vacuum	$\mathbf{B} = \mu_0 \mathbf{H}$	$\mathbf{B} = \mathbf{H}$
Biot-Savart law	$\mathbf{H} = \frac{I}{4\pi R^3} [ds \mathbf{R}]$	$\mathbf{H} = \frac{I}{cR^3} [ds \mathbf{R}]$
Circulation of the vector \mathbf{H}	$\oint H_s ds = \Sigma I$	$\oint H_s ds = \frac{4\pi}{c} \Sigma I$
Gauss's theorem for \mathbf{B}		$\oint B_n dS = 0$
Ampere's law	$\mathbf{F} = I [ds \mathbf{B}]$	$\mathbf{F} = \frac{1}{c} I [ds \mathbf{B}]$
Lorentz force	$\mathbf{F} = q [\mathbf{vB}]$	$\mathbf{F} = \frac{q}{c} [\mathbf{vH}]$
Energy of magnetic field of a current	$U = \frac{LI^2}{2}$	$U = \frac{1}{c^2} \frac{LI^2}{2}$
Density of magnetic field energy	$u = \frac{\mu H^2}{2}$	$u = \frac{1}{8\pi} \mu H^2$
Density of displacement current (definition)	$\mathbf{j}_{dis} = \frac{\partial \mathbf{D}}{\partial t}$	$\mathbf{j}_{dis} = \frac{1}{4\pi} \frac{\partial \mathbf{D}}{\partial t}$

	SI	Gaussian system
Maxwell's equations in the integral form	$\oint E_s ds = - \int \frac{\partial \mathbf{B}}{\partial t} d\mathbf{S}$ $\oint H_s ds = \int \mathbf{j} d\mathbf{S} + \int \frac{\partial \mathbf{D}}{\partial t} d\mathbf{S}$	$\oint E_s ds = - \frac{1}{c} \int \frac{\partial \mathbf{B}}{\partial t} d\mathbf{S}$ $\oint H_s ds = \int \mathbf{j} d\mathbf{S} + \frac{1}{c} \int \frac{\partial \mathbf{D}}{\partial t} d\mathbf{S}$
Maxwell's equations in the differential form	$\text{curl } \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t}$ $\text{div } \mathbf{B} = 0$ $\text{curl } \mathbf{H} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t}$ $\text{div } \mathbf{D} = \rho$	$\text{curl } \mathbf{E} = - \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}$ $\text{div } \mathbf{B} = 0$ $\text{curl } \mathbf{H} = \frac{4\pi}{c} \mathbf{j} + \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t}$ $\text{div } \mathbf{D} = 4\pi\rho$
Velocity of electromagnetic waves	$v = \frac{1}{\sqrt{\epsilon\mu}}$	$v = \frac{c}{\sqrt{\epsilon\mu}}$
Relationship between the amplitudes of the vectors \mathbf{E} and \mathbf{H} in an electromagnetic wave		$E_0 \sqrt{\epsilon} = H_0 \sqrt{\mu}$
Poynting's vector	$\mathbf{S} = [\mathbf{E}\mathbf{H}]$	$\mathbf{S} = \frac{c}{4\pi} [\mathbf{E}\mathbf{H}]$
Density of momentum of electromagnetic field	$\mathbf{g} = \frac{1}{c^2} [\mathbf{E}\mathbf{H}]$	$\mathbf{g} = \frac{1}{4\pi c} [\mathbf{E}\mathbf{H}]$
Permittivity of a vacuum	ϵ_0	1
Permeability of a vacuum	μ_0	1

Supplements

S.1 Superconductivity (to Sec. 3.7)

The appearance of the microscopic theory of superconductivity relates to 1957-1958.

It is impossible to explain the mechanism of superconductivity within the confines of the classical conceptions, so that it is a victory of the quantum theory. Its essence consists in that although Coulomb forces of repulsion act between electrons, nevertheless in solids, apart from them, forces of attraction also appear between electrons. These forces are due to the fact that electrons can exchange phonons, i.e. quanta of elastic oscillations of a body. This attraction results in the formation of bound pairs of electrons near a Fermi energy surface.

The quantum laws lead to the fact that these pairs form a so-called Bose condensate having properties of superfluidity. Since these pairs of electrons have an electric charge, their superfluidity is equivalent to superconductivity [see Lynton, E. A. *Superconductivity*, 3rd ed. London, Methuen & Co. (1969); Bardeen, J. and Schrieffer, J. R. Recent Developments in Superconductivity. In: Gorter, C.J. (ed.). *Progress in Low Temperature Physics*, Vol. III. Amsterdam, North-Holland (1961); Rose-Innes, A. C. and Rhoderick, F. H. *Introduction to Superconductivity*. New York, Pergamon Press (1969)].

S.2 Antiferromagnetism and Ferrites (to Sec. 5.12)

In addition to ferromagnetism, there also exists antiferromagnetism. If in ferromagnetics the exchange forces between the atoms tend to set up the spins of all the atoms parallel to one another, then in antiferromagnetics they tend to set up the spins of adjacent atoms antiparallel to one another, which naturally results in a reduction of the permeability of a body μ . Antiferromagnetism manifests itself most vividly in that upon lowering of the temperature beginning with the antimagnetic Curie point, below which ordered orientation

of the spins is observed, a sharp drop in the magnetic susceptibility of the antiferromagnetic is noted.

Recent years have seen ferrites acquiring a great practical importance. These are magnetic materials with a very high electric resistance, which is important for application in high-frequency equipment. Ferrites are oxides (or salts) of metals. A ferrite molecule, for example, apart from Fe also includes an ion of a bivalent metal (this may also be Fe^{++}). In ferrites, similar to antiferromagnetics, the directions of the spins of adjacent atoms are antiparallel to one another (below the Curie point), but since the magnetic moments of adjacent atoms (unlike antiferromagnetics), do not numerically equal one another, the resultant magnetization does not equal zero. Thus, from the macroscopic viewpoint, ferrites are ferromagnetics [see Kittel, *C. Introduction to Solid State Physics*. New York, Wiley (1971); Mattis, D.C. *The Theory of Magnetism*. New York, Harper and Row (1965)].

S.3 Dispersive Media. Spatial Dispersion (to Sec. 7.2)

Formula (VIa) of the text for the energy density u holds for an electromagnetic field that is constant in time and may be applied to varying fields only for conditions when we may disregard the dispersion of the medium, i.e. the dependence of ε and μ on the frequency ω of the varying field (see Sec. 7.11). For the general case of a field having the frequency ω , however, the density of the electromagnetic energy in the medium is expressed by the equation*

$$u = \frac{1}{8\pi} \left[\frac{d(\omega\varepsilon)}{d\omega} \tilde{E}^2 + \frac{d(\omega\mu)}{d\omega} \tilde{H}^2 \right] \quad (\text{VIc})$$

where the tilde over E^2 and H^2 signifies the values of the squares of the *real* (and not expressed in the complex form) field intensities E and H averaged over the period of the field. For example, if $E = E_0 \cos \omega t$, where E_0 is independent of the time, then $\tilde{E}^2 = \frac{1}{2} E_0^2$. When the dependence of ε and μ on ω can be disregarded, then Eq. (VIc) coincides with Eq. (VIa) of the text.

The dependence of the permittivity ε on the field frequency is called the frequency dispersion. Great attention is being given at present to the so-called spatial dispersion. Its essence is that the polarization of a medium \mathbf{P} and, consequently, the displacement $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$ at a given point of the medium depend in the general case not only on the intensity of the electric field \mathbf{E} at the same point (as is adopted

* See, for example, Landau, L. D. and Lifshits, E. M. *Electrodinamika sploshnykh sred* (Electrodynamics of Continuous Media). Moscow, Gostekhizdat (1957), § 62.

in the conventional theory where it is assumed that $\mathbf{D} = \epsilon\mathbf{E}$, but also on the field \mathbf{E} at adjacent points of the medium or, which is the same, on the spatial derivatives of the vector \mathbf{E} . The reason for such a dependence can be seen most clearly on the example of plasma, i.e. a greatly ionized gas. If the mean thermal velocity of the free electrons in plasma is v , then during the period of one oscillation of the field having a frequency ω they travel over a path $a = v/\omega$. If a is of the order of magnitude of the wavelength λ of the field, then during the period of one oscillation an electron will visit regions of space having different fields and, consequently, the resultant displacement of the electrons determining the polarization \mathbf{P} of the medium will depend not only on the value of the vector \mathbf{E} at the given point, but also on its derivatives. The generalized dependence of the displacement \mathbf{D} on the intensity \mathbf{E} in an isotropic medium has the form

$$\mathbf{D} = \epsilon\mathbf{E} + \delta_1 \nabla^2 \mathbf{E} + \delta_2 \text{grad div } \mathbf{E} \quad (\text{A})$$

From considerations of symmetry, there can be no terms containing the first derivatives of the field in an expansion of the type of (A) for an isotropic medium. The values of the coefficients δ_1 and δ_2 in Eq. (A) for plasma can be determined from the kinetic equation for the electrons of plasma by calculating the deviations of the distribution of the electrons in space and the velocities due to the electric field from their equilibrium distribution in the absence of a field.

The consideration of the spatial dispersion can be formulated in such a way that the permittivity ϵ or, more exactly, the tensor of the permittivity ϵ_{ij} in the equations of a field is assumed to depend not only on the field frequency ω , but also on the wave vector \mathbf{k} [see Agranovich, V. M. and Ginzburg, V. L.: *Kristallogoptika s uchetom prostranstvennoi dispersii i teoriya eksitonov* (Crystal Optics with Account of Spatial Dispersion and the Theory of Excitons). Moscow, Nauka (1965)].

S.4 Anisotropic Media (to Sec. 7.2)

The consideration of the propagation of electromagnetic waves in anisotropic media becomes appreciably more complicated owing to the fact that the permittivity ϵ (and also the permeability μ) for such media is not a scalar, but a tensor of the second rank. Hence, the relationship between \mathbf{D} and \mathbf{E} in accordance with Eq. (2.16) of the text acquires the form $D_i = \epsilon_{ij}E_j$, where summation over the subscript j encountered twice is meant. It is obvious, particularly, that for anisotropic media the wavelength λ or the wave vector \mathbf{k} depends not only on the wave frequency ω , but also on the direction of propagation of the wave relative to the axes of symmetry of the medium (see Agranovich, V. M. and Ginzburg, V.L., *op. cit.*).

S.5 Vavilov-Cerenkov Effect (to Sec. 7.9)

The radiation of an oscillator, as in general the radiation of electromagnetic waves, is connected with the accelerated (periodic for an oscillator) motion of electric charges. An electric charge uniformly travelling with a constant velocity emits no electromagnetic waves. There is one exception to this general law, however—a uniformly travelling electric charge emits electromagnetic waves if its velocity v exceeds the velocity of light in the medium in which it is propagating. Naturally, no material body can travel with a velocity exceeding that of light in a vacuum c . The phase velocity of light (or in the general case of an electromagnetic wave), however, in a medium is $v_{ph} = c/n$, where n is the refractive index of the medium depending on the wave frequency. Hence, when $n > 1$, the observance of the condition $v > c/n$ needed for this radiation is possible.

The above phenomenon was discovered in the thirties and is called the *Vavilov-Cerenkov effect*, or *Cerenkov radiation*.

We can indicate as an example of Cerenkov radiation the bright blue luminescence of water placed in an atomic reactor due to the passage through the water of the fast electrons appearing upon the decay of atomic nuclei. This radiation has nothing in common with chemiluminescence. Another example of Cerenkov radiation—the radiation of plasma waves—is briefly described in the following section S.6 [see Bolotovskiy, B. M. *Svechenie Vavilova-Cerenkova* (Vavilov-Cerenkov Radiation). Moscow, Nauka (1964)].

S.6 Plasma (to Sec. 7.12)

In recent years, special interest has been attached to studying the propagation of waves in a kind of a conducting medium that in general is not mentioned in the book—in plasma, i.e. in a greatly ionized gas. These investigations are very important for geophysics and astrophysics because the Earth's ionosphere (the upper layers of the atmosphere), the solar atmosphere, nebulae, and to a considerable extent interplanetary and interstellar space are plasma. In laboratory conditions, plasma plays a very important part in gas discharges and, moreover, its properties have a dominating significance for investigations aimed at carrying out controlled thermonuclear reactions. It is planned to conduct these reactions in a high-temperature plasma placed in a strong magnetic field that should stabilize it and thermally insulate it to a considerable extent from the walls of the vessel containing it.

The electromagnetic processes that can occur in plasma are quite distinctive. We shall mention, for example, that for a greatly rarefied, so-called collisionless plasma, in which the collisions between its particles (electrons and ions) may be completely disregarded, the

dependence of the permittivity ε on the field frequency ω is determined by the equation

$$\varepsilon = 1 - \frac{\omega_0^2}{\omega^2} \quad (\text{B})$$

Here ω_0 is the so-called plasma frequency equal to

$$\omega_0 = \sqrt{\frac{4\pi e^2 N}{m}}$$

where e = charge of an electron

m = mass of an electron

N = number of free electrons in a unit volume.

Thus, when $\omega < \omega_0$ the permittivity takes on negative values. When $\omega = \omega_0$, i.e. $\varepsilon(\omega) = 0$, longitudinal electromagnetic waves, in which the vector \mathbf{E} follows the direction of propagation of the wave and the vector $\mathbf{H} = 0$, may propagate in plasma in addition to the conventional transverse waves. These longitudinal waves are called *plasma waves*.

The circumstance that the frequency ω of the longitudinal waves must comply with the condition $\varepsilon(\omega) = 0$ directly follows from the equation given on p. 514 of the text, i.e.

$$\frac{\varepsilon}{c} \frac{\partial^2 \mathbf{E}}{\partial t^2} = -\frac{c}{\mu} (\text{grad div } \mathbf{E} - \nabla^2 \mathbf{E}) \quad (\text{C})$$

For transverse waves, $\text{div } \mathbf{E} = 0$, and from Eq. (C) we directly get the wave equation (7.83). For a longitudinal wave, if it, for example, is propagating in the direction of the z -axis and the vector \mathbf{E} is directed along the same axis, we have, as is easily seen,

$$\text{grad div } \mathbf{E} = \nabla^2 \mathbf{E} = n_z \frac{\partial^2 E_z}{\partial z^2}$$

i.e. the right-hand side of Eq. (C) becomes equal to zero. Consequently, when $\partial^2 \mathbf{E} / \partial t^2 \neq 0$ it is essential that ε also become equal to zero.

It is evident that the customary dependence of the wavelength constant k on the wave frequency ω , i.e. $k = \omega \sqrt{\varepsilon \mu} / c$ [see Eq. (7.87) of the text] following for transverse waves from Eq. (C) cannot be applied to longitudinal waves. To find this relationship, we must take into consideration the spatial dispersion of the medium (plasma in the given case) that was mentioned in Sec. S.3.

We must also note that Cerenkov radiation of plasma waves occurs upon the uniform motion of an electric charge in plasma. This radiation does not occur with transverse waves because for them according to Eq. (B) we have $\varepsilon < 1$ and, consequently, the refractive index $n = \sqrt{\varepsilon}$

is less than unity. Thus, the condition for Cerenkov radiation $v > c/n$ (see Sec. S.5), where v is the velocity of a radiating electron, cannot be observed for transverse waves. For longitudinal plasma waves, we have $n > 1$, so that Cerenkov radiation occurs; the condition $v > c/n$ is observed for a considerable part of the free electrons of high-temperature plasma. But since plasma contains particles capable of emitting a wave of a given kind, then they can also absorb it. This leads to appreciable damping of the longitudinal waves propagating in plasma even in the absence of collisions.

It is also significant that the energy of beams of fast electrons introduced from outside into plasma rapidly dissipates owing to the emission by them of plasma waves according to the laws of Cerenkov radiation.

We cannot stop here to treat a number of very interesting and distinctive electromagnetic phenomena in plasma and refer our reader to special books on the subject [for example, Ginzburg, V. L. *Rasprostranenie elektromagnitnykh voln v plazme* (The Propagation of Electromagnetic Waves in Plasma). Moscow, Nauka (1967); Ginzburg, V. L. and Rukhadze, A. A. *Volny v magnitoaktivnoi plazme* (Waves in Magnetoactive Plasma). Moscow, Nauka (1975); and Artsimovich, L.A. *Elementarnaya fizika plazmy* (Elementary Physics of Plasma), 3rd ed. Moscow, Atomizdat (1969).

Solutions of Problems

1. (Page 37). Owing to symmetry, the vector \mathbf{E} is parallel (or antiparallel) to \mathbf{r} and is a function only of r . Let us consider a cylinder with the altitude 1 and the radius r . Since \mathbf{E} is parallel to \mathbf{r} , the flux of the vector \mathbf{E} through the base of the cylinder equals zero. Hence, the application of Gauss's law to the cylinder gives

for $r > a$

$$\pm 2\pi r E = 4\pi q'$$

for $r < a$

$$2\pi r E = 0$$

(here a is the radius of the charged infinite cylinder), whence we get Eq. (1.16).

The jump of the vector \mathbf{E} when passing through the surface of the cylinder is

$$E_e - E_i = \frac{2q'}{a} - 0 = 4\pi\sigma$$

because $q' = 2\pi a\sigma$.

2. (Page 37). Owing to symmetry, the vector \mathbf{E} is parallel (or antiparallel) to \mathbf{R} and is a function only of R . The application of Gauss's law to a sphere with the radius R gives

for $R > a$

$$\pm E \cdot 4\pi R^2 = 4\pi q$$

for $R < a$

$$E \cdot 4\pi R^2 = 0$$

(here a is the radius of the charged sphere), whence we get Eq. (1.17).

The jump of the vector \mathbf{E} when passing through the surface of the sphere is

$$E_e - E_i = \frac{q}{a^2} - 0 = 4\pi\sigma$$

because $q = \sigma \cdot 4\pi a^2$.

3. (Page 37). Owing to symmetry, the vector \mathbf{E} is parallel (or antiparallel) to \mathbf{R} and is a function only of R . The application of Gauss's law to a sphere with the radius R gives

for $R > a$

$$\pm E \cdot 4\pi R^2 = 4\pi \cdot \frac{4}{3} \pi a^3 \rho = 4\pi q$$

for $R < a$

$$\pm E \cdot 4\pi R^2 = 4\pi \cdot \frac{4}{3} \pi R^3 \rho = 4\pi \frac{R^3}{a^3} q$$

(because $q = \frac{4}{3} \pi a^3 \rho$), whence we get Eqs. (1.18) and (1.19).

The continuity of the vector \mathbf{E} when passing through the surface of the sphere follows from the fact that for $R = a$ we have

$$\mathbf{E}_e = \frac{4}{3} \pi \rho R = \mathbf{E}_i$$

4. (Page 41). According to Eq. (1.16) the field of the outer cylinder in the space of the capacitor equals zero. Hence, in this space, the field is induced only by the inner cylinder and is determined by Eq. (1.16). We can convince ourselves that the charges per unit length of the capacitor plates are equal in magnitude and opposite in sign by applying Gauss's law to a closed surface, part of which passes through the outer metal cylinder and part of which is confined between two parallel cross sections perpendicular to the axis of the capacitor. The electric vector flux through such a surface equals zero.

5. (Page 41). The solution of this problem is similar to that of the preceding one.

6. (Page 56). In Eq. (1.40) we choose the x -axis (for the plane) and the radius-vector \mathbf{r} perpendicular to the cylinder axis, respectively, as the path of integration. Introducing the value of the vector \mathbf{E} from Eqs. (1.15) and (1.16), we get for a plane

$$\varphi - \varphi_0 = - \int 2\pi\sigma \cdot dx = - 2\pi\sigma x$$

for a cylinder: $r > r_1$

$$\varphi - \varphi_0 = - \int_{r_1}^r \frac{2q'}{r} dr = - 2q' \ln \frac{r}{r_1}$$

$r < r_1$

$$\varphi - \varphi_0 = 0$$

7. (Page 56.) Choosing the radius-vector \mathbf{R} in Eq. (1.41) as the path of integration we get on the basis of Eq. (1.18) for points outside the sphere:

$$\varphi_e = \int_R^\infty \frac{q}{R^2} dR = \frac{q}{R}$$

Hence on the surface of the sphere $\varphi = \varphi_0 = q/a$. Using Eqs. (1.40) and (1.19) we get for points inside the sphere

$$\varphi_1 - \varphi_0 = - \int_a^R \frac{4}{3} \pi R \rho \, dR = 2\pi\rho \left(\frac{a^2}{3} - \frac{R^2}{3} \right)$$

which coincides with the equation in the text when the value of φ_0 is introduced.

8. (Page 59). It follows from Eq. (1.22) that the potential difference across the plates of a plane capacitor is $\varphi_2 - \varphi_1 = 4\pi qd/S$.

The field in the space between the plates of a cylindrical or spherical capacitor is induced only by the charge of the inner plate. Hence, in accordance with the results obtained in solving Problem 6, for a cylindrical capacitor we have

$$\varphi_2 - \varphi_1 = - 2q' \ln \frac{r_2}{r_1}$$

where $q' = q/l$, and for a spherical capacitor

$$\varphi_2 - \varphi_1 = \frac{q}{R_2} - \frac{q}{R_1} = \frac{(R_1 - R_2) q}{R_1 R_2}$$

Introducing these expressions into Eq. (1.52) and selecting a positive sign for C as usual [the sign of Eq. (1.52) depends on which of the plates is considered to be the first one and which the second], we get the required results.

9. (Page 64.) It follows from Eqs. (1.49) and (A.43) that

$$\mathbf{E} = - \operatorname{grad}_a \frac{(\mathbf{pR})}{R^3} = - \frac{1}{R^3} \operatorname{grad}_a (\mathbf{pR}) - (\mathbf{pR}) \operatorname{grad}_a \left(\frac{1}{R^3} \right)$$

Taking into consideration that on the basis of Eqs. (A.8), (A.9), and (A.11), we have

$$\operatorname{grad}_a (\mathbf{pR}) = \mathbf{p}$$

and

$$\operatorname{grad}_a \left(\frac{1}{R^3} \right) = \frac{\partial}{\partial R} \left(\frac{1}{R^3} \right) \cdot \operatorname{grad}_a R = - \frac{3\mathbf{R}}{R^5}$$

we obtain Eq. (1.61).

In a spherical coordinate system, the components of the constant vector \mathbf{p} at the arbitrary point R , θ , and α equal, respectively, $p_R = p \cos \theta$, $p_\theta = -p \sin \theta$, and $p_\alpha = 0$, so that $\mathbf{pR} = pR \cos \theta$.

Introducing this into Eq. (1.61) we get Eq. (1.62).

10. (Page 69.) Adopting the point P as the origin of coordinates, we have

$$R^2 = x^2 + y^2 + z^2$$

$$\frac{\partial R}{\partial x} = \frac{x}{R}$$

$$\frac{\partial}{\partial x} \left(\frac{1}{R} \right) = \frac{\partial}{\partial R} \left(\frac{1}{R} \right) \cdot \left(\frac{\partial R}{\partial x} \right) = - \frac{x}{R^3}$$

$$\frac{\partial^2}{\partial x^2} \left(\frac{1}{R} \right) = - \frac{1}{R^3} - x \frac{\partial}{\partial x} \left(\frac{1}{R^3} \right) = - \frac{1}{R^3} + \frac{3x^2}{R^5}$$

We get similar expressions for

$$\frac{\partial^2}{\partial y^2} \left(\frac{1}{R} \right) \text{ and } \frac{\partial^2}{\partial z^2} \left(\frac{1}{R} \right)$$

Consequently

$$\nabla^2 \left(\frac{1}{R} \right) = -\frac{3}{R^3} + 3 \frac{x^2 + y^2 + z^2}{R^5} = 0$$

11. (Page 69.) It follows from considerations of symmetry that the potential φ depends only on the coordinate x and that, consequently, the Laplace equation (1.65) becomes

$$\frac{\partial^2 \varphi}{\partial x^2} = -4\pi\rho$$

Inside the plate, $\rho = \text{const}$, and, therefore,

$$\varphi_i = -2\pi\rho x^2 + Ax + B$$

while outside the plate, $\rho = 0$, and, consequently,

$$\varphi_e = Cx + D$$

where A , B , C , and D are arbitrary integration constants. Assuming that $A = B = 0$ and determining the values of the constants C and D from the condition of continuity of φ and $\partial\varphi/\partial x$ on the surfaces of the plate, i.e. when $x = \pm a$, we get the required values of φ_i and φ_e , which help us to determine the value of the vector \mathbf{E} by Eq. (1.59).

To pass over to an infinite charged *plane*, it is sufficient to pass over to the limit $a \rightarrow 0$ and $\rho \rightarrow \infty$ in the obtained formulas, assuming in this transition that the charge per unit of surface area of the plate remains constant, i.e. $2a\rho = \sigma = \text{const}$.

12. (Page 70.) It follows from considerations of symmetry that the potential φ depends only on the distance R from the centre of the charged sphere. Assuming in Eq. (A.21) that $f(R) = \varphi$, we get from Eq. (1.63)

$$\frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 \frac{\partial \varphi}{\partial R} \right) = -4\pi\rho$$

Inside the sphere, $\rho = \text{const}$, and, therefore

$$\varphi_i = -\frac{2\pi\rho}{3} R^2 + \frac{A}{R} + B$$

while outside the sphere, $\rho = 0$, and, therefore,

$$\varphi_e = \frac{C}{R} + D$$

where A , B , C , and D are arbitrary integration constants. For φ to remain finite when $R = 0$, it is essential that $A = 0$. For φ to equal zero when $R = \infty$, it is essential that $D = 0$. The values of the constants B and C are determined from the condition of continuity of φ and $\partial\varphi/\partial R$ on the surface of the sphere $R = a$ and lead to Eq. (1.51).

13. (Page 81.) We introduce a cylindrical coordinate system whose z -axis is directed to the right and passes through the charge q , while the plane $z = 0$ coincides with the surface of the conductor. The distance from the arbitrary point P with the coordinates r and z to the charge $+q$ and $-q$ will be, respectively,

$$r_1 = \sqrt{r^2 + (z - d)^2}, \quad \text{and} \quad r_2 = \sqrt{r^2 + (z + d)^2}$$

The potential of the field in the right-hand half-space will evidently be

$$\varphi = q \left(\frac{1}{r_1} - \frac{1}{r_2} \right)$$

Introducing the values of φ , r_1 and r_2 into the expression $E_z = -\partial\varphi/\partial z$, differentiating, and then assuming that $z = 0$, we get the value of the normal component of the field at the surface of the conductor, whence by Eq. (1.20) we find σ .

We can see by direct integration that the total charge induced on the conductor equals $-q$.

14. (Page 96.) We use Eq. (1.101). Upon the surface distribution of the charge, this entire charge is at the constant potential $\varphi = q/a$. Hence,

$$U = \frac{1}{2} \int \sigma \varphi dS = \frac{1}{2} \varphi q = \frac{q^2}{2a}$$

Upon the volume distribution of the charge, we have

$$\sigma = 0, \quad \rho = \frac{q}{\frac{4}{3}\pi a^3}, \quad \text{and} \quad \varphi = 2\pi\rho \left(a^2 - \frac{R^2}{3} \right)$$

where R is the distance from the centre of the sphere (see Problem 7). Consequently,

$$U = \frac{1}{2} \int \rho \varphi dV = \frac{1}{2} \int_0^a \rho \cdot 2\pi\rho \left(a^2 - \frac{R^2}{3} \right) \cdot 4\pi R^2 dR$$

whence after integration we get the required result.

15. (Page 96.) Expressing \mathbf{E}_1 and \mathbf{E}_2 in Eq. (1.112) through the gradients of the corresponding potentials φ_1 and φ_2 , we get

$$U_{12} = \frac{1}{4\pi} \int (\nabla\varphi_1) (\nabla\varphi_2) dV$$

Let us separate from the total field a small sphere S encircling the charge q_2 . Ignoring the fraction of the mutual energy localized inside this sphere and using Green's theorem (A.52), we can write

$$U_{12} = \frac{1}{4\pi} \oint_S \varphi_1 \frac{\partial\varphi_2}{\partial n} dS - \frac{1}{4\pi} \int_{V'} \varphi_1 \nabla^2 \varphi_2 dV$$

In the space V' outside the sphere S , we have $\nabla^2 \varphi_2 = 0$. Consequently, the last integral vanishes. If the charges q_1 and q_2 are sufficiently remote from

each other, then the potential φ_1 can be considered constant on the surface of the sphere S and equal to q_1/R . Hence,

$$U_{12} = \frac{\varphi_1}{4\pi} \oint \frac{\partial \varphi_2}{\partial n} dS = - \frac{q_1}{4\pi R} \oint E_{2n} dS$$

Taking into account that in the given case the normal must be considered directed *into* the sphere S , and using Gauss's law (1.11), we get the required result.

16. (Page 121.) By definition ($i = 1, 2$), we have

$$\tan \beta_i = \frac{E_{it}}{E_{in}}$$

Using Eqs. (2.24) and (2.25), we get the required result.

17. (Page 121.) The statements made follow directly from Eq. (2.25) for a slot parallel to \mathbf{E} , and from Eq. (2.24) (when $\sigma = 0$) for a slot perpendicular to \mathbf{E} . At the end portions of the slot, however, the surfaces bounding it converge at an angle or, at any rate, have a considerable curvature, and the direction of a normal to them sharply changes so that the field in the adjacent portions of the slot is far from being uniform.

18. (Page 121.) The conditions for the unambiguity of the solution of the complete system (A) of equations for an electrostatic field given in the present problem differ from those indicated on p. 119 in that here the distribution of the charges σ on the surface of the conductors is not given for them. Hence, in the formula used there

$$\int_V \epsilon E''^2 dV = - \oint_S D_n'' \varphi'' dS$$

all the surfaces of the conductors must be included in the surface S . But on each such surface, either $\varphi'' = 0$ (problem A) or $\varphi'' = \text{const}$ (problem B). In the latter case, we can put φ'' outside the integral, and we get

$$\oint D_n'' dS = \oint D_n dS - \oint D_n' dS = 4\pi q - 4\pi q' = 0$$

Hence, all the integrals over these surfaces vanish in both cases, and the proof of the unambiguity of the solution given on p. 119f remains in force.

19. (Page 121.) According to Eqs. (2.24) and (2.26), the electric displacement D will have the constant value $D = 4\pi\sigma$, where σ is the density of the charge on the capacitor plates, in the entire space between the plates. The field intensity in the first and second layers will be $E_1 = 4\pi\sigma/\epsilon_1$ and $E_2 = 4\pi\sigma/\epsilon_2$, respectively. Finally, the potential difference between the plates will be $\varphi_2 - \varphi_1 = E_1 d_1 + E_2 d_2$. It is now easy to find C from the relationship $q = S\sigma$ and Eq. (1.52).

20. (Page 183.) It follows from considerations of symmetry that the resultant \mathbf{F} of the tensile stresses applied to the dielectric should be perpendicular to its surface, i.e. parallel to the z -axis. Hence,

$$F = F_z = \int_S T_{zz} dx dy$$

where integration is performed over the surface of the dielectric $z = 0$, and T_{zz} is the value of the corresponding stress tensor component from the *outer* side of

the surface of the dielectric. Passing over to the cylindrical coordinates z , r , and α , we get

$$F = \int_0^{\infty} T_{zz} 2\pi r \, dr$$

Assuming in Eq. (2.38) that $\epsilon_1 = 1$ and $\epsilon_2 = \epsilon$, we see that the field outside the dielectric equals the sum of the field $q\mathbf{R}/R^3$ of the charge q and the field $\frac{1-\epsilon}{1+\epsilon} \frac{q\mathbf{R}'}{R'^3}$ of the fictitious charge $\frac{1-\epsilon}{1+\epsilon} q$ placed at the point P' that is the image of the point where the charge q is in the flat surface of the dielectric. Hence, at the outer surface of the dielectric, we have

$$E_z = -\frac{qz_0}{R^3} + \frac{1-\epsilon}{1+\epsilon} \frac{qz_0}{R^3} = -\frac{2\epsilon qz_0}{(1+\epsilon)R^3}$$

$$E_r = \frac{qr}{R^3} + \frac{1-\epsilon}{1+\epsilon} \frac{qr}{R^3} = \frac{2qr}{(1+\epsilon)R^3}$$

$$E_\alpha = 0$$

where $R = \sqrt{r^2 + z_0^2}$.

Since in a vacuum $\mathbf{T}'' = 0$, and therefore $\mathbf{T} = \mathbf{T}'$, then on the basis of Eq. (2.122)

$$T_{zz} = \frac{1}{4\pi} (E_z^2 - \frac{1}{2} E_r^2) = \frac{1}{8\pi} (E_z^2 - E_r^2)$$

Consequently,

$$F = \frac{q^2}{(1+\epsilon)^2} \int_0^{\infty} r \, dr \frac{(\epsilon^2 z_0^2 - r^2)}{(z_0^2 + r^2)^3} = \frac{\epsilon - 1}{4(\epsilon + 1)} \frac{q^2}{z_0^2}$$

We invite our reader to obtain the same result by determining the resultant of the stresses applied to an arbitrarily small sphere surrounding the charge q . **21.** (Page 195.) Owing to the absence of charges inside a homogeneous conductor [Eq. (3.22)], we can repeat our reasoning which in Problem 12 led us to the formula $\varphi = \frac{C}{R} + D$. This will be the potential inside the capacitor, while the potential difference across its plates will be

$$\varphi_1 - \varphi_2 = C \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \quad (\text{a})$$

The field \mathbf{E} will be directed along the radius of the sphere ($E = |E_R|$), and

$$E_R = -\frac{\partial \varphi}{\partial R} = \frac{C}{R^2}, \quad j_R = \kappa E_R = \frac{\kappa C}{R^2}$$

Finally, the current flowing through the capacitor, i.e. through the spherical surface having the arbitrary radius R concentric to its plates (provided that $R_1 < R < R_2$) will be

$$I = \oint j_R dS = \kappa C \oint \frac{dS}{R^2} = 4\pi C\kappa \quad (\text{b})$$

Dividing Eq. (a) by Eq. (b) by terms, we get the required value of the resistance R on the basis of Eq. (3.1).

22. (Page 195.) By definition ($i = 1, 2$)

$$\tan \beta_i = \frac{j_{it}}{j_{in}}$$

Using Eqs. (3.20), (3.13), and (1.37), we get the required result.

23. (Page 195.) Let us use the subscript 1 for the conductor and the subscript 2 for the dielectric surrounding it. Since $j_{2n} = 0$, then owing to Eq. (3.20) we also have $j_{1n} = 0$. Hence, on the basis of Eq. (3.13), we have $E_{1n} = 0$ and $D_{1n} = 0$. Therefore, Eq. (2.23) gives the required result:

$$D_{2n} = 4\pi\sigma$$

The vectors \mathbf{E} and \mathbf{D} stop being perpendicular to the surface of the conductor because their tangential components no longer equal zero as in the absence of a current. The latter circumstance follows from Eqs. (2.25) and (3.13).

24. (Page 200.) Damaging of the insulation means earthing of it through a certain unknown resistance r . In the first case, the current I induced by the battery e.m.f. \mathcal{E}_{ext} branches at the point of damage to the insulation into two currents I_1 and I_2 that flow through the resistances r and $(c-x)s$ into the earth (s is the resistance of a unit length of the telegraph line). Applying Kirchhoff's first and second laws (3.16) and (3.30) to this case, we get (assuming that the resistance of the earth and also that of the earthing at the stations equal zero):

$$I = I_1 + I_2$$

$$Ixs + I_1r = \mathcal{E}_{\text{ext}}, \quad -I_1r + I_2(c-x)s = 0$$

We get similar equations for the second case. For the third case, we have only one equation, because here only one closed circuit remains, so that $I_2'' = 0$ and $I_1'' = I''$. We succeed in excluding seven unknowns, i.e. $I_1, I_2, I_1', I_2', r, \mathcal{E}_{\text{ext}}$, and s from the seven equations obtained. As a result, we get

$$x = \frac{(bk - ck')(1 + p)}{k - k'}$$

where

$$k = \frac{I''}{I - I''}, \quad k' = \frac{I''}{I' - I''}, \quad \text{and} \quad p = \sqrt{\frac{c - b}{bk - ck'}}$$

25. (Page 222.) For a rectilinear current, the vector $[ds \mathbf{R}]$ in Eq. (4.4) has the same direction for all the current elements (with a fixed observation point). Therefore, the numerical value of the vector \mathbf{H} equals the sum of the numerical values of the integrand in Eq. (4.4)

$$H = \frac{I}{c} \int \frac{ds \sin(ds, \mathbf{R})}{R^3}$$

If r is the length of a perpendicular dropped from the observation point onto the straight line of the current, and α is the angle between r and R , then

$$R = \frac{r}{\cos \alpha} \text{ and } ds \sin (ds, \mathbf{R}) = R d\alpha$$

Therefore,

$$H = \frac{I}{c} \int_{-\pi/2}^{\pi/2} \frac{d\alpha}{R} = \frac{I}{rc} \int_{-\pi/2}^{\pi/2} \cos \alpha d\alpha = \frac{2I}{cr}$$

It is very easy to convince oneself of the truth of the statements made in the text on the shape and direction of the lines of force.

26. (Page 222.) The radius-vector from an arbitrary current element ds to a point on the axis of the current is $\mathbf{R} = \mathbf{R}_0 + \mathbf{d}$ (if we consider the vector \mathbf{R}_0 to be directed from the circumference to the centre), therefore $[ds \mathbf{R}] = [ds \mathbf{R}_0] + [ds \cdot \mathbf{d}]$. If we insert this result into Eq. (4.4), then owing to symmetry of the problem, the integral with respect to $[ds \cdot \mathbf{d}]$ will vanish because the vector $[ds \cdot \mathbf{d}]$ is parallel to \mathbf{R}_0 . The vector $[ds \mathbf{R}_0]$, however, is parallel to the vector $-\mathbf{d}$ and does indeed form a right-handed system with the direction of the current. Finally, owing to \mathbf{R}_0 and \mathbf{d} being perpendicular to each other, we have $R^2 = R_0^2 + d^2$. Thus, the absolute value of the vector \mathbf{H} is

$$H = \frac{I}{c} \oint \frac{ds R_0}{(R_0^2 + d^2)^{3/2}} = \frac{I 2\pi R_0^2}{c(R_0^2 + d^2)^{3/2}}$$

27. (Page 226.) According to the rules of vector multiplication, we have

$$[ds_2 [ds_1 \mathbf{R}_{12}]] = ds_1 (ds_2 \mathbf{R}_{12}) - \mathbf{R}_{12} (ds_1 ds_2)$$

We introduce this into Eq. (4.6) and integrate over the contours L_1 and L_2 of both currents. The integral of the first term is [cf. Eq. (A.10)]

$$\begin{aligned} \oint_{L_1} \oint_{L_2} ds_1 \frac{(ds_2 \mathbf{R}_{12})}{R_{12}^3} &= - \oint_{L_1} ds_1 \oint_{L_2} \left(ds_2 \operatorname{grad}_a \frac{1}{R_{12}} \right) = \\ &= - \oint_{L_1} ds_1 \oint_{L_2} \left(\frac{\partial}{\partial s_2} \left(\frac{1}{R_{12}} \right) \right) ds_2 = 0 \end{aligned}$$

because the integrand of the second integral is a total differential. Consequently, the resultant force \mathbf{f}_{12} with which the entire first current acts on the second one is

$$\mathbf{f}_{12} = - \frac{I_1 I_2}{c^2} \oint_{L_1} \oint_{L_2} \frac{\mathbf{R}_{12} (ds_1 ds_2)}{R_{12}^3}$$

We get the corresponding expression for \mathbf{f}_{21} by changing the places of the subscripts 1 and 2. Taking into account that $\mathbf{R}_{12} = -\mathbf{R}_{21}$, we get

$$\mathbf{f}_{12} = -\mathbf{f}_{21}$$

28. (Page 226.) Let us consider the difference between Eq. (4.6) for the force F_{12} and its expression given by Ampère. Omitting the factor $I_1 I_2 / c^2$, we get the following expression for this difference (compare with the preceding problem):

$$\begin{aligned} & \frac{ds_1}{R_{12}^3} (\mathbf{R}_{12} ds_2) - \frac{\mathbf{R}_{12}}{R_{12}^3} (ds_1 ds_2) - \frac{3\mathbf{R}_{12}}{R_{12}^5} (ds_1 \mathbf{R}_{12}) (ds_2 \mathbf{R}_{12}) + \\ & + \frac{2\mathbf{R}_{12}}{R_{12}^3} (ds_1 ds_2) = \frac{ds_1}{R_{12}^3} (\mathbf{R}_{12} ds_2) + \frac{\mathbf{R}_{12}}{R_{12}^3} (ds_1 ds_2) - \\ & - \frac{3\mathbf{R}_{12}}{R_{12}^5} (ds_1 \mathbf{R}_{12}) (ds_2 \mathbf{R}_{12}) \end{aligned}$$

Our task consists in proving that the integral of this expression over the closed contour of the current I_1 equals zero. That this is indeed true follows from the fact that the above expression coincides with the last term of Eq. (4.10) if we assume in the latter that

$$f(R_{12}) = -\frac{1}{R_{12}^3} \text{ and } \Phi(R_{12}) = 0$$

Indeed, taking into account that $d\mathbf{R}_{12} = -ds_1$ (see p. 225), we have [see also Eq. (A.9)]

$$\begin{aligned} -d \left(\mathbf{R}_{12} ds_2 \mathbf{R}_{12} \frac{1}{R_{12}^3} \right) &= ds_1 \frac{(ds_2 \mathbf{R}_{12})}{R_{12}^3} - \frac{\mathbf{R}_{12} \{ ds_2 (-ds_1) \}}{R_{12}^3} - \\ - \mathbf{R}_{12} (ds_2 \mathbf{R}_{12}) \left(\text{grad}_q \frac{1}{R_{12}^3}, ds_1 \right) &= \frac{ds_1 (ds_2 \mathbf{R}_{12})}{R_{12}^3} + \frac{\mathbf{R}_{12} (ds_2 ds_1)}{R_{12}^3} - \\ - \frac{3\mathbf{R}_{12}}{R_{12}^5} (ds_2 \mathbf{R}_{12}) (\mathbf{R}_{12} ds_1) \end{aligned}$$

29. (Page 229.) Let us conduct through the point P inside the cylinder a number of meridian planes dividing the surface of the cylinder into a number of rectangular strips (Fig. 107). The width of opposite strips is $ds_1 = r_1 d\varphi$ and $ds_2 = r_2 d\varphi$,

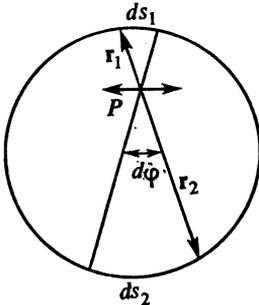


Fig. 107

respectively. The currents flowing through these strips are proportional to their width:

$$dI_1 = k ds_1 = kr_1 d\varphi \text{ and } dI_2 = k ds_2 = kr_2 d\varphi$$

Hence, the field intensities dH_1 and dH_2 induced by each of these rectilinear current strips at the point P equal one another [see Eq. (4.5)]:

$$dH_1 = \frac{2dI_1}{cr_1} = \frac{2k d\varphi}{c} = \frac{2dI_2}{cr_2} = dH_2$$

Since, in addition, the vectors $d\mathbf{H}_1$ and $d\mathbf{H}_2$ are directed oppositely to each other, then the fields of each pair of opposite strips mutually compensate each other.

30. (Page 240.) Owing to symmetry, the vector \mathbf{H} is a function only of r . To find the component H_α (we introduce the cylindrical coordinates r , α , and z), we use Eq. (4.39), in which we take a circle of the radius r concentric to the current as the closed curve L . We get

$$H_\alpha 2\pi r = \frac{4\pi}{c} \int_S j_n dS$$

For $r \geq r_0$

$$\int j_n dS = I$$

for $r < r_0$ (since $j_n = \frac{I}{\pi r_0^2}$)

$$\int j_n dS = \frac{I}{\pi r_0^2} \pi r^2 = I \frac{r^2}{r_0^2}$$

whence we find the values of $H_{e\alpha}$ and $H_{i\alpha}$. As regards the component H_z , it equals zero because the intensity of the field of each of the current elements according to Eq. (4.12) is perpendicular to the direction of the current z . Finally, it follows from Eqs. (4.36) and (A.22) that $rH_r = \text{const}$, which owing to the finite nature of the vector \mathbf{H} when $r = 0$ can take place only when $H_r = 0$.

31. (Page 248.) Let us introduce the cylindrical system of coordinates z , r , and α whose z -axis coincides with the axis of the cylindrical conductor. The magnetic lines of force of the field of each of the rectilinear current filaments into which the current being considered can be divided are circles whose plane is perpendicular to the z -axis. This circumstance, and also the symmetry of the problem give us grounds to assume that the magnetic lines of force of the field of the current being considered will also be circles concentric to the cylindrical conductor, i.e. that $H_z = H_r = 0$ and $H = H_\alpha$. Owing to the symmetry of the problem, H_α can depend on the coordinate r , so that the equation $\text{div } \mathbf{H} = 0$ will be satisfied. Further, owing to the absence of *space* currents, $\text{curl } \mathbf{H} = 0$. On the basis of Eq. (A.32) and the assumptions made relative to the components of the vector \mathbf{H} , this equation is transformed into the equation $\partial(rH_\alpha)/\partial r = 0$, whence $H_\alpha = a/r$, where a is a constant.

Inside the cylinder, the value of the constant in this expression should equal zero because otherwise H_α on the axis of the conductor would become equal to infinity when $r = 0$. Hence, inside the cylinder, $H = 0$, i.e. there is no field. The value of the constant outside the cylinder is determined from the boundary condition (4.51), which in our case acquires the form

$$(H_\alpha)_{r=r_0} = \frac{a}{r_0} = \frac{4\pi i}{c}$$

where r_0 = radius of the hollow cylindrical conductor

i = surface density of the current related to the total current I flowing through the cylinder by the equation $2\pi r_0 i = I$.

Consequently, $a = 2I/c$, and the field intensity outside the cylinder is $H = H_\alpha = 2I/cr$, Q.E.D.

According to the general theorem, the solution of the system of equation (B) which we have obtained is the only one.

32. (Page 319.) According to the results of Problem 30, the magnetic lines of the current being considered when $\mu = \mu' = \mu'' = 1$ are concentric circles. If portions of a field that are inhomogeneous in a magnetic respect are divided by coaxial cylindrical surfaces, then this circular symmetry of the magnetic lines cannot be violated. The use of Eq. (4.39) remaining in force for an arbitrary medium leads in our case to the same values of the vector \mathbf{H} as in Problem 30.

33. (Page 323.) See the solution of Problem 17.

34. (Page 370.) From the symmetry of the problem and by analogy with the electric field of a dielectric sphere polarized by a homogeneous external field [see Eq. (2.43)], it follows that the field \mathbf{H}_i inside the magnet will be homogeneous, while the field \mathbf{H}_e outside the magnet will be the field of a dipole. Accordingly, we presume that the scalar potentials of the internal and external magnetic fields are

$$\varphi_i = -zH_i \text{ and } \varphi_e = \frac{z\mu_m}{R^3} \quad (\text{a})$$

it being assumed that the z -axis is selected along the direction of the permanent magnetization \mathbf{M}_0 of the magnet. From the condition of continuity of the potential on the surface of the magnet, we get

$$-H_i = \frac{\mu_m}{a^3} \quad (\text{b})$$

where a = radius of the magnet

μ_m = magnetic moment.

Further, the condition of continuity of the normal component of the magnetic induction on the surface of the magnet has the form (when $R = a$)

$$B_{iR} = \mu_i H_{iR} + 4\pi M_{0R} = B_{eR} = \mu_e H_{eR} \quad (\text{c})$$

The radial components of the magnetic field are

$$H_{iR} = -\frac{\partial \varphi_i}{\partial R} = H_i \cos \theta \text{ and } H_{eR} = -\frac{\partial \varphi_e}{\partial R} = \frac{2\mu_m}{R^3} \cos \theta$$

where θ is the polar angle measured from the z -axis.

It therefore follows from Eq. (c) that

$$\mu_i H_i + 4\pi M_0 = \frac{2\mu_e \mu_m}{a^3} \quad (\text{d})$$

From Eqs. (b) and (d), we finally get

$$\mu_m = \frac{4\pi M_0 a^3}{\mu_i + 2\mu_e}$$

Thus, in accordance with Eq. (5.134), the potential φ_e and, consequently, the external field \mathbf{H}_e of a spherical permanent magnet are inversely proportional to the quantity $\mu_i + 2\mu_e$.

35. (Page 402.) We apply Eq. (6.14) in which we assume that

$$\mathcal{E}_{\text{ext}} = 0 \text{ and } \mathcal{E}'_{\text{ind}} = \mathcal{E}'_{\text{ind},e} + \mathcal{E}'_{\text{ind},l}$$

where $\mathcal{E}'_{\text{ind},l} = -L'(dI'/dt)$ is the self-induced e.m.f. The general solution of the non-homogeneous differential equation obtained in this way, i.e.

$$I'R' + L' \frac{dI'}{dt} = \Psi'_0 \omega \sin \omega t \quad (\text{a})$$

is expressed by the sum of an arbitrary partial solution of this equation and the general solution $a \exp\left(-\frac{R'}{L'} t\right)$ of the corresponding homogeneous equation

$$I'R' + L' \frac{dI'}{dt} = 0$$

In a sufficiently long interval of time, the last term of this sum becomes as small as desired owing to attenuation, and only the first term remains.

We shall look for the partial solution of the non-homogeneous equation in the form

$$I' = I'_0 \sin(\omega t - \varphi)$$

Introducing this expression for I' into our equation (a), expanding $\sin(\omega t - \varphi)$ and $\cos(\omega t - \varphi)$, and equating the coefficients of $\sin \omega t$ and $\cos \omega t$ in both sides of the equation, we have

$$I'_0 R' \cos \varphi + L' I'_0 \omega \sin \varphi = \Psi'_0 \omega \text{ and } -I'_0 R' \sin \varphi + L' I'_0 \omega \cos \varphi = 0$$

whence we get the required value of I'_0 .

To find the work needed to maintain the rotation of the contour, we shall note that the potential function of the current in the external magnetic field, according to Eq. (5.44), is (in practical units)

$$U' = -I'\Psi' = -I'\Psi'_0 \cos \omega t$$

Consequently, during the element of time dt , the forces of the magnetic field must do the work [see Eq. (6.20)]

$$-(dU)_I = I'\Psi'_0 d \cos \omega t = -\omega I'\Psi'_0 \sin \omega t dt$$

The expenditure of work from outside should be equal in magnitude and opposite in sign. Hence, the expenditure of external work during one period is

$$\omega I'_0 \Psi'_0 \int_0^T \sin(\omega t - \varphi) \sin \omega t dt = \frac{1}{2} \omega I'_0 \Psi'_0 \cos \varphi T$$

Using here the values of $\cos \varphi$ and I'_0 and recalculating per unit time, we get the expression given in the text.

36. (Page 445.) Since according to Sec. 6.11 the presence of a voltage applied from outside is equivalent to the presence in the circuit of the corresponding extraneous e.m.f., then Eq. (6.111) in our case will acquire the form

$$I'R' = \mathcal{E}'_{\text{ind}} + \varphi_1 - \varphi_2 + \mathcal{E}'_{\text{cd}}$$

or [cf. Eq. (6.105)]

$$R'I' = \frac{q'}{c'} - L' \frac{dI'}{dt} + \mathcal{E}'_0 \cos \omega t$$

Differentiating this equation with respect to t and using Eq. (6.104), we finally get

$$L' \frac{d^2 I'}{dt^2} + R' \frac{dI'}{dt} + \frac{1}{C'} I' = -\omega \mathcal{E}'_0 \sin \omega t$$

The general solution of this non-homogeneous equation is expressed by the sum of the general solution of the corresponding homogeneous equation (6.107) and an arbitrary partial solution of the non-homogeneous equation being considered. The first addend is the natural oscillations of the contour whose amplitude owing to attenuation with time tends to zero. As regards the partial solution of the equation being considered, we can seek it in the form

$$I' = I'_0 \cos(\omega t - \varphi)$$

Introducing this expression into our equation and equating the coefficients of $\sin \omega t$ and $\cos \omega t$ in the right-hand and left-hand sides of the equation (cf. Problem 35), we get two relationships for determining I'_0 and $\tan \varphi$.

37. (Page 512.) It follows from Eq. (7.59) that for harmonic oscillations of the oscillator, the mean square of the current is

$$\bar{I}^2 = \frac{1}{l^2} \overline{\left(\frac{\partial p}{\partial t}\right)^2} = \frac{\omega^2}{l^2} p_0^2 \overline{\cos^2 \omega t} = \frac{\omega^2 p_0^2}{2l^2}$$

Using the relationship $\omega = 2\pi c/\lambda$ and introducing the expression obtained for p_0^2 into Eq. (7.80), we get

$$\bar{I}^2 = \frac{2}{3c} \left(\frac{2\pi l}{\lambda}\right)^2 \bar{I}^2$$

Q.E.D.

38. (Page 512.) Assume that the magnetic moment of the system is

$$\boldsymbol{\mu}(t) = \boldsymbol{\mu}_0 \cos \omega t$$

where $\boldsymbol{\mu}_0$ is a constant vector. In the wave zone, i.e. when $R \gg c/\omega = \lambda/2\pi$, the first term of Eq. (7.63) can be disregarded in comparison with the second one, so that

$$\mathbf{A} = -\frac{\omega}{cR^2} [\boldsymbol{\mu}_0 \mathbf{R}] \sin \omega \left(t - \frac{R}{c}\right)$$

In a spherical system of coordinates with its centre at a dipole and whose axis is directed along the vector $\boldsymbol{\mu}_0$, we have

$$A_R = A_\theta = 0 \text{ and } A_\alpha = -\frac{\omega}{cR} \mu_0 \sin \theta \cdot \sin \omega \left(t - \frac{R}{c}\right)$$

Since the scalar potential of the field of the magnetic dipole equals zero, then

$$\mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}$$

and

$$E_R = E_\theta = 0, E_\alpha = \frac{\omega^2}{c^2 R} \mu_0 \sin \theta \cdot \cos \omega \left(t - \frac{R}{c} \right)$$

Further, calculating $\mathbf{H} = \text{curl } \mathbf{A}$ with the aid of Eq. (A.31) and limiting ourselves to senior terms whose denominator contains R to the first power, we get $H_R = H_\alpha = 0$ and $H_\theta = E_\alpha$.

39. (Page 594.) It follows from Maxwell's equation (II) that a static electric field is a potential not only in stationary media, but also in moving ones:

$$\text{curl } \mathbf{E} = 0 \text{ and } \mathbf{E} = -\nabla \varphi$$

From Eqs. (8.16) and (8.19), we find that

$$\mathbf{j}_1 = \rho \mathbf{u} + \kappa_1 \left(-\nabla \varphi_1 + \left[\frac{\mathbf{u}}{c} \mathbf{B} \right] \right) \text{ and } \mathbf{j}_2 = -\kappa_2 \nabla \varphi_2$$

where the subscripts 1 and 2 relate to the iron sphere and the electrolyte, respectively. In the expressions for \mathbf{j}_1 , we can replace the magnetic induction \mathbf{B} in the rotating sphere with the value \mathbf{B}_0 of this induction in a *stationary* sphere because the relative difference between \mathbf{B} and \mathbf{B}_0 should be of the order of magnitude of u/c , while in the expression for \mathbf{j}_1 this difference is multiplied, in turn, by u/c , i.e. results in terms of the order of magnitude u^2/c^2 , which we disregard. The induction \mathbf{B}_0 in the sphere, according to the initial conditions, is constant in magnitude and is directed along the axis of rotation, so that

$$[\mathbf{uB}_0] = [[\boldsymbol{\omega r}] \mathbf{B}_0] = \omega B_0 \mathbf{r}$$

where \mathbf{r} is the vector distance of a point from the axis of rotation. Hence,

$$\mathbf{j}_1 = \rho \mathbf{u} - \kappa_1 \nabla \varphi_1 + \frac{\kappa_1 \omega}{c} B_0 \mathbf{r}$$

Introducing the expressions for \mathbf{j}_1 and \mathbf{j}_2 into the equation

$$\text{div } \mathbf{j} = 0$$

which on the basis of the continuity equation holds for any system of steady currents, we get

$$-\nabla^2 \varphi_1 + \frac{2\omega}{c} B_0 = 0 \text{ and } \nabla^2 \varphi_2 = 0 \quad \backslash \quad (\text{A})$$

because $\text{div } \mathbf{r} = 2$ and $\text{div } (\rho \mathbf{u}) = 0$. Indeed, in the spherical system of coordinates R , θ , and α , the vector $\rho \mathbf{u}$ has only one component ρu_α differing from zero, and the numerical value of this component owing to the axial symmetry of the problem does not depend on the angle α . Hence, it follows on the basis of Eq. (A.20) that $\text{div } (\rho \mathbf{u}) = 0$.

On the surface of the sphere, i.e. when $R = a$, first, the electric potential must be continuous, i.e.

$$\varphi_1 = \varphi_2 \quad (\text{B})$$

and, second, the normal component of the current density must be continuous, i.e.

$$j_{1R} = j_{2R}$$

The last equation can be written as follows (when $R = a$):

$$-\kappa_1 \frac{\partial \varphi_1}{\partial R} + \frac{\kappa_1 \omega}{c} B_0 R \sin^2 \theta = -\kappa_2 \frac{\partial \varphi_2}{\partial R} \quad (\text{C})$$

because the component of the vector \mathbf{r} in the direction of the radius-vector \mathbf{R} equals $r \sin \theta = R \sin^2 \theta$. Finally, it is essential that the potential be finite everywhere and tend to zero at infinity.

The expressions for φ_1 and φ_2 given in the text, as can easily be seen, comply with Eqs. (A), (B), (C) and the condition for infinity; their combination determines the potential unambiguously.

Name Index

- Abraham, M., 169, 420, 457, 547
Agranovich, V. M., 653
Ampere, A. M., 226, 291, 666
Artsimovich, L. A., 656
- Bardeen, J., 651
Bolotovskiy, B. M., 654
Born, M., 523
- Coulomb, C. A., 290
Curie, P., 341
- Debye, P., 148
Dellenbach, 579
- Eikhenvald, A. A., 583, 595
Einstein, A., 425, 576, 606, 607
- Faraday, M., 457, 572
Fizeau, A. I., 600
Foucault, J. B. L., 383
Fresnel, A. J., 513, 600
- Ginzburg, V. L., 653, 656
Green, G., 513
- Heaviside, O., 275
Helmholtz, H. L., 177, 205
Hertz, H. R., 456, 595
- Jeans, J., 177
- Kamerlingh Onnes, H., 216
Kittel, C., 150, 652
Kochin, N. E., 474
Kohn, W., 420
- Landau, L. P., 652
Landsberg, G. S., 513
Langevin, P., 351
Laplace, P. S., 66
Lebedev, P. N., 539
- Libin, Z., 446
Lifshits, E. M., 652
Lorentz, H. A., 130, 231, 474, 486,
578, 579, 600
Lynton, E. A., 651
- Mattis, D. C., 343, 345, 652
Maxwell, J. C., 169, 177, 178, 420,
421, 423, 438, 456, 457, 572
Michelson, A. A., 383
- Neumann, F., 513
- Oersted, H. C., 291
- Panofsky, W. K. H., 70
Pauli, W., 579
Phillips, M., 70
Poisson, S. D., 66
- Rhoderick, F. H., 651
Roentgen, W. K., 595
Rose-Innes, A. C., 651
Rukhadze, A. A., 656
- Schrieffer, J. R., 651
Sommerfeld, A., 160, 210, 383
Stratton, J. A., 167, 367
- Tolman, R. C., 207, 210
- Umov, N. A., 465
- Vavilov, S. I., 383
- Weber, W. E., 291
Weiss, P., 348
Wilson, C., 595, 596
Winer, O., 513
Wolf, E., 523
- Zeeman, P., 600
Ziman, J. M., 454

Subject Index

- Abundance, field sources, 43
- Acceleration, influence on medium motion, 587
- Accumulator, lead-acid, 206
- Ampere, 185
 - international, 300f
- Angle,
 - critical, 523
 - incidence, 523
 - reflection, 523
 - refraction, 523
- Angular momentum,
 - atom, 346
 - electromagnetic, 540ff
 - electrons, 332f
 - orbital, 344
 - spin, 344
- Antiferromagnetism, 651f
- Atom(s),
 - angular momentum, 346
 - magnetic field, 305
 - magnetic moment, 332f, 343f, 355
 - magnetic properties, 305

- Barriers, conditional, 270f, 273f
- Bodies, charged,
 - interaction, 23
 - ponderomotive forces on, 28

- Capacitance,
 - capacitor, 59
 - cylindrical, 59f
 - parallel-plate, 59, 649
 - spherical, 60
 - isolated conductor, 57
 - isolated sphere, 57
 - units, 57

- Capacitor,
 - capacitance, 59
 - charge, 59

- Capacitor,
 - cylindrical, 41
 - capacitance, 59f
 - in uniform magnetic field, 539ff
 - electric field, 39ff
 - energy, 89f, 101
 - parallel-plate,
 - capacitance, 59, 649
 - with dielectric, 121f
 - electric field, 39f
 - energy, 100
 - finite, 40
 - force of plate attraction, 100
 - infinite, 39f
 - plane, *see* Capacitor, parallel-plate
 - potential difference, 101
 - in quasistationary current circuit, 441ff
 - spherical, 41
 - capacitance, 60
 - charge, 59
 - resistance, 195
 - in varying current circuit, 440

- Centimetre, 57

- Charge(s),
 - bound, 113
 - density in dielectric, 136f
 - potential, 113ff
 - capacitor, 59
 - density, 42, 577ff
 - surface, 33
 - true, 134
 - in uniformly polarized dielectric, 115

- distribution,
 - and electric field, 45
 - in steady current field, 192

- energy, 87f, 94ff
 - in external field, 88f, 111f
 - interaction, 86ff
 - intrinsic, 94f
 - potential, 104
 - total, 94f

- Charge(s),
 electric moment, 54, 109f
 fixed, electric field, 46
 force of interaction, 23f
 forces on, 96, 231
 free, 113
 ponderomotive forces on, 168
 potential, 113
 magnetic, 290f
 bound, 359
 density, 358f, 364f
 fictitious, 271f
 permanent, 359
 density, 358, 364f
 forces on, 360f, 364
 moving, force on, 231
 point, 47f, 52
 definition, 23, 25f
 electric field, 32, 80
 energy, 86ff
 field intensity in dielectrics, 122ff
 field potential, 52
 mutual energy, 87f, 96
 potential, 648
 in dielectrics, 122ff
 system instability, 104ff
 potential, 110f
 scalar potential, 493f
 sign, 25
 space, 52
 potential, 70ff
 surface, 33, 52
 electric displacement, 118f
 potential, 75f
 test, 27f
 force on, 182
 true, *see* Charge(s), free
 units, 24f
 volume density, 42
 volume distribution, 41f, 98
- Circuit, energy transformations, 202ff, 396f
- Coefficient, *see also* Constant, Factor
 Fresnel drag, 600
 Hall, 232f
 polarizability, dielectric, 116f
- Conductivity, 189
 metal, 211f
 and temperature, 215
- Conductor(s),
 capacitance, 57
 classes, 205
 cylindrical, skin effect, 450ff
 definition, 38
 in electric field, 38ff
 electrostatic shielding, 58
 first kind, 205
 ideal, 550
- Conductor(s)
 potential, 56ff
 second kind, 205
 self-inductance, 259
 and current frequency, 454
 total current, 452
 total energy, 95f
- Constant, *see also* Coefficient, Factor
 Boltzmann, 145
 dielectric, *see* Permittivity, dielectrics
 electromagnetic, 223f
 dimension, 296
 experimental determination, 295f
 value, 296, 490
 Planck, 214
 wavelength, 516
 complex, 529
- Constraints, non-electrostatic, 107
- Coulomb, 25
- Curl, 628, 632f
 surface, 245
- Current(s),
 alternating,
 conductor resistance and frequency, 450
 non-quasistationary, 561
 voltage, 430ff
 amplitude, 445
 maximum, 446
 closed,
 magnetic field, 285, 289f
 magnetic field intensity, 222, 227
 conduction, 303, 435, 438
 density, 417
 moving media, 582, 588
 stationary media, 582
 vector potential, 306
 in conductor, total, 452
 convection, 582
 definition, 185
 density, 211, 230, 312, 435, 449, 577ff
 definition, 188f
 displacement, 435f, 438f, 463
 density, 436, 437, 459, 649
 in dielectric, 438ff
 surface density, 459
 elementary, 278
 field, 282
 potential function, 281
 elements, 219ff
 interaction, 222ff
 energy,
 liberated, 187
 magnetic, 392f
 magnetic field, 649
 mutual, 406
 potential, 254f, 258, 260ff

- Current(s),
 energy
 proper, 406
 total, 406
 fast-varying,
 approximate theory, 559ff
 in cable, 550ff
 calculation by telegraph equation,
 563f
 properties, 549ff
 filaments, 194f, 226f
 flow through metal, 206f
 heat liberation, 185f, 201f, *see also*
 Heat, Joule
 in homogeneous magnetic medium,
 319
 induction in moving conductors,
 377ff
 infinite rectilinear, magnetic field
 intensity, 222
 interaction,
 magnetic energy, 403ff
 ponderomotive, 252ff
 line,
 definition, 226
 magnetic field intensity, 227
 magnetic moment, 288ff
 ponderomotive interaction, 252ff
 vector potential, 236
 lines, 194
 loops,
 interaction forces, 225f
 mechanical interaction, 254f
 magnetic field, 218ff, 285, 289f
 energy, 649
 intensity, 360
 total, 404
 magnetic moment, 278ff, 284f
 molecular, 303f, 435
 density, 308f, 329
 and magnetic moment, 304
 space, 308f
 surface, 309
 vector potential, 306ff
 potential function, 250f, 254f, 258f,
 321
 quasilinear, 197
 quasistationary,
 definition, 387
 interaction forces, 390f
 magnetic field, 492
 ring,
 magnetic field intensity, 222
 magnetic moment, 325
 self-inductance, 407ff
 space, 251
 density, 308f
- Current(s),
 steady,
 electric field, 191f
 in homogeneous cylindrical con-
 ductor, 190
 magnetic field, 245
 surface, 228, 243ff
 density, 243f, 556
 magnetic field intensity, 246f
 and vector potential, 247
 thermionic, 66
 density, 66ff
 unit power, 191
 units, 185
 varying,
 differential equations, 389
 distribution over conductor sec-
 tion, 446ff
 electromagnetic field, 387
 energy transformations in, 391ff
 induction interaction, 390
 magnetic moment, 500f
 vector potential, 500f
 vector potential, 236ff, 282ff, 319
 volume, mutual inductance, 260
- Decrement, logarithmic, 444
- Density,
 charge, 33, 42, 115, 134, 577ff
 current, 66ff, 188f, 211, 230, 243f,
 308f, 312, 329, 417, 435f, 439,
 449, 459, 556, 577ff
 electromagnetic field momentum, 650
 energy, 92, 154ff, 405, 412, 415, 517,
 568f, 649, 652
 force, 98, 162f, 168, 175, 227, 323f,
 360f, 364, 380, 419ff, 544, 547f
 magnetic charge, 358f, 364f
- Derivatives, spatial, 636
- Diamagnetism, 335ff
- Dielectrics, 108ff
 charge density, 115, 136f
 classes, 137ff
 displacement current in, 438ff
 electric field, 115, 595f
 energy, 150ff
 intensity, 122ff
 energy density, 155
 forces on, density, 162f, 168
 free energy, 161
 homogeneous,
 Coulomb's law, 125f
 electric field, 122ff
 point charge potential, 122ff
 isotropic, electric displacement, 118
 moving in electromagnetic field,
 595ff

- Dielectrics,**
 permittivity, 118f, 148f, 167
 polarizability, 116f, 138
 polarization, 112f, 116, 144, 146ff
 and field intensity, 125ff
 and temperature, 144
 ponderomotive forces in, 162ff
 with quasi-elastic dipoles,
 electric field, 155
 permittivity, 148f
 and density, 149
 and temperature, 149
 residual polarization, 302f
 with rigid dipoles,
 energy of electric field, 158
 fraction of polarizability, 149
 permittivity, 149f
 and density, 149f
 and temperature, 149f
 work of external forces, 158f
 solid,
 permittivity, 167
 striction stresses, 177
 structure, 108
- Differentiation, spatial,** 642
- Dipole(s), 55**
 coefficient of elasticity, 139
 elastic energy density, 157
 energy in external field, 89, 102
 field intensity, 64f
 forces, 98f, 102ff
 magnetic, 272, 281f, 501
 scalar field potential, 280
 moment, 279f
 moment, 109f, 139f
 quasi-elastic, 139
 rigid, polarization, 148
- Displacement, electric, 117f**
 definition, 648
 electric vector, 648
 and field intensity, 653
 Gauss's theorems 649
 isotropic dielectrics, 118
 lines, 120f
 point charge field, 648
 surface charges, 118f
- Dispersion,**
 frequency, 652
 spatial, 652
- Divergence, 620**
 definition, 622
 surface, 45
 volume, 45
- Domain, Weiss, 354**
- Effect;**
 Barnett, 347
 diamagnetic, 335, 338
- Effect,**
 Einstein-de Haas, 347
 Hall, 231ff
 paramagnetic, 333
 skin, 446ff, 549f
 anomalous, 454
 cylindrical conductors, 450ff
 normal, 454
 Vavilov-Cerenkov, 654
- Electric field, 27ff**
 capacitors, 39ff
 charge, 32, 46, 80
 and charge distribution, 45
 charged surfaces, 33ff
 in conductors, 38
 currents, 191f
 cylinder, 37
 charged, 39, 56
 in dielectrics, 115, 122ff, 155, 595ff
 energy, 92, 150ff
 density, 156f, 649
 free, 160, 567
 mutual density, 156, 157
 equations, 134, 237
 experimental study, 28
 induction, 426, 593f
 infinite plane, 35ff, 56
 intensity, *see* Intensity, electric field
 macroscopic, mean, 146
 ponderomotive forces, 180ff
 postulates of macroscopic theory,
 152, 159f
 radial, in rotating magnet, 592f
 sphere, 37f, 56
 charged, 39
 uniformly polarized, 127ff
 spherical surface, 37
 stationary, boundary conditions, 245
 stress tensor, 175ff
 superposition principle, 26
 work of forces, 45ff, 342f
- Electrodes, polarization, 85f**
- Electromagnets, field, 369**
- Electromotive forces,**
 contact, 204ff
 work, 205f
 definition, 198
 extraneous, 196ff
 work, 391
 induced, 378f, 388f, 424f, 589f
- Electron(s),**
 angular momentum, 332f
 forces on in magnetic field, 293f
 free, classical theory, 213
 kinetic energy, 214f
 free path, 213, 215
 heat capacity, 212f, 215

- Electron(s)**,
 kinetic energy in magnetic field, 339
 342
 Lorentz force on, 293f, 377f
 magnetic moment, 292
 motion in conductor, 208ff
 motion in constant magnetic field,
 233f
 precession, 330f, 334
 properties, 291f
 quantum theory, 213f
 spin, 292f
 velocity, 210f
Electrostriction, 159, 178
Emission, thermionic, 207
Energy,
 capacitor, 89f, 100f
 charge, 86ff, 94ff, 104, 111f
 circulation in static electromagnetic
 field, 540
 conductors, charged, 95f
 currents, 187, 254f, 258, 260ff, 392f,
 403ff, 649
 density, 156f, 405, 412, 415, 517,
 568f, 649, 652
 space, 462
 true, 155
 in vacuum, 412
 volume, 92, 154
 dipole, 89, 102
 electric field, 92, 150ff, 156f, 567
 electromagnetic, 573f, 652
 total, 442
 electrostatic field, 164
 field, localization, 461f
 free, 159
 density, 159, 568f
 electric field, 160, 567
 ferromagnetics, 564ff
 magnetic field, 413
 unit volume of dielectric, 161
 intrinsic,
 charge, 94f
 charged sphere, 96
 kinetic, electron, 339, 342
 magnetic, 392f, 403ff, 414f
 density, 405, 412
 magnetic field, 251, 412f
 potential, 104, 145, 251, 254f, 258,
 260ff, 276f, 281
 total field, 92, 154
 transformations,
 in circuit, 202ff
 in circuit opening and closing,
 396f
 in varying current field, 391ff
 units, 187f
 zero-point, 214
- Equation(s)**, *see also* Formula(s)
 Ampere's 226,
 Bessel, 451
 Boltzmann, 160
 continuity, 193, 433, 579f
 differential form, 433f
 d'Alembert, 473
 solution, 474ff
 electric field,
 differential, 447
 microscopic, 134
 electromagnetic field, 426
 macroscopic, 455ff
 microscopic, 577
 electromagnetic field potential,
 differential, 470ff
 field,
 microscopic, 134
 in moving media, 578
 force line, 61f
 Laplace, 66, 69, 78
 Lorenz-Lorentz, 143f
 magnetic field, 317
 differential, 239, 245f, 312ff, 357,
 447
 in magnetics, 325ff
 steady currents, 239, 245f
 magnetic field lines, differential, 262
 Maxwell's, 456ff
 differential, 650
 application, 580
 integral form, 650
 Poisson, 66, 473
 differential, 237
 telegraph, 561
 applicability to cylindrical cable,
 561ff
 varying currents, differential, 389
 wave, 473
 solution, 474ff
- Ether**, 62
Experiment(s),
 Eikhenvald's, 583ff, 596f
 Tolman's, 207ff
 Wilson's, 595f
- Factor**, *see also* Coefficient, Constant
 damping, 558f, 562
 reflection, 525, 532
Farad, 57
Ferrites, 652
Ferroelectrics, 150
 permittivity, 150
Ferromagnetics, 302, 315
 free energy, 564ff
 density, 356
 hysteresis, 356

- Ferromagnetics**,
 idealized, 356f, *see also* Magnet(s),
 permanent
 magnetic susceptibility, 315f
 magnetization, 346ff, 351ff
 magnetized state, 352f
 permanent magnetism, 315
 real, 357
 residual magnetization, 302f
 spontaneous magnetization, 353
- Ferromagnetism**,
 explanation, 348, 356
 and temperature, 353
- Field**,
 concept, 575
 currents, 282, 436
 energy, 406
 electric, *see* Electric field
 electromagnetic,
 advanced potentials, 484ff
 basic vectors in moving medium,
 585f
 delayed potentials, 482ff
 electric vector, 427
 fundamental equations, 426, 455ff
 microscopic, equations, 577
 momentum, 535ff
 density, 537
 varying currents, 387
 electromagnets, 369
 electrostatic,
 energy, 164
 intensity, 61
 potential, 50ff
 in vacuum, 23f
 fast-varying, heat losses, 556ff
 force,
 conservative, 46
 potential, 46
 magnetic, *see* Magnetic field
 magnetic dipole, scalar potential, 280
 magnetized magnetics, 303
 molecular, 348f, 355
 explanation, 355f
 non-circuital, 241
 oscillator, 504ff
 plane monochromatic wave, 529f
 point charge, potential, 52
 potential, 49, 241
 propagation velocity, 489f
 quasistationary, 490
 electric intensity, 491
 scalar, 610f
 sinks, 623
 solenoidal, 242
 sources, 43, 623
 abundance, 43
 strength, 43, 623
- Field**,
 space charge, potential, 52
 stationary, voltage, 427
 surface charge, potential, 52
 topological and physical properties
 269f
 total, 92
 electric energy, 92, 154
 varying,
 scalar potential, 472, 481f
 vector potential, 472, 481f
 vector, 610
 vortex, 242
 vortex-free, 241
- Filaments, current**, 194f, 226f
- Flux**,
 electric vector, 30ff
 electromagnetic energy, 464
 quasistationary varying currents,
 466f
 steady current, 466
 magnetic, 249f
 magnetic induction, 320ff
 vector, through surface, 616ff
- Force(s)**,
 Abraham, density, 547
 charge interaction, 23f
 in closed current system, 285ff.
- Coriolis**,
 applied to electron, 331
 on currents, 361
 in magnetic field, 319ff
- current loop interaction, 225f
 on current loop in magnetic field, 222
 on dielectric, density, 162f, 168
 on dipole, 98f, 102ff
 on discontinuity surface, 181
 electric, work, 47f, 187
 electromotive, *see* Electromotive
 force(s)
 on electron, 293f
 exchange, 355f
 generalized, 100, 102f
 lines, *see* Lines, force
 Lorentz, 231, 293f, 377f, 649
 on magnetic dipole, 281f
 on magnetics, 323ff
 on moving charge, 231
 ponderomotive, *see* Ponderomotive
 force(s)
 quasistationary current interaction,
 390f
 striction, 177
 on surface, charges, 96
- Formula(s)**, *see also* Equation(s)
 Fresnel, 524
 Green's, 70, 78, 643

- Formula, Thomson, 443
 Frames, reference, inertial, 383
 Frequency, 398
 cyclic, 398
 plasma, 655
 Function,
 Langevin, 351f
 potential, current, 250f, 254f, 258f,
 281, 321, 392f

 Gradient,
 lines, 613
 scalar, 612f
 Gyromagnetic phenomena, 345ff

 Heat,
 hysteresis, 571
 Joule, 201f
 determination, 463
 quantity liberated, 391, 442, 453
 in surface layer of conductor, 556
 unit quantity, 202
 losses, fast-varying fields, 556ff
 Peltier, 201
 Thomson, 201
 Hypothesis, Ampere's, 291
 Hysteresis, 315, 570f
 ferromagnetics, 356,
 heat, 571

 Index, refractive,
 and permittivity, 526f
 relative, 523
 Inductance, *see also* Self-inductance
 definition, 406
 mutual, 253f
 currents, 260
 and permeability, 321f
 Induction,
 electrostatic, 57
 magnetic, 320, 487f
 effective, 585, 605
 experimental determination, 322f
 and magnetic field intensity, 649
 vector, 312
 unipolar, 590f
 Integration,
 over surface, 617f
 over volume, 621
 Intensity, electric field, 27f, 231, 487f,
 see also Vector(s), electric
 between charged planes, 648
 in dielectrics, 122ff
 dipole, 64f
 effective, 141ff, 605
 and reference frame, 605

 Intensity, electric field,
 point charge, 648
 total, 425
 Intensity, magnetic field, 219f
 absolute unit, 298f
 circulation along contour, 269f,
 continuity, 229
 currents, 222, 227, 246f, 360
 definition, 649
 fictitious magnetic charge, 271f
 in magnetics, 312, 360
 measurement, 220
 permanent magnets, 360, 364
 and permeability, permanent mag-
 nets, 366ff
 Interaction,
 charged bodies, 23
 charges, 23f
 Invariance, gauge, 488

 Joule, 395

 Laplacian, 66
 Law,
 Ampere's, 226, 649
 Biot-Savart, 221, 649
 conservation,
 angular momentum, 541
 electricity, 193
 energy, 463
 momentum, 536
 total momentum, 549
 Coulomb's, 23ff, 648
 fictitious magnetic charges, 296
 generalized, 122, 125f
 magnetic charges, 361
 Curie's, 341, 345
 electromagnetic induction, 385
 in moving conductors, 379
 Gauss's, 32f, 114
 Joule's, 187, 191, 211
 Kirchhoff's,
 first, 192
 second, 199
 varying currents, 385f
 Lenz's, 395
 Ohm's, 184, 186, 189f, 211, 428, 431,
 445, 559
 alternating currents, 560
 differential form, 190, 197
 generalized, 197, 198
 moving conductors, 588
 varying currents, 385f
 Layer, double,
 electrical, 81ff
 magnetic, 272, 279
 potential, 273

- Light,
 dispersion, 526
 polarization plane, 513
 pressure, 532ff
 reflection,
 from metal surface, 530ff
 in moving dielectric, 600ff
 from moving mirror, 602
 refraction, in moving dielectric, 600ff
 velocity, 490
 in moving medium, 600
- Lines,
 current, 194
 displacement, 120f
 field, magnetic, 262
 force, 121
 electric, 61ff
 and magnetic, 263
 magnetic, 64, 221f, 262ff
 unclosed, 264ff
 gradient, 613
- Loops,
 current, mechanical interaction, 254f
 rectangular, in homogeneous magnetic field, 251f
- Machine, unipolar, 590ff
- Magnet(s),
 and currents in homogeneous medium, 363
 cylindrical, and solenoid, 310
 permanent, 302, 356f
 field, 358ff
 field intensity, 360, 364
 and permeability, 366ff
 forces on, 371ff
 poles, 360
 scalar potential, 360
- Magnetic field,
 atoms, 305
 boundary conditions, 314
 currents, 218ff, 245, 285, 289f, 404
 492
 energy, 649
 cylindrical solenoid, 247f
 energy density, 415, 568, 649
 equations,
 complete system, 317
 differential, 312ff, 357
 force lines, 262ff
 free energy, 413
 density, 568
 induction, 320, 437
 experimental determination, 322f
 intensity, *see* Intensity, magnetic field
 internal energy, 413
 line current contour, 369f
- Magnetic field,
 in magnetics, total energy, 412
 in moving dielectric, 595f
 permanent magnets, 358ff
 ponderomotive forces, 415ff
 potential energy, 251
 quasistationary, magnetic energy, 405
 scalar potential, 270f, 367ff
 sheets, 274f
 solenoid, 247f
 open, 369f
 sources, 291
 stress system, 423
 stress tensor, 421ff
 in superconductors, 216f
 vector potential, 306, 308, 318f
 vortex, topology, 268ff
 vortex space, 242
- Magnetic properties, explanation, 294
- Magnetics,
 definition, 302
 energy, 412
 field, 303
 forces on in magnetic field, 323ff
 magnetization, 304f
- Magnetism, absolute unit, 271
- Magnetization,
 diamagnetics, and electron spin, 345
 ferromagnetics, 346, 351ff
 induced, 356, 363
 and magnetic field intensity, 314ff,
 649
 magnetics, 304f
 monatomic gases, 330ff
 paramagnetics, 340f, 345ff, 413
 permanent, 302, 356
 cause, 365
 residual, 302
 slowly moving media, 583
 stationary media, 583
 vector, 304
- Method, images, 79f
- Molecule(s),
 dipole, potential energy, 145
 magnetic properties, 305
 polarizability, 138
 quasi-elastic, 138f
 rigid, 138
- Moment,
 dipole, 109f, 139f
 magnetic, 279
 electric, 54
 system of charges, 109f
 magnetic, 304
 atom, 332f, 343f, 355
 calculation, 341f
 current, 278ff, 284f, 288f, 325, 500f
 electron, 292

- Moment**,
 forces on closed currents, 288
 orbital, 344
 spin, 344
- Momentum**,
 angular, *see* Angular momentum
 electromagnetic field, 535ff
 density, 650
 volume density, 537
- Nabla**, 636
- Ohm**, 185
 international, 300f
- Operator**,
 Hamiltonian, 636
 Laplacian, 637
- Oscillations**,
 electric, in circuit with capacitor,
 444ff
 harmonic, undamped, 504f
- Oscillator**, 497
 emission, 510ff
 field, electric and magnetic, 504ff
 Hertzian, 498f
 resistance, 512
 magnetic, 501
 moment, 498, 502
 radiation, 510ff
 and wavelength, 511
 wave zone, 507f
- Paramagnetics**,
 in magnetic field, 324f
 magnetic susceptibility, 315f, 527
 and temperature, 341, 351
 magnetization, 340f, 346f, 413
 and electron spin, 345
 saturation, 341, 351
- Paramagnetism**, 338ff, 350f
- Parameter**, degeneracy, 214
- Permeability**, 316
 vacuum, 650
 virtual displacement of bodies, 417f
- Permittivity**,
 air, 490
 complex, 528
 dielectrics, 118f, 148ff, 167
 ferroelectrics, 150
 and field frequency, 655
 metals, 440
 and polarizability, 143
 and temperature, 148f
 vacuum, 650
- Phase shift**,
 current and electromotive force 399
 current and voltage, 446
- Piezoelectricity**, 116
- Plasma**,
 electromagnetic processes, 654f
 frequency, 655
- Point**, Curie, 353
- Polarizability**,
 dielectric, 138
 molecules, 138
- Polarization**,
 dielectric, 112f, 116, 125ff, 144, 146ff
 and electric vector, 648
 magnetic, 304, *see also* Magnetization
 rigid dipoles, 148
 saturation, 148
 slowly moving media, 583
 stationary media, 583
 and surface density of bound charges,
 648
 vector direction, 116f
- Poles**, magnet, 360
- Ponderomotive force(s)**, 96ff
 on charged body, 28
 on current-carrying conductor in
 magnetic field, 380
 density, 168, 323f, 360f, 364, 419ff,
 533, 544, 547f
 determination, 99ff
 in dielectrics, 162ff
 electric field, 180ff
 magnet interaction in homogeneous
 field, 364
 magnetic field, 415ff
 on permanent magnetic charges, 360f,
 364
 on permanent magnets, 371ff
 surface density, 98
 volume density, 98, 175, 227
 work, 248ff, 381f, 390
- Potential**,
 charge, 70ff, 75f, 110f, 113ff, 648
 and charge density, 65f
 charged hollow cylinder field, 56
 charged infinite plane field, 56
 charged sphere field, 56
 conductor, 56ff
 difference, 50ff
 capacitor, 101
 discontinuity surfaces, 72ff, 84
 double electrical layer, 81ff
 closed, 83
 open, 83f
 double magnetic layer, 273
 electromagnetic field,
 advanced, 484ff
 delayed, 482ff
 i ntegral expressions, 481

- Potential,
 electrostatic field, 50ff
 function,
 current, 250f, 254f, 258f, 321, 392f
 elementary current, 281
 in homogeneous dielectrics, 122ff
 point charge field, 52
 scalar, 270f, 280, 360, 367ff, 472,
 481f, 493f
 space charge field, 52
 surface charge field, 52
 units, 54
 vector, 236ff, 247, 282ff, 306ff
 318f, 322, 472, 481f, 495ff, 500f
- Power, current, 191
- Precession,
 electrons, 330f, 334
 Larmor, 335
- Product,
 scalar triple, 609
 vector triple, 609
- Protons, properties, 291f
- Quantities,
 electrical, dimensions in absolute
 system, 298
 magnetic, dimensions in absolute
 system, 298
 physical,
 local changes, 165f
 macroscopic, 131
 material changes, 165f
 microscopic, 130f
- Radiation, Cerenkov, 654, 655f
- Resistance, 185
 Hertzian oscillator, 512
 ohmic, 453f
 to fast-varying currents, 562
 pure metals at low temperatures,
 216
 radiation, 512
 real, 453
 fast-varying current, 453
 quasistationary varying current,
 453
 unit, practical, 185
- Resistivity, 189
- Ring, vortex, 268
- Self-inductance, 397
 conductor, 259f, 454
 external, 409f
 internal, 409f
 and permeability, 321f
 ring current, 407ff
 unit cable length, 410f
- Self-induction, in varying current cir-
 cuit, 397ff
- Sheets, magnetic, 272ff
 field, 274f
 potential energy, 276f, 281
- Solenoid, 247
 current, 310
 magnetic field, 247f
 open, field, 369f
 replacement with magnet, 277f
- Space,
 multiple-connected, 268
 properties, 62
- Spin, electron, 292f
- State, quasistationary, conditions, 490ff
- Strength,
 double electrical layer, 81ff
 field sources, 43
- Stress tensor, 172
 components, 172f
 electric field, 175ff
 electromagnetic field, 544
 magnetic field, 421ff
 Maxwell's, 177, 182
- Stresses,
 Maxwellian, 179
 striction, in liquids and solids, 177f
- Superconductivity, 216f, 651
- Surfaces,
 discontinuity,
 electric vector, 43
 electromagnetic vectors, 458f
 forces on, 181
 potential, 72ff, 84
 vectors, 245
 equipotential, 611
 level, 611
- Susceptibility, 340f
 diamagnetic, 345
 determination, 335ff
 magnetic, 315
 diamagnetics, 315f, 527
 ferromagnetics, 315f
 and field intensity, 315f
 paramagnetics, 315f, 341, 351, 527
- Tensor(s),
 energy-momentum, 538
 stress, *see* Stress tensor
 striction, 177
 symmetrical, 174
- Theorem,
 Boltzmann's, 145, 338ff
 Earnshaw, 106f
 Gauss's, 619ff
 electric displacement, 649
 magnetic induction, 649
 mathematical formulation, 622

- Theorem**
 Green's, 70f, 642ff
 Larmor's, 330ff
 Poynting's, 464
 Stokes's, 630
 uniqueness, 125, 467ff, 472
- Theory(ies),**
 electron, 575
 long-range interaction, 571ff
 short-range interaction, 572f
 slowly moving media, 576
 thermodynamic, of superconductivity, 217
 Weiss's, 348f, 353
- Time, effective, 477**
- Transformer, elementary theory, 400ff**
- Triplet, orthogonal, 609**
- Tubes, field, 63**
- Units,**
 absolute, 24, 54, 57, 185, 187, 271, 294ff, 298f
 electromagnetic, 297ff
 electrostatic, 223, 297ff
 Gaussian, 25, 223, 294ff
 practical, 25, 54, 57, 185, 187f, 299f, 395
 international, 300f
 rational system, 301
 symmetrical system, *see* Units, absolute
- Vector(s),**
 axial, 609
 circulation, 47, 626ff
 discontinuity surfaces, 245
 electric, 27, *see also* Intensity, electric field
 amplitude, 448f
 boundary condition, 43
 complex expression, 449
 discontinuity surfaces, 43
 divergence, 43
 flux, 30ff
 and potential, 134
 varying electromagnetic field, 427
 and volume charge distribution, 42
 electric moment, 495, 501
 electromagnetic,
 boundary conditions, 459f
 discontinuity surfaces, 458f
 field, and reference frame, 603ff
 Hertz, 501f
 magnetic field intensity, circulation, 240
 magnetic induction, 312
 magnetization, 304
 polar, 609
- Vector(s),**
 Poynting's, 464, 466, 510, 650
 direction in wave, 518
 scalar product, 608
 surface curl, 245
 surface sources, 45
 vector product, 608
- Velocity,**
 electromagnetic waves, 650
 light, 490
 in moving medium, 600
- Volt, 54**
- Voltage,**
 alternating current, 430ff
 definition, 186
 stationary field, 427
- Watt, 187f**
- Wave(s),**
 definition, 509
 electric field intensity, 598
 electromagnetic,
 field vectors, 517
 propagation in anisotropic media, 653
 propagation in cable, 555
 velocity, 650
 frequency, 521
 light, propagation in moving dielectric, 597ff
 monochromatic, 514
 in metal, 528f
 phase velocity, 518
 plane, 514, 517
 field vectors, 514f
 propagation in dielectric, 516
 plasma, 655f
 propagation,
 in conductors, 627f
 damping factor and frequency, 563
 phase velocity and frequency, 563
 reflected, 520
 reflection in dielectrics, 520ff
 refracted, 520
 refraction in dielectrics, 520ff
 spherical, 509
 velocity, 516
- Wavelength, 509**
- Work,**
 contact electromotive forces, 205f
 electric field forces, 45ff, 342f
 in dielectric with rigid dipoles, 158f
 in point charge field, 47f
 electric forces, 187
 extraneous electromotive forces, 391
 magnetic field forces, 248ff, 381
 steady current, 390
 total, 382

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The Theory of Elasticity

**Yu. AMENZADE, Corr. Mem. Azerb. SSR
Acad. Sci.**

This book contains relevant data from tensor analysis (the exposition of the fundamentals of the theory of elasticity is given at contemporary advanced level and in modern form), the plane problems of theory of elasticity are considered with the help of the method of functions of complex variables and the method of integral transforms. The book deals with the theory of rotation and bending of prismatic bodies, Hertz contact problem and certain axi-symmetric problems. It also contains theory of propagation of elastic waves in an infinite medium and surface waves of Rayleigh, etc. Examples of the theory of bending of thin sheets are given. The textbook excels in clarity and originality and is illustrated by numerous examples.

The textbook is intended for university students.

Solid State Physics

G. EPIFANOV, D. Sc.

This book deals with the fundamentals of solid state physics including the mechanical, thermal, electric and magnetic properties of solids, the contact, thermoelectric and galvanomagnetic phenomena in such bodies. The book may serve as a textbook in the study of the special course on solid state physics and of various subjects such as the physical foundations of electrical engineering, the physical foundations of microelectronics, etc. In addition it may prove to be helpful to readers of a wide range of engineering professions desiring to refresh their knowledge of the fundamentals of solid state physics.

The Special Theory of Relativity

V. UGAROV, Cand. Sc.

This is the second revised and enlarged edition of the book which was first published in 1969 under the same title. It is written as a textbook to be used both by university students and teachers as well as high school teachers. Although the general layout of the book has not been changed, the principles of the theory are given in more details now and greater attention is paid to the four-dimensional treat. Different ways of presenting the special theory of relativity are described. Sections devoted to the methodology and history of the special theory of relativity are added. The chapter on electrodynamics is enlarged. Finally a paper written by Academician V. L. Ginzburg and entitled "Who Created the Special Theory of Relativity and How it Was Done" is included as an addendum.

1

